Cat. Dor4.

Com-04
A/N. 70

Reference

09-10

(85)

AMARSINGH COLLEGE

LIGHT MORE LIGHT
A.S. COLLEGE

Checked
1976

Library

Class No. R510
Book No. T64P V. II
Acc. No. 8385

PRACTICAL MATHEMATICS

UNIVERSITY OF LONDON
EXAMINATION PAPERS

The Examination Papers of the University of London are now published by the University direct, and are obtainable from the Publications Branch, University of London, Senate House, London, W.C.1.

A complete list of papers and their cost, along with a detailed list of numerous collected sets of papers, will be sent post free upon application.

PRACTICAL MATHEMATICS

A TEXTBOOK COVERING THE SYLLABUS OF THE B.Sc. EXAMINATIONS IN THIS SUBJECT AND SUITABLE FOR ADVANCED CLASSES IN TECHNICAL COLLEGES

BY

LOUIS TOFT, M.Sc.

Late Head of the Mathematical Department
Royal Technical College, Salford

AND

A. D. D. McKAY, M.A.

Head of the Mathematical Department
Dundee Technical College

VOLUME II

LONDON
SIR ISAAC PITMAN & SONS, LTD.

First edition 1931
Reprinted 1935
Reprinted 1939
Reprinted 1940
Second edition 1942
Reprinted 1943
Revised reprint 1944
Reprinted 1946
Third edition 1952

A.S. college
Acc no 8385
510

SIR ISAAC PITMAN & SONS, LTD.
PITMAN HOUSE, PARKER STREET, KINGSWAY, LONDON, W.C.2
THE PITMAN PRESS, BATH
PITMAN HOUSE, LITTLE COLLINS STREET, MELBOURNE
27 BECKETTS BUILDINGS, PRESIDENT STREET, JOHANNESBURG

ASSOCIATED COMPANIES

PITMAN PUBLISHING CORPORATION
2 WEST 45TH STREET, NEW YORK

SIR ISAAC PITMAN & SONS (CANADA), LTD.
(INCORPORATING THE COMMERCIAL TEXT BOOK COMPANY)
PITMAN HOUSE, 381-383 CHURCH STREET, TORONTO

MADE IN GREAT BRITAIN AT THE PITMAN PRESS, BATH
E2—(T.611)

PREFACE TO THIRD EDITION

WE have accepted the advice of many teachers and have divided the original *Practical Mathematics* into two volumes so as to meet the needs of students preparing for an Engineering degree of the University of London. The present volume deals mainly with the subject-matter included in Part II of the syllabus in this subject. In order to cover this syllabus adequately we have added a considerable amount of new matter—in particular, Chapters treating of the Complex Variable, Vector and Scalar Products, and Partial Differential Equations—and have extended the treatment of Ordinary Differential Equations, Applied Mathematics, etc. We have also included a chapter on Nomograms. This volume should also prove suitable for classes doing work in Mathematics of degree standard in Technical Colleges.

The abbreviation (U.L.) following an example indicates that it is taken from an examination paper of the University of London. The author and publishers thank the Senate of the University and the University of London Press for permission to reprint these examples.

Thanks are also due to Mr. R. A. Downs, B.Sc., A.M.I.Mech.E. for preparing the illustrations taken from the original volume and to Mr. F. X. Merrick, B.Sc. for help with the same.

L. T.
A. D. D. McK.

AMAR SINGH COLLEGE
Acc. No.
LIBRARY, SRINAGAR.

CONTENTS

Acc. No.
LIBRARY. SRINAGAR.

Acc no 8385

CHAPTER I

DETERMINANTS—FINITE DIFFERENCES

1. Meaning of Determinants. If three quantities x, y, z are so related that

$$a_2x + b_2y + c_2z = 0$$

and

$$a_3x + b_3y + c_3z = 0$$

then, solving for x and y in terms of z in the usual way, we obtain

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{c_2a_3 - c_3a_2} = \frac{z}{a_2b_3 - a_3b_2} \quad . \qquad (I.1)$$

If, further, a third relation $a_1x + b_1y + c_1z = 0$ is also satisfied by x, y, z, we deduce by substitution from (I.1)

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0 \qquad (I.2)$$

We can, however, express this result in a more compact form as follows. The expression $b_2c_3 - b_3c_2$ is written as

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

and by analogy $c_2a_3 - c_3a_2$ is written as

$$\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}$$

and $a_2b_3 - a_3b_2$ as

$$\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

so that the relation (I.2) now becomes

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$$

and usually this is written

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \quad . \quad . \quad . \qquad (I.3)$$

The expression

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

consisting of two rows and two columns, is called a *determinant* of the second *order*, each of the two terms in its expansion $b_2c_3 - b_3c_2$ being the product of two quantities; similarly the expression

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

consisting of three rows and three columns, is called a determinant of the third order, each term in its expansion being the product of three quantities. The quantities a_1, a_2, b_1, b_2, etc., are called the *elements* of the determinant. It will be noted that each term in the expansion of the determinant contains one and only one element from each row, and one and only one element from each column. A determinant of the nth order consists of n rows and n columns, and its expansion contains $\underline{|n}$ terms since the suffixes 1, 2, 3, . . . n can be assigned to the n letters $a, b, c,$ . . . in $\underline{|n}$ possible ways.

EXAMPLE 1

Expand $\begin{vmatrix} 15 & 7 \\ 11 & 4 \end{vmatrix}$ and $\begin{vmatrix} 3p - 4q & 11p - 5q \\ 2p - q & 8p + q \end{vmatrix}$

$$\begin{vmatrix} 15 & 7 \\ 11 & 4 \end{vmatrix} = 15 \times 4 - 11 \times 7 = 60 - 77 = -17$$

$$\begin{aligned} \begin{vmatrix} 3p - 4q & 11p - 5q \\ 2p - q & 8p + q \end{vmatrix} &= (3p - 4q)(8p + q) - (2p - q)(11p - 5q) \\ &= 24p^2 - 29pq - 4q^2 - 22p^2 + 21pq - 5q^2 \\ &= 2p^2 - 8pq - 9q^2 \end{aligned}$$

EXAMPLE 2

Solve the equation $\begin{vmatrix} 2x & 5 \\ 9 & x + 3 \end{vmatrix} = \begin{vmatrix} 5 & 4 \\ 13 & 3x \end{vmatrix}$

Expanding both sides of the equation, we obtain

$$2x^2 + 6x - 45 = 15x - 52$$

$$\therefore \quad 2x^2 - 9x + 7 = 0$$

$$\therefore \quad (2x - 7)(x - 1) = 0$$

$$\therefore \quad x = 3{\cdot}5 \text{ or } 1$$

2. Expansion of Determinants of the Third Order. From the last article we have

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) \qquad (1.4)$$

and by a slight rearrangement of terms on the right-hand side we find that

$$\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \qquad (\text{I}.5)$$

and $\Delta = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$ (I.6)

Let A_1 denote the determinant obtained by ignoring the row and the column in which a_1 lies, i.e.

$$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

A_2 will then mean $\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$; B_1 will mean $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$

and so on. A_1, A_2, etc., are called the *minors* of a_1, a_2, etc., in the original determinant. We thus obtain from (I.5)

$$\Delta = a_1A_1 - b_1B_1 + c_1C_1 \qquad . \qquad . \qquad (\text{I}.7)$$

and from (I.6) $\qquad \Delta = a_1A_1 - a_2A_2 + a_3A_3 \qquad . \qquad . \qquad$ (I.8)

EXAMPLE 1

Expand $\begin{vmatrix} 3 & 4 & 5 \\ 5 & 3 & 4 \\ 4 & 5 & 3 \end{vmatrix}$

The determinant $= 3\begin{vmatrix} 3 & 4 \\ 5 & 3 \end{vmatrix} - 4\begin{vmatrix} 5 & 4 \\ 4 & 3 \end{vmatrix} + 5\begin{vmatrix} 5 & 3 \\ 4 & 5 \end{vmatrix}$

$= 3(9 - 20) - 4(15 - 16) + 5(25 - 12)$

$= 36$

Otherwise, the determinant $= 3\begin{vmatrix} 3 & 4 \\ 5 & 3 \end{vmatrix} - 5\begin{vmatrix} 4 & 5 \\ 5 & 3 \end{vmatrix} + 4\begin{vmatrix} 4 & 5 \\ 3 & 4 \end{vmatrix}$

$= 3(9 - 20) - 5(12 - 25) + 4(16 - 15)$

$= 36$

This method of expansion by minors is of great importance in the case of determinants of order higher than the third, but determinants of the third order may often be conveniently expanded by a method known as the *Rule of Sarrus*, which we shall now explain.

Write down the three columns of the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and repeat the first two columns.

$$\begin{array}{ccccc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}$$

The products of the elements on the lines running to the right from top to bottom give the positive terms in the expansion, and the products of the elements on the lines running to the right from bottom to top give the negative terms. Thus

$$\Delta = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1 \qquad (1.9)$$

EXAMPLE 2

Expand (1) $\begin{vmatrix} 2 & 5 & 7 \\ -3 & 8 & 6 \\ 9 & -2 & 11 \end{vmatrix}$ and (2) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

(1) $\begin{array}{ccccc} 2 & 5 & 7 & 2 & 5 \\ -3 & 8 & 6 & -3 & 8 \\ 9 & -2 & 11 & 9 & -2 \end{array}$

The determinant $= 2.8.11 + 5.6.9 + 7(-3)(-2) - 9.8.7 - (-2)6.2 - 11(-3)5$

$= 176 + 270 + 42 - 504 + 24 + 165$

$= 173$

(2) $\begin{array}{ccccc} a & h & g & a & h \\ h & b & f & h & b \\ g & f & c & g & f \end{array}$

The determinant $= abc + hfg + ghf - gbg - ffa - chh$

$= abc + 2fgh - af^2 - bg^2 - ch^2$

(This expansion occurs in two-dimensional co-ordinate geometry; it is the *discriminant* of the general equation of the second degree, namely, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.)

EXAMPLE 3

Solve the equation $\begin{vmatrix} x-1 & -6 & 2 \\ -6 & x-2 & -4 \\ 2 & -4 & x-6 \end{vmatrix} = 0$

Using the method of expansion by minors, we have

$$(x-1)(x^2-8x+12-16)+6(-6x+36+8)+2(24-2x+4)=0$$

which reduces to $x^3-9x^2-36x+324=0$

By the Remainder Theorem we find that $x = 6$ is a root; hence we can write the equation as

$$x^2(x-6)-3x(x-6)-54(x-6)=0$$

i.e. $$(x-6)(x^2-3x-54)=0$$

or $$(x-6)(x+6)(x-9)=0$$

$\therefore$ $$x = 6, -6, \text{ or } 9$$

3. **Properties of Determinants.** We shall establish certain properties of a determinant of the third order, but the reader should note that these are capable of application to a determinant of any order.

(i) THE VALUE OF A DETERMINANT REMAINS UNALTERED WHEN THE ROWS ARE CHANGED TO COLUMNS AND THE COLUMNS TO ROWS. In Art. 2 we showed that the expansion of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

can be expressed as $a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)$. Now this is also the expansion of the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

It follows that the two determinants are equivalent.

(ii) IF TWO ROWS (OR TWO COLUMNS) OF A DETERMINANT ARE INTERCHANGED, THE SIGN OF THE DETERMINANT IS ALTERED.

For $\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = b_1(a_2c_3-a_3c_2)-a_1(b_2c_3-b_3c_2)+c_1(b_2a_3-b_3a_2)$

$$= - a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$$

$$= - [a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$$

$$= - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(iii) IF TWO ROWS (OR TWO COLUMNS) OF A DETERMINANT ARE IDENTICAL, THE DETERMINANT VANISHES.

Let
$$\Delta = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix}$$

By interchanging the first two columns, we alter the sign of the determinant [by (ii)]; but we have still the same determinant after the interchange. Hence, $\Delta = - \Delta$; whence $\Delta = 0$.

(iv) IF THE ELEMENTS OF ANY ROW (OR ANY COLUMN) OF A DETERMINANT BE EACH MULTIPLIED BY THE SAME FACTOR, THE DETERMINANT IS MULTIPLIED BY THAT FACTOR.

Let
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then
$$\begin{vmatrix} pa_1 & pb_1 & pc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = pa_1A_1 - pb_1B_1 + pc_1C_1$$

$$= p(a_1A_1 - b_1B_1 + c_1C_1)$$

$$= p\Delta$$

A_1, B_1, C_1 being the minors of a_1, b_1, c_1 in Δ.

(v) IF EACH ELEMENT OF ANY ROW (OR COLUMN) OF A DETERMINANT CONSISTS OF THE ALGEBRAIC SUM OF r TERMS, THE DETERMINANT IS EQUIVALENT TO THE SUM OF r OTHER DETERMINANTS IN EACH OF WHICH THE ELEMENTS CONSIST OF SINGLE TERMS.

For
$$\begin{vmatrix} a_1 + l_1 - m_1 & b_1 & c_1 \\ a_2 + l_2 - m_2 & b_2 & c_2 \\ a_3 + l_3 - m_3 & b_3 & c_3 \end{vmatrix}$$

$$= (a_1 + l_1 - m_1).A_1 - (a_2 + l_2 - m_2).A_2 + (a_3 + l_3 - m_3).A_3$$

$$= (a_1.A_1 - a_2.A_2 + a_3.A_3) + (l_1A_1 - l_2.A_2 + l_3.A_3) - (m_1.A_1 - m_2.A_2 + m_3.A_3)$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} l_1 & b_1 & c_1 \\ l_2 & b_2 & c_2 \\ l_3 & b_3 & c_3 \end{vmatrix} - \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}$$

A_1, A_2, A_3 being the minors of a_1, a_2, a_3 in the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(vi) IF THE ELEMENTS OF ANY ROW (OR COLUMN) ARE INCREASED OR DIMINISHED BY EQUIMULTIPLES OF THE CORRESPONDING ELEMENTS OF ANY OTHER ROW (OR COLUMN), THE VALUE OF THE DETERMINANT IS UNALTERED. For, using the results (iv) and (v), we find

$$\begin{vmatrix} a_1 + u.b_1 + v.c_1 & b_1 & c_1 \\ a_2 + u.b_2 + v.c_2 & b_2 & c_2 \\ a_3 + u.b_3 + v.c_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + u \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + v \begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

since the other two determinants vanish [by (iii)].

The result established in (vi) is often of great assistance in the simplification of determinants.

EXAMPLE 1

$$\text{Evaluate } \Delta = \begin{vmatrix} 3 & 5 & 8 & 3 \\ 2 & -3 & -5 & -10 \\ 4 & 1 & 6 & 7 \\ 5 & -4 & -2 & -4 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 3 & 5 & 8-3-5 & 3-3 \\ 2 & -3 & -5-2-(-3) & -10-2 \\ 4 & 1 & 6-4-1 & 7-4 \\ 5 & -4 & -2-5-(-4) & -4-5 \end{vmatrix}$$

[subtracting sum of first two columns from third column, and subtracting first column from fourth column]

$$= \begin{vmatrix} 3 & 5 & 0 & 0 \\ 2 & -3 & -4 & -12 \\ 4 & 1 & 1 & 3 \\ 5 & -4 & -3 & -9 \end{vmatrix}$$

$$= 3\begin{vmatrix} 3 & 5 & 0 & 0 \\ 2 & -3 & -4 & -4 \\ 4 & 1 & 1 & 1 \\ 5 & -4 & -3 & -3 \end{vmatrix} \text{[taking out common factor of elements of fourth column]}$$

$= 0$ (since two columns are identical)

EXAMPLE 2

Solve the equation $\begin{vmatrix} p+x & q+x & r+x \\ q+x & r+x & p+x \\ r+x & p+x & q+x \end{vmatrix} = 0$

We have, by adding the sum of the second and third columns to the first column,

$$\begin{vmatrix} p+q+r+3x & q+x & r+x \\ p+q+r+3x & r+x & p+x \\ p+q+r+3x & p+x & q+x \end{vmatrix} = 0$$

$$\therefore (p+q+r+3x)\begin{vmatrix} 1 & q+x & r+x \\ 1 & r+x & p+x \\ 1 & p+x & q+x \end{vmatrix} = 0$$

$$\therefore (p+q+r+3x)\begin{vmatrix} 1 & q+x & r+x \\ 0 & r-q & p-r \\ 0 & p-q & q-r \end{vmatrix} = 0 \text{ [taking first row from second and third in turn]}$$

$$\therefore (p+q+r+3x)\,[(r-q)(q-r)-(p-q)(p-r)] = 0$$

$$\therefore x = -\frac{p+q+r}{3}, \text{ provided that } (p-q)(r-p) \neq (q-r)^2$$

4. Solution of a System of Linear Equations by means of Determinants. We shall first obtain a solution in the case of three equations

$$a_1x + b_1y + c_1z + k = 0 \quad . \quad . \quad . \quad (I.10)$$

$$a_2x + b_2y + c_2z + l = 0 \quad . \quad . \quad . \quad (I.11)$$

$$a_3x + b_3y + c_3z + m = 0 \quad . \quad . \quad . \quad (I.12)$$

Multiplying equation (I.10) by A_1, equation (I.11) by $-A_2$, equation (I.12) by A_3, where A_1, A_2, A_3 are the minors of a_1, a_2, a_3 in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and adding the results, we obtain

$$(a_1A_1 - a_2A_2 + a_3A_3)x + (b_1A_1 - b_2A_2 + b_3A_3)y$$
$$+ (c_1A_1 - c_2A_2 + c_3A_3)z$$
$$+ (kA_1 - lA_2 + mA_3) = 0 \ . \qquad (I.13)$$

Now $a_1A_1 - a_2A_2 + a_3A_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$b_1A_1 - b_2A_2 + b_3A_3 = \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = 0 \text{ (by Art. 3 (iii));}$$

$$c_1A_1 - c_2A_2 + c_3A_3 = \begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} = 0 \text{ (by Art. 3 (iii));}$$

$$kA_1 - lA_2 + mA_3 = \begin{vmatrix} k & b_1 & c_1 \\ l & b_2 & c_2 \\ m & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 & k \\ b_2 & c_2 & l \\ b_3 & c_3 & m \end{vmatrix}$$

(since two interchanges of columns leave the determinant unaltered in value).

It follows then from (I.13) that

$$x = -\frac{\begin{vmatrix} b_1 & c_1 & k \\ b_2 & c_2 & l \\ b_3 & c_3 & m \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

or

$$\frac{-x}{\begin{vmatrix} b_1 & c_1 & k \\ b_2 & c_2 & l \\ b_3 & c_3 & m \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

By similar methods we find

$$\frac{y}{\begin{vmatrix} c_1 & k & a_1 \\ c_2 & l & a_2 \\ c_3 & m & a_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \text{ and } \frac{-z}{\begin{vmatrix} k & a_1 & b_1 \\ l & a_2 & b_2 \\ m & a_3 & b_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

The solution of the system can, therefore, be written

$$\frac{-x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{-z}{\Delta_3} = \frac{1}{\Delta_0} \quad . \quad . \quad . \quad (I.14)$$

where Δ_0 is the determinant formed by the coefficients when we omit the column of absolute terms, Δ_1 the determinant formed by the coefficients when we omit the column of terms involving x, and so on, the columns following one another in cyclic order.

EXAMPLE 1

Solve by means of determinants the equations

$$7x - 12y + 5z + 18 = 0$$
$$3x + 8y - 2z - 40 = 0$$
$$10x - 7y + 4z - 11 = 0$$

The solution is $\dfrac{-x}{\Delta_1} = \dfrac{y}{\Delta_2} = \dfrac{-z}{\Delta_3} = \dfrac{1}{\Delta_0}$

where $\Delta_0 = \begin{vmatrix} 7 & -12 & 5 \\ 3 & 8 & -2 \\ 10 & -7 & 4 \end{vmatrix} = 7(32 - 14) + 12(12 + 20) + 5(-21 - 80)$

$$= 7 \times 18 + 12 \times 32 - 5 \times 101 = 5$$

$$\Delta_1 = \begin{vmatrix} -12 & 5 & 18 \\ 8 & -2 & -40 \\ -7 & 4 & -11 \end{vmatrix} = -12(22 + 160) - 5(-88 - 280) + 18(32 - 14)$$

$$= -12 \times 182 + 5 \times 368 + 18 \times 18 = -20$$

$$\Delta_2 = \begin{vmatrix} 5 & 18 & 7 \\ -2 & -40 & 3 \\ 4 & -11 & 10 \end{vmatrix} = 5(-400 + 33) - 18(-20 - 12) + 7(22 + 160)$$

$$= -5 \times 367 + 18 \times 32 + 7 \times 182 = 15$$

$$\Delta_3 = \begin{vmatrix} 18 & 7 & -12 \\ -40 & 3 & 8 \\ -11 & 10 & -7 \end{vmatrix} = 18(-21 - 80) - 7(280 + 88) - 12(-400 + 33)$$

$$= -18 \times 101 - 7 \times 368 + 12 \times 367 = 10$$

Hence, $\dfrac{-x}{-20} = \dfrac{y}{15} = \dfrac{-z}{10} = \dfrac{1}{5}$

and, therefore, $x = 4$, $y = 3$, $z = -2$

The solution of a system of four or more linear equations can be effected by a method similar to the one employed above. There is a point to be noted in the matter of signs. In order to preserve cyclic order we changed the determinant

$$\begin{vmatrix} k & b_1 & c_1 \\ l & b_2 & c_2 \\ m & b_3 & c_3 \end{vmatrix}$$

into its equivalent

$$\begin{vmatrix} b_1 & c_1 & k \\ b_2 & c_2 & l \\ b_3 & c_3 & m \end{vmatrix}$$

the process requiring two interchanges of columns, and accordingly leaving the determinant unaltered. If, however, this determinant had been of even order we should have required an odd number of interchanges of columns to bring k, l, . . . from the first column to the last, and in consequence the determinant would have been altered in sign, and no negative sign would have appeared in front of x in the final solution. Similar arguments can be applied to the cases of the other unknowns. It follows that the solution of a system of n linear equations in x, y, z, u, v, w . . . is

$$\frac{(-1)^n x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{(-1)^n z}{\Delta_3} = \frac{u}{\Delta_4} = \ \ldots = \frac{1}{\Delta_0} \quad . \qquad \text{(I.15)}$$

where Δ_0 is the determinant formed by the coefficients with the column of absolute terms omitted, Δ_1 the determinant formed by the coefficients with the column of x— coefficients omitted, and so on, the columns following one another in cyclic order.

The solution of the system of four linear equations

$$a_1x + b_1y + c_1z + d_1u + k = 0 \quad . \quad . \qquad \text{(I.16)}$$

$$a_2x + b_2y + c_2z + d_2u + l = 0 \quad . \quad . \qquad \text{(I.17)}$$

$$a_3x + b_3y + c_3z + d_3u + m = 0 \quad . \quad . \qquad \text{(I.18)}$$

$$a_4x + b_4y + c_4z + d_4u + n = 0 \quad . \quad . \qquad \text{(I.19)}$$

is, therefore, $$\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{u}{\Delta_4} = \frac{1}{\Delta_0} \qquad (1.20)$$

EXAMPLE 2

Solve the equations

$$5x - 3y + 7z + 2u = -15 \qquad (1)$$

$$3x + 8y - 6z - 9u = 31 \qquad (2)$$

$$8x - y + z - 10u = 1 \qquad (3)$$

$$x + 3y + 5z + 12u = 5 \qquad (4)$$

The value of x is given by $\dfrac{x}{\Delta_1} = \dfrac{1}{\Delta_0}$

$$\text{where } \Delta_0 = \begin{vmatrix} 5 & -3 & 7 & 2 \\ 3 & 8 & -6 & -9 \\ 8 & -1 & 1 & -10 \\ 1 & 3 & 5 & 12 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -18 & -18 & -58 \\ 0 & -1 & -21 & -45 \\ 0 & -25 & -39 & -106 \\ 1 & 3 & 5 & 12 \end{vmatrix}$$ [taking 1st row -5 times 4th row, 2nd row $-$ 3 times 4th row, 3rd row $-$ 8 times 4th row]

$$= \begin{vmatrix} 18 & 18 & 58 \\ 1 & 21 & 45 \\ 25 & 39 & 106 \end{vmatrix}$$

$$= 2\begin{vmatrix} 9 & 0 & 29 \\ 1 & 20 & 45 \\ 25 & 14 & 106 \end{vmatrix}$$ [taking out common factor of 1st row and then subtracting 1st column from 2nd column]

$$= 2[9(2\,120 - 630) + 29(14 - 500)]$$

$$= 2[-684] = -1\,368$$

$$\Delta_1 = \begin{vmatrix} -3 & 7 & 2 & 15 \\ 8 & -6 & -9 & -31 \\ -1 & 1 & -10 & -1 \\ 3 & 5 & 12 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 4 & 32 & 18 \\ 0 & 2 & -89 & -39 \\ -1 & 1 & -10 & -1 \\ 0 & 8 & -18 & -8 \end{vmatrix}$$ [taking 1st row $-$ 3 times 3rd row, 2nd row + 8 times 3rd row, 4th row + 3 times 3rd row]

$$= -\begin{vmatrix} 4 & 32 & 18 \\ 2 & -89 & -39 \\ 8 & -18 & -8 \end{vmatrix}$$

$$= -8\begin{vmatrix} 1 & 16 & 9 \\ 1 & -89 & -39 \\ 2 & -9 & -4 \end{vmatrix}$$ [taking out common factor of 1st row, and of 3rd row, and then of 1st column]

$$= -8\begin{vmatrix} 1 & 16 & 9 \\ 0 & -105 & -48 \\ 0 & -41 & -22 \end{vmatrix}$$ [taking 2nd row − 1st row, and 3rd row − twice 1st row]

$$= -8 \times 3\begin{vmatrix} 35 & 16 \\ 41 & 22 \end{vmatrix} = -24(770 - 656) = -24 \times 114 = -2\,736$$

$$\therefore \quad x = \frac{\Delta_1}{\Delta_0} = \frac{-2\,736}{-1\,368} = 2$$

We can obtain y, z, u, in like manner, but it is easier to substitute the value of x now obtained in three of the given equations, and thereby reduce the system to three linear equations.

Substituting in equations (1), (3), and (4), we obtain—

$$-3y + 7z + 2u + 25 = 0$$
$$-y + z - 10u + 15 = 0$$
$$3y + 5z + 12u - 3 = 0$$

The solution of these equations is—

$$\frac{-y}{\begin{vmatrix} 7 & 2 & 25 \\ 1 & -10 & 15 \\ 5 & 12 & -3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & 25 & -3 \\ -10 & 15 & -1 \\ 12 & -3 & 3 \end{vmatrix}} = \frac{-u}{\begin{vmatrix} 25 & -3 & 7 \\ 15 & -1 & 1 \\ -3 & 3 & 5 \end{vmatrix}} = \frac{1}{\begin{vmatrix} -3 & 7 & 2 \\ -1 & 1 & -10 \\ 3 & 5 & 12 \end{vmatrix}}$$

and on expansion of the determinants this becomes

$$\frac{-y}{656} = \frac{z}{984} = \frac{-u}{328} = \frac{1}{-328}$$

whence $y = 2$, $z = -3$, $u = 1$

5. **Elimination.** In Art. 1 we deduced the condition that the three equations

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

should be satisfied by the same values of x, y, z (assumed not all zero). The relation we obtained, viz.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

is called the *eliminant* of the given equations, and the process of finding it is called *eliminating x, y, z between the equations*. From the point of view of co-ordinate geometry this eliminant expresses the condition that the three planes, represented by the three given equations, should have a common line of intersection.

A system of n linear equations in n unknowns will, in general, give unique values for the unknowns; if, however, the number of equations is greater than the number of unknowns, then values for the unknowns which will simultaneously satisfy all the equations cannot, in general, be found. In the case where such values can be found, only n of the equations are independent, and the system is said to be *consistent*.

Thus, if values of x, y, z can be found to satisfy simultaneously the four equations

$$a_1x + b_1y + c_1z + k = 0 \qquad \text{(I.21)}$$

$$a_2x + b_2y + c_2z + l = 0 \qquad \text{(I.22)}$$

$$a_3x + b_3y + c_3z + m = 0 \qquad \text{(I.23)}$$

$$a_4x + b_4y + c_4z + n = 0 \qquad \text{(I.24)}$$

the system is consistent. We have now to obtain the condition that this should be so—in other words, we require the *eliminant* of the four equations.

Multiply equation (I.21) by K, (I.22) by $-L$, (I.23) by M, and (I.24) by $-N$ (where K, L, M, N are the minors of k, l, m, n in the determinant)

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & k \\ a_2 & b_2 & c_2 & l \\ a_3 & b_3 & c_3 & m \\ a_4 & b_4 & c_4 & n \end{vmatrix}$$

and add the results.

$$\therefore x(a_1K - a_2L + a_3M - a_4N) + y(b_1K - b_2L + b_3M - b_4N)$$
$$+ z(c_1K - c_2L + c_3M - c_4N) + kK - lL + mM - nN = 0$$

The coefficients of x, y, and z in this equation vanish by reason of the property proved in Art. 3 (iii); hence

$$kK - lL + mM - nN = 0$$

The eliminant is therefore

$$\begin{vmatrix} a_1 & b_1 & c_1 & k \\ a_2 & b_2 & c_2 & l \\ a_3 & b_3 & c_3 & m \\ a_4 & b_4 & c_4 & n \end{vmatrix} = 0 \qquad . \quad . \quad . \qquad (1.25)$$

EXAMPLE 1

Eliminate a, b, and c from the equations $p(b-c) = a$; $q(c-a) = b$; $r(a-b) = c$

We write the equations—

$$a - pb + pc = 0$$
$$qa + b - qc = 0$$
$$ra - rb - c = 0$$

The eliminant is then $\begin{vmatrix} 1 & -p & p \\ q & 1 & -q \\ r & -r & -1 \end{vmatrix} = 0$

i.e. $\begin{vmatrix} 1 & -p & p \\ 1 & 1 & -q \\ -1 & -r & -1 \end{vmatrix} = 0$ [adding sum of 2nd and 3rd columns to 1st column]

i.e. $\begin{vmatrix} 1 & -p & p \\ 0 & 1+p & -(p+q) \\ 0 & -(r+p) & (p-1) \end{vmatrix} = 0$ [subtracting 1st row from 2nd row, and adding 1st row to 3rd row]

i.e.
$$(p+1)(p-1) - (r+p)(p+q) = 0$$

whence
$$pq + qr + rp + 1 = 0$$

EXAMPLE 2

Eliminate x, y, z, λ from the equations

$$\frac{ax + hy + gz}{l} = \frac{hx + by + fz}{m} = \frac{gx + fy + cz}{n} = -\lambda$$

and $\qquad lx + my + nz = 0$

We write the equations

$$ax + hy + gz + l\lambda = 0$$
$$hx + by + fz + m\lambda = 0$$
$$gx + fy + cz + n\lambda = 0$$
$$lx + my + nz + 0 = 0$$

The required eliminant is therefore $\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0$

6. **Product of Two Determinants.** Consider the determinant

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$$

By the property established in Art. 3 (v), this determinant

$$= \begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 \\ a_2\alpha_1 & a_2\alpha_2 \end{vmatrix} + \begin{vmatrix} a_1\alpha_1 & b_1\beta_2 \\ a_2\alpha_1 & b_2\beta_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & a_1\alpha_2 \\ b_2\beta_1 & a_2\alpha_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & b_1\beta_2 \\ b_2\beta_1 & b_2\beta_2 \end{vmatrix}$$

$$= a_1a_2 \begin{vmatrix} \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 \end{vmatrix} + \alpha_1\beta_2 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \beta_1\alpha_2 \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} + \beta_1\beta_2 \begin{vmatrix} b_1 & b_1 \\ b_2 & b_2 \end{vmatrix}$$

$$= 0 + \alpha_1\beta_2 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} - \beta_1\alpha_2 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + 0$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} . (\alpha_1\beta_2 - \alpha_2\beta_1)$$

$$\therefore \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \quad . \qquad \text{(I.26)}$$

Thus the product of the two determinants of the second order

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \text{ and } \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}$$

is a determinant of the second order, viz.

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$$

the method of formation of the elements of this latter determinant being evident.

EXAMPLE 1

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1^2 + b_1^2 & a_1a_2 + b_1b_2 \\ a_2a_1 + b_2b_1 & a_2^2 + b_2^2 \end{vmatrix}$$

$$= (a_1^2 + b_1^2)(a_2^2 + b_2^2) - (a_1a_2 + b_1b_2)^2 = (a_1b_2 - a_2b_1)^2$$

Consider next the determinant

$$\Delta = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

$$(L_1) \quad (L_2) \quad (L_3) \quad (M_1) \quad (M_2) \quad (M_3) \quad (N_1) \quad (N_2) \quad (N_3)$$

Using the property of Art. 3 (v) we can expand this determinant as the sum of 27 determinants. If the columns be denoted by L_1, L_2, L_3, M_1, M_2, M_3, N_1, N_2, N_3 as shown, it is not difficult to see that the determinants such as (L_2, M_2, N_1), where the suffixes of two of the letters L, M, N are equal, all vanish. The determinants which do not vanish can be obtained by giving to the three letters L, M, N the suffixes 1, 2, 3 arranged in all possible ways. These determinants are, therefore, *six* in number

$$\begin{aligned} \therefore \Delta &= (L_1M_2N_3) + (L_1M_3N_2) + (L_2M_3N_1) + (L_2M_1N_3) \\ &\qquad + (L_3M_1N_2) + (L_3M_2N_1) \\ &= \alpha_1\beta_2\gamma_3 \,.\, \Delta_1 - \alpha_1\gamma_2\beta_3 \,.\, \Delta_1 + \beta_1\gamma_2\alpha_3 \,.\, \Delta_1 - \beta_1\alpha_2\gamma_3 \,.\, \Delta_1 \\ &\qquad + \gamma_1\alpha_2\beta_3 \,.\, \Delta_1 - \gamma_1\beta_2\alpha_3 \,.\, \Delta_1 \end{aligned}$$

$$\text{where } \Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{aligned} &= \Delta_1 (\alpha_1\beta_2\gamma_3 - \alpha_1\beta_3\gamma_2 - \alpha_2\beta_1\gamma_3 + \alpha_2\beta_3\gamma_1 + \alpha_3\beta_1\gamma_2 - \alpha_3\beta_2\gamma_1) \\ &= \Delta_1 \left[\alpha_1 \begin{vmatrix} \beta_2 & \gamma_2 \\ \beta_3 & \gamma_3 \end{vmatrix} - \alpha_2 \begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_3 & \gamma_3 \end{vmatrix} + \alpha_3 \begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix}\right] \\ &= \Delta_1 \times \Delta_2, \text{ where } \Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \end{aligned}$$

Thus $$\Delta_1 \times \Delta_2 = \Delta$$

EXAMPLE 2

Express as a determinant

$$\begin{vmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{vmatrix}^2$$

$$\begin{vmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{vmatrix} \times \begin{vmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{vmatrix} = \begin{vmatrix} 0 + z^2 + y^2 & 0 + 0 + xy & 0 + zx + 0 \\ 0 + 0 + xy & z^2 + 0 + x^2 & zy + 0 + 0 \\ 0 + zx + 0 & yz + 0 + 0 & y^2 + x^2 + 0 \end{vmatrix}$$

$$= \begin{vmatrix} y^2 + z^2 & xy & zx \\ xy & z^2 + x^2 & yz \\ zx & yz & x^2 + y^2 \end{vmatrix}$$

EXAMPLE 3

Evaluate as a determinant $\begin{vmatrix} 2 & 3 & 1 \\ 3 & 2 & -2 \\ 4 & -4 & 3 \end{vmatrix} \times \begin{vmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \\ -1 & 2 & 2 \end{vmatrix}$

$$\text{The product} = \begin{vmatrix} 2+9-2 & 4+3+3 & -2+6+2 \\ 3+6+4 & 6+2-6 & -3+4-4 \\ 4-12-6 & 8-4+9 & -4-8+6 \end{vmatrix}$$

$$= \begin{vmatrix} 9 & 10 & 6 \\ 13 & 2 & -3 \\ -14 & 13 & -6 \end{vmatrix}$$

$$= \begin{vmatrix} 35 & 14 & 0 \\ 13 & 2 & -3 \\ -40 & 9 & 0 \end{vmatrix}$$ [taking 1st row + twice 2nd row, and 3rd row − twice 2nd row]

$$= 3\begin{vmatrix} 35 & 14 \\ -40 & 9 \end{vmatrix} = 3(315 + 560) = 3 \times 875 = 2\,625$$

7. **Interpolation by the Method of Finite Differences.** Suppose we have a given series of values of $y = f(x)$ for the series of values of x given by a, $a + h$, $a + 2h$, $a + 3h$, . . . $a + nh$, where n is any whole number and a and h are constants. The following table defines the quantities known as finite differences—

Values of x	Values of y	First Differences	Second Differences	Third Differences	Fourth Differences	Fifth Differences	Etc.
a	y_0						
		Δy_0					
$a + h$	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
$a + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
$a + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		etc.
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
$a + 4h$	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$		$\Delta^5 y_2$	
$a + 5h$	y_5		$\Delta^2 y_4$		$\Delta^4 y_3$		
		Δy_5		$\Delta^3 y_4$		$\Delta^5 y_3$	etc.
$a + 6h$	y_6		$\Delta^2 y_5$		$\Delta^4 y_4$		
		Δy_6		$\Delta^3 y_5$		$\Delta^5 y_4$	
etc.	etc.				$\Delta^4 y_5$		

Each number in a difference column represents the difference between the two numbers adjacent to it in the next column on the left. Thus

$$\Delta y_n = y_{n+1} - y_n$$

and $$\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n$$

or, in general $$\Delta^r y_n = \Delta^{r-1} y_{n+1} - \Delta^{r-1} y_n$$

where r and n are integers.

The quantities in the difference columns are known as *finite differences*, and are referred to as first, second, third, etc., differences. It is possible to express any of the values of y in terms of any other of the values of y and the differences. Thus

$$\begin{aligned} y_3 &= y_2 + \Delta y_2 = y_1 + \Delta y_1 + \Delta y_2 = y_1 + \Delta y_1 + \Delta y_1 + \Delta^2 y_1 \\ &= y_1 + 2\Delta y_1 + \Delta^2 y_1 \end{aligned}$$

We shall express y_n in terms of y_0 and differences involving y_0, writing any expression such as

$$y + n\Delta y + \frac{n(n-1)}{\underline{|2}} \Delta^2 y + \frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{\underline{|3}} \Delta^3 y + \ldots + \Delta^n y$$

in the convenient symbolical form $(1 + \Delta)^n y$.

From the method of formation of the differences, we have

$$\begin{aligned} y_1 &= y_0 + \Delta y_0 = (1 + \Delta) y_0 \\ y_2 &= y_1 + \Delta y_1 \\ &= y_0 + \Delta y_0 + \Delta y_0 + \Delta^2 y_0 \\ &= y_0 + 2\Delta y_0 + \Delta^2 y_0 = (1 + \Delta)^2 y_0 \\ y_3 &= y_2 + \Delta y_2, \text{ or since } \Delta y_2 = \Delta y_1 + \Delta^2 y_1 \\ y_3 &= y_0 + 2\Delta y_0 + \Delta^2 y_0 + \Delta y_0 + \Delta^2 y_0 + \Delta^2 y_0 + \Delta^3 y_0 \\ &= y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0 = (1 + \Delta)^3 y_0 \end{aligned}$$

The reader will see that the coefficients of the terms involving y_0 in these relations being 1, 1; 1, 2, 1; 1, 3, 3, 1 respectively are those

Acc no 8385

of the terms of the expanded forms of $x + a$, $(x + a)^2$, $(x + a)^3$, respectively, and by analogy with the binomial theorem, that

$$y_n = y_0 + n\Delta y_0 + \frac{n\,.\,\overline{n-1}}{\underline{|2}}\,\Delta^2 y_0 + \frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{\underline{|3}}\,\Delta^3 y_0 + \ldots + \Delta^n y_0 = (1 + \Delta)^n y_0 \qquad \text{(I.27)}$$

Writing Δy for y, and $\Delta^{r+1} y_0$ for $\Delta^r y_0$, we have also

$$\Delta y_n = \Delta y_0 + n\Delta^2 y_0 + \frac{n\,.\,\overline{n-1}}{\underline{|2}}\,\Delta^3 y_0 + \frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{\underline{|3}}\,\Delta^4 y_0 + \ldots$$

EXAMPLE 1

Find an expression for the $(n + 1)$th term of the sequence of numbers 5, 9, 19, 38, 74, 140, 254, 439, and continue the sequence to include another term.

Tabulating as below, we find the differences up to $\Delta^5 y$.

Values of y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
$y_0 = 5$					
	4				
$y_1 = 9$		6			
	10		3		
$y_2 = 19$		9		5	
	19		8		0
$y_3 = 38$		17		5	
	36		13		0
$y_4 = 74$		30		5	
	66		18		0
$y_5 = 140$		48		5	
	114		23		
$y_6 = 254$		71			
	185				
$y_7 = 439$					

We can carry on the sequence to any required stage by continuing the noughts downwards in the $\Delta^5 y$ column, and making corresponding additions to the other columns. Thus, to find the value of y_8, we add another nought in the $\Delta^5 y$ column, and another 5 to the $\Delta^4 y$ column. The number we must add in the $\Delta^3 y$ column is then $23 + 5$, i.e. 28; the number in the $\Delta^2 y$ column $28 + 71$, i.e. 99; the number in the Δy column $99 + 185$, i.e. 284; and the number in the y column $284 + 439$, i.e. 723. Hence, the next term in the sequence is 723.

We shall here find the general term, using (I.27), and find from this the value of y_8.

$$y_n = (1 + \Delta)^n y_0$$

$$= y_0 + n\Delta y_0 + \frac{n\,.\,\overline{n-1}}{\underline{|2}}\,\Delta^2 y_0 + \frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{\underline{|3}}\,\Delta^3 y_0 + \ldots$$

$$= 5 + 4n + 3n\,.\,\overline{n-1} + \tfrac{1}{2}\,n\,.\,\overline{n-1}\,\overline{n-2} + \frac{5}{24}\,n\,\overline{n-1}\,\overline{n-2}\,\overline{n-3}$$

the other terms vanishing because all differences after $\Delta^4 y$ are zero. This reduces to

$$y_n = \frac{1}{24}\,(5n^4 - 18n^3 + 91n^2 + 18n + 120)$$

By giving to n the values 8, 9, 10, etc., in turn, the values of y_8, y_9, y_{10}, etc., can be calculated. It is often easier to calculate y_n from $y_n = (1 + \Delta)^n y_0$.

Thus $\quad y_8 = (1 + \Delta)^8 y_0$

$$= y_0 + 8\Delta y_0 + \frac{8\,.\,7}{\underline{|2}}\,\Delta^2 y_0 + \frac{8\,.\,7\,.\,6}{\underline{|3}}\,\Delta^3 y_0 + \frac{8\,.\,7\,.\,6\,.\,5}{\underline{|4}}\,\Delta^4 y_0$$

$$= 5 + 8 \times 4 + \frac{8\,.\,7}{\underline{|2}}\,.\,6 + \frac{8\,.\,7\,.\,6}{\underline{|3}}\,.\,3 + \frac{8\,.\,7\,.\,6\,.\,5}{\underline{|4}}\,.\,5$$

$$= 5 + 32 + 168 + 168 + 350$$

$$= 723 \text{ as before.}$$

If the terms in the left-hand column are calculated from a polynomial in x of degree n for integral values of x increasing by equal amounts, the numbers in the $(n + 1)$th difference column are all zero. If the function is not a polynomial, the differences in any column will not all be zero and the values found by applying (I.27) will not be exact. In such a case difference columns are calculated until all the differences in one column are nearly zero, or until irregularities appear amongst them. These irregularities are sometimes caused by inaccuracies in the values of the x's or y's, and consequently these and subsequent differences should be ignored.

The above example is an example on *extrapolation*, as the value found extends the range of the given values. The next is an example on interpolation.

The reader is warned against extending a range of experimental results by extrapolation. In general, empirical formulae should only be applied inside the range of values of the dependent variable covered by the experiments from which the results were obtained.

EXAMPLE 2

The left-hand column in the following table gives values of e^x for $x = 1{\cdot}7$, $1{\cdot}8$, $1{\cdot}9$, $2{\cdot}0$, $2{\cdot}1$, $2{\cdot}2$, $2{\cdot}3$. Find the value of $e^{1\cdot85}$

Values of y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
$y_0 = 5{\cdot}4739$						
	0·5757					
$y_1 = 6{\cdot}0496$		0·0606				
	0·6363		0·0063			
$y_2 = 6{\cdot}6859$		0·0669		0·0007		
	0·7032		0·0070		0·0001	
$y_3 = 7{\cdot}3891$		0·0739		0·0008		0
	0·7771		0·0078		0·0001	
$y_4 = 8{\cdot}1662$		0·0817		0·0009		
	0·8588		0·0087			
$y_5 = 9{\cdot}0250$		0·0904				
	0·9492					
$y_6 = 9{\cdot}9742$						

Since when $x = 1{\cdot}85$, $y_n = y_{1\cdot5}$, we have

$$y_{1\cdot5} = (1 + \Delta)^{1\cdot5} y_0$$

$$= y_0 + 1{\cdot}5\Delta y_0 + \frac{1{\cdot}5 \times 0{\cdot}5}{\underline{|2}}\Delta^2 y_0 + \frac{1{\cdot}5 \times 0{\cdot}5 \times (-\,0{\cdot}5)}{\underline{|3}}\Delta^3 y_0$$

the other terms being so small as to be negligible.

$$\therefore\ y_{1\cdot5} = 5{\cdot}4739 + 1{\cdot}5 \times 0{\cdot}5757 + 0{\cdot}375 \times 0{\cdot}0606 - 0{\cdot}0625 \times 0{\cdot}0063 + \text{negligible terms}$$

$$= 6{\cdot}3598$$

From tables or by calculation $e^{1\cdot85} = 6{\cdot}3598$.

A shorter method would have been to look upon y_1 as our first value, and proceed thus

$$y_{0\cdot5} = (1 + \Delta)^{0\cdot5}\, y_1$$

$$= y_1 + 0{\cdot}5\Delta y_1 - \frac{0{\cdot}5 \times 0{\cdot}5}{\underline{|2}}\Delta^2 y_1 + \frac{0{\cdot}5 \times 0{\cdot}5 \times 1{\cdot}5}{\underline{|3}}\Delta^3 y_1 + \text{terms which are negligible}$$

$$= 6{\cdot}0496 + \frac{1}{2} \times 0{\cdot}6363 - \frac{1}{8} \times 0{\cdot}0669 + \frac{1}{16} \times 0{\cdot}0070 + \ldots$$

$$= 6{\cdot}3598 \text{ as before.}$$

EXAMPLE 3

If the difference symbols Δ, Δ^2 be defined by the equations

$$\Delta f(x) = f(x + h) - f(x)$$

$$\Delta^2 f(x) = \Delta f(x + h) - \Delta f(x)$$

show that $(1 + 2\Delta + \Delta^2) f(x) = f(x + 2h)$

Quote the corresponding formula for $f(x+nh)$ and using this as an interpolation formula, calculate cosh 1·088 to six figures from the following table.

(U.L.)

We give the table below with the difference columns added, but shall first prove the above formula. Arranging values of $f(x)$, $f(x+h)$, and $f(x+2h)$ in a table thus

$$\begin{array}{ccc} f(x) & & \\ & \Delta f(x) & \\ f(x+h) & & \Delta^2 f(x) \\ & \Delta f(x+h) & \\ f(x+2h) & & \end{array}$$

we see that
$$\begin{aligned} f(x+2h) &= f(x+h) + \Delta f(x+h) \\ &= \{f(x) + \Delta f(x)\} + \{\Delta f(x) + \Delta^2 f(x)\} \\ &= f(x) + 2\Delta f(x) + \Delta^2 f(x) \\ &= (1 + 2\Delta + \Delta^2) f(x) = (1+\Delta)^2 f(x) \end{aligned}$$

also
$$f(x+nh) = (1+\Delta)^n f(x)$$

x	$y = \cosh x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1·0	1·54308					
		0·12544				
1·1	1·66852		0·01670			
		0·14214		0·00141		
1·2	1·81066		0·01811		0·00022	
		0·16025		0·00163		$-$0·00007
1·3	1·97091		0·01974		0·00015	
		0·17999		0·00178		
1·4	2·15090		0·02152			
		0·20151				
1·5	2·35241					

$f(x+nh) = (1+\Delta)^n f(x)$ and taking the value of x as $x = 1$, $h = 0{\cdot}1$, and $x + nh = 1{\cdot}088$, we have $1 + nh = 1{\cdot}088$ or $nh = 0{\cdot}088$, or $n = 0{\cdot}88$.

$$\therefore \quad f(1{\cdot}088) = (1+\Delta)^{0\cdot 88} f(1) = (1+\Delta)^{0\cdot 88} y_0 \text{ where } y_0 = 1{\cdot}54308$$

$$\begin{aligned} \therefore \cosh 1{\cdot}088 = \Big(1 &+ 0{\cdot}88\,\Delta + \frac{0{\cdot}88(0{\cdot}88 - 1)}{\underline{|2}}\Delta^2 + \frac{0{\cdot}88(-0{\cdot}12)(-1{\cdot}12)}{\underline{|3}}\Delta^3 \\ &+ \frac{0{\cdot}88(-0{\cdot}12)(-1{\cdot}12)(-2{\cdot}12)}{\underline{|4}}\Delta^4 \\ &+ \frac{0{\cdot}88(-0{\cdot}12)(-1{\cdot}12)(-2{\cdot}12)(-3{\cdot}12)}{\underline{|5}}\Delta^5\Big) y_0 \\ = y_0 &+ 0{\cdot}88\Delta y_0 - 0{\cdot}0528\Delta^2 y_0 + 0{\cdot}019712\Delta^3 y_0 - 0{\cdot}010447\Delta^4 y_0 \\ &+ 0{\cdot}006519\Delta^5 y_0 \end{aligned}$$

$$= 1{\cdot}54308 + 0{\cdot}88 \times 0{\cdot}12544 - 0{\cdot}0528 \times 0{\cdot}01670 + 0{\cdot}019712 \times 0{\cdot}00141 - 0{\cdot}010447 \times 0{\cdot}00022 - 0{\cdot}006519 \times 0{\cdot}00007$$

$$= 1{\cdot}65261$$

EXAMPLE 4

The following values of the pressure P lb/ft² and the temperature T degrees Fahrenheit of steam at maximum density are given. Find the pressure when the temperature is 147°F.

T	P	ΔP	$\Delta^2 P$	$\Delta^3 P$	$\Delta^4 P$
131	327·0				
		87·3			
140	414·3		19·0		
		106·3		3·2	
149	520·6		22·2		0·3
		128·5		3·5	
158	649·1		25·7		0·9
		154·2		4·4	
167	803·3		30·1		− 0·4
		184·3		4·0	
176	987·6		34·1		
		218·4			
185	1 206·0				

The numbers in the last column fluctuate in such a manner as to indicate that they are largely affected by errors of observation in measuring the values of P and T. On this account we ignore differences beyond the third.

Proceeding as before, we write $P = f(T + nh)$ where $T = 131$ and h = the common difference in the T column = 9. Then, when $131 + nh = 147$,

$n = \dfrac{147 - 131}{9} = \dfrac{16}{9}$, so that $P = (1 + \Delta)^{16/9} P_0$ where $P_0 = 327$, $\Delta P_0 = 87{\cdot}3$, $\Delta^2 P_0 = 19{\cdot}0$, etc.

$$\therefore P = P_0 + \frac{16}{9}\Delta P_0 + \frac{\frac{16}{9} \times \frac{7}{9}}{\underline{|2}}\Delta^2 P_0 + \frac{\frac{16}{9}\left(\frac{7}{9}\right)\left(-\frac{2}{9}\right)}{\underline{|3}}\Delta^3 P_0$$

$$= 327{\cdot}0 + \frac{16}{9} \times 87{\cdot}3 + \frac{56}{81} \times 19{\cdot}0 - \frac{112}{729 \times 3} \times 3{\cdot}2$$

$$= 327{\cdot}0 + 155{\cdot}2 + 13{\cdot}136 - 0{\cdot}164 = 495{\cdot}172$$

or, correct to four significant figures, $P = 495{\cdot}2$ lb/ft².

As a rough check on this value, we calculate the value of P by proportion.

$P = 414{\cdot}3 + \dfrac{147 - 140}{149 - 140} \times (520{\cdot}6 - 414{\cdot}3)$. This gives $P = 497{\cdot}0$, a value which is obviously too large, since $\dfrac{dP}{dT}$ is increasing and the graph between P (vertical) and T (horizontal) is therefore concave upwards. We have no simple means of measuring the error in a value calculated by means of the interpolation formula.

It is sometimes convenient to let $y_0 = f(x)$ represent the middle of the series of values of y. This is done in the following table, the differences being found exactly as before, the symbol δ being used to denote differences in the upper half of the table and Δ those in the lower half. The suffixes are negative in the upper half and positive in the lower half, the two halves being treated exactly as if they were two distinct tables each similar to those above.

Values of x	Values of y	Differences in the Values in the Preceding Column					
		1st	2nd	3rd	4th	5th	6th
·	·						
·	·						
x_{-5}	y_{-5}						
		δy_{-4}					
x_{-4}	y_{-4}		$\delta^2 y_{-3}$				
		δy_{-3}		$\delta^3 y_{-2}$			
x_{-3}	y_{-3}		$\delta^2 y_{-2}$		$\delta^4 y_{-1}$		
		δy_{-2}		$\delta^3 y_{-1}$		$\delta^5 y_0$	
x_{-2}	y_{-2}		$\delta^2 y_{-1}$		$\delta^4 y_0$		
		δy_{-1}		$\delta^3 y_0$			
x_{-1}	y_{-1}		$\delta^2 y_0$				
		δy_0					
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		
		Δy_3		$\Delta^3 y_2$			
x_4	y_4		$\Delta^2 y_3$				
		Δy_4					
x_5	y_5						
·	·						
·	·						
·	·						

The relation $y_n = (1 + \Delta)^n y_0$ holds as before for the lower half. For the upper half, since $y_0 = y_{-1} + \delta y_0$, then $y_{-1} = y_0 - \delta y_0 = (1 - \delta)y_0$, and generally

$$y_{-n} = (1 - \delta)^n y_0 \qquad (1.28)$$

Interpolation may be carried out with this scheme exactly as in Examples 1 and 2 above, interpolation in the lower half being carried out exactly as in those examples. Interpolation in the upper half is carried out with the use of negative indices and the terms y_0, δy_0, $\delta^2 y_0$, etc. This method of tabulation has no advantage over the previous method for interpolation, but it is particularly useful when finding the value of $\frac{dy}{dx}$, when $x = x_0$ (Art. 8).

8. **Value of $\frac{dy}{dx}$ by Finite Differences.** Let us now expand y_{-1}, y_{-2}, etc., . . . y_1, y_2, etc., in terms of y_0. Assume $y_0 = f(x)$. Then

$$y_1 = f(x+h) = f(x) + hf'(x) + \frac{h^2}{\underline{|2}}f''(x) + \frac{h^3}{\underline{|3}}f'''(x) + \frac{h^4}{\underline{|4}}f^{(4)}(x) + \frac{h^5}{\underline{|5}}f^{(5)}(x) + \frac{h^6}{\underline{|6}}f^{(6)}(x) + \dots$$

$$y_{-1} = f(x-h) = f(x) - hf'(x) + \frac{h^2}{\underline{|2}}f''(x) - \frac{h^3}{\underline{|3}}f'''(x) + \frac{h^4}{\underline{|4}}f^{(4)}(x) - \frac{h^5}{\underline{|5}}f^{(5)}(x) + \frac{h^6}{\underline{|6}}f^{(6)}(x) + \dots$$

$$y_2 = f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{\underline{|2}}f''(x) + \frac{8h^3}{\underline{|3}}f'''(x) + \frac{16h^4}{\underline{|4}}f^{(4)}(x) + \frac{32h^5}{\underline{|5}}f^{(5)}(x) + \frac{64h^6}{\underline{|6}}f^{(6)}(x) + \dots$$

$$y_{-2} = f(x-2h) = f(x) - 2hf'(x) + \frac{4h^2}{\underline{|2}}f''(x) - \frac{8h^3}{\underline{|3}}f'''(x) + \frac{16h^4}{\underline{|4}}f^{(4)}(x) - \frac{32h^5}{\underline{|5}}f^{(5)}(x) + \frac{64h^6}{\underline{|6}}f^{(6)}(x) + \dots$$

$$y_3 = f(x+3h) = f(x) + 3hf'(x) + \frac{9h^2}{\underline{|2}} f''(x) + \frac{27h^3}{\underline{|3}} f'''(x)$$

$$+ \frac{81h^4}{\underline{|4}} f^{(4)}(x) + \frac{243h^5}{\underline{|5}} f^{(5)}(x) + \frac{729h^6}{\underline{|6}} f^{(6)}(x) + \ldots$$

$$y_{-3} = f(x-3h) = f(x) - 3hf'(x) + \frac{9h^2}{\underline{|2}} f''(x) - \frac{27h^3}{\underline{|3}} f'''(x)$$

$$+ \frac{81h^4}{\underline{|4}} f^{(4)}(x) - \frac{243h^5}{\underline{|5}} f^{(5)}(x) + \frac{729h^6}{\underline{|6}} f^{(6)}(x) + \ldots$$

Neglecting terms beyond those of the sixth degree in h, we have

$$y_1 - y_{-1} = 2\left(hf'(x) + \frac{h^3}{\underline{|3}} f'''(x) + \frac{h^5}{\underline{|5}} f^{(5)}(x)\right). \qquad (I.29)$$

$$y_2 - y_{-2} = 2\left(2hf'(x) + \frac{8h^3}{\underline{|3}} f'''(x) + \frac{32h^5}{\underline{|5}} f^{(5)}(x)\right) \qquad (I.30)$$

$$y_3 - y_{-3} = 2\left(3hf'(x) + \frac{27h^3}{\underline{|3}} f'''(x) + \frac{243h^5}{\underline{|5}} f^{(5)}(x)\right) \qquad (I.31)$$

Multiplying through by unity in (I.29), by a in (I.30), and by b in (I.31), and adding the results,

$$(y_1 - y_{-1}) + a(y_2 - y_{-2}) + b(y_3 - y_{-3})$$

$$= 2hf'(x)(1 + 2a + 3b) + \frac{2h^3}{\underline{|3}} f'''(x)(1 + 8a + 27b)$$

$$+ \frac{2h^5}{\underline{|5}} f^{(5)}(x)(1 + 32a + 243b) \qquad (I.32)$$

If, now, we choose a and b so that $1 + 8a + 27b = 0$ and $1 + 32a + 243b = 0$, we shall obtain an expression for $f'(x)$ in terms of the ordinates $y_{-3}\ y_{-2} \ldots y_2\ y_3$. Solving the equations for a and b we obtain $a = -\frac{1}{5}$, $b = \frac{1}{45}$. Substituting in (I.32)

$$(y_1 - y_{-1}) - \frac{1}{5}(y_2 - y_{-2}) + \frac{1}{45}(y_3 - y_{-3}) = \frac{4}{3}hf'(x)$$

$$\therefore\ f'(x) = \frac{3}{4h}\left\{(y_1 - y_{-1}) - \frac{1}{5}(y_2 - y_{-2}) + \frac{1}{45}(y_3 - y_{-3})\right\} \qquad (I.33)$$

This formula enables us to find the value of $f'(x)$, or $\dfrac{dy}{dx}$ from tabulated values of $f(x)$ without using the method of differences as below.

EXAMPLE 1

The following numbers give the angle θ (radians) turned through by a shaft, the interval of time between two successive positions being 0·02 second: 0·052, 0·105, 0·168, 0·242, 0·327, 0·408, 0·489. Find the angular velocity at the instant when $\theta = 0{\cdot}242$.

Let the values of θ be $\theta_{-3}, \theta_{-2}, \theta_{-1}, \theta_0, \theta_1, \theta_2, \theta_3$. $h = 0{\cdot}02$. The angular velocity is $\dfrac{d\theta}{dt} = f'(\theta)$, t being the independent variable. We have then by (I.33)

$$f'(\theta) = \frac{3}{0{\cdot}08}\left\{(0{\cdot}327 - 0{\cdot}168) - \frac{1}{5}(0{\cdot}408 - 0{\cdot}105) + \frac{1}{45}(0{\cdot}489 - 0{\cdot}052)\right\}$$

$\therefore$ Angular velocity $= 4{\cdot}054$ radians per second.

In order to obtain an interpolation formula in terms of differences, we have

$$\left.\begin{aligned} y_1 &= y_0 + \Delta y_0 \\ y_{-1} &= y_0 - \delta y_0 \end{aligned}\right\} \quad \therefore y_1 - y_{-1} = \Delta y_0 + \delta y_0$$

$$\left.\begin{aligned} y_2 &= y_0 + 2\Delta y_0 + \Delta^2 y_0 \\ y_{-2} &= y_0 - 2\delta y_0 + \delta^2 y_0 \end{aligned}\right\} \quad \begin{aligned} \therefore y_2 - y_{-2} = 2(\Delta y_0 + \delta y_0)& \\ + (\Delta^2 y_0 - \delta^2 y_0)& \end{aligned}$$

$$\left.\begin{aligned} y_3 &= y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0 \\ y_{-3} &= y_0 - 3\delta y_0 + 3\delta^2 y_0 - \delta^3 y_0 \end{aligned}\right\} \quad \begin{aligned} \therefore y_3 - y_{-3} &= 3(\Delta y_0 + \delta y_0) \\ &+ 3(\Delta^2 y_0 - \delta^2 y_0) \\ &+ (\Delta^3 y_0 + \delta^3 y_0) \end{aligned}$$

Substituting these values in (I.33), we have

$$\begin{aligned} f'(x) = \frac{3}{4h}\Big\{ &\Delta y_0 + \delta y_0 - \frac{2}{5}(\Delta y_0 + \delta y_0) - \frac{1}{5}(\Delta^2 y_0 - \delta^2 y_0) \\ &+ \frac{1}{15}(\Delta y_0 + \delta y_0) + \frac{1}{15}(\Delta^2 y_0 - \delta^2 y_0) + \frac{1}{45}(\Delta^3 y_0 + \delta^3 y_0)\Big\} \end{aligned}$$

which reduces to

$$f'(x) = \frac{1}{h}\left\{\frac{\Delta y_0 + \delta y_0}{2} - \frac{\Delta^2 y_0 - \delta^2 y_0}{10} + \frac{\Delta^3 y_0 + \delta^3 y_0}{60}\right\} \quad \text{(I.34)}$$

EXAMPLE 2

Work out Example 1 above by the difference method.

t	θ	First Differences	Second Differences	Third Differences
0	0·052			
		0·053		
0·02	0·105		0·010	
		0·063		0·001
0·04	0·168		0·011	
		0·074		
0·06	0·242			
		0·085		
0·08	0·327		−0·004	
		0·081		0·004
0·10	0·408		0	
		0·081		
0·12	0·489			

Using (I.34),

$$f'(\theta) = \frac{1}{0{\cdot}02}\left(\frac{0{\cdot}085 + 0{\cdot}074}{2} - \frac{-0{\cdot}004 - 0{\cdot}011}{10} + \frac{0{\cdot}004 + 0{\cdot}001}{60}\right)$$

$$= 4{\cdot}054 \text{ as before.}$$

9. To Find $\dfrac{d^2y}{dx^2}$ by Finite Differences. The method adopted here is substantially that of the last section. We may assume that y_0 is zero, for if it is not zero we can make it zero (by deducting the quantity y_0 from every y) without altering the values of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ or any of the tabulated differences. Taking terms as far as that involving $h^6f^{(6)}(x)$, we have

$$f(x+h) = hf'(x) + \frac{h^2}{\underline{|2}}f''(x) + \frac{h^3}{\underline{|3}}f'''(x) + \frac{h^4}{\underline{|4}}f^{(4)}(x) + \frac{h^5}{\underline{|5}}f^{(5)}(x) + \frac{h^6}{\underline{|6}}f^{(6)}(x)$$

$$f(x-h) = -hf'(x) + \frac{h^2}{\underline{|2}}f''(x) - \frac{h^3}{\underline{|3}}f'''(x) + \frac{h^4}{\underline{|4}}f^{(4)}(x) - \frac{h^5}{\underline{|5}}f^{(5)}(x) + \frac{h^6}{\underline{|6}}f^{(6)}(x)$$

$$\therefore \quad f(x+h)+f(x-h) = 2\left(\frac{h^2}{\underline{|2}}f''(x)+\frac{h^4}{\underline{|4}}f^{(4)}(x)+\frac{h^6}{\underline{|6}}f^{(6)}(x)\right) \tag{I.35}$$

Similarly,

$$f(x+2h)+f(x-2h)$$
$$= 2\left(\frac{4h^2}{\underline{|2}}f''(x)+\frac{16h^4}{\underline{|4}}f^{(4)}(x)+\frac{64h^6}{\underline{|6}}f^{(6)}(x)\right) \tag{I.36}$$

and $f(x+3h)+f(x-3h)$

$$= 2\left(\frac{9h^2}{\underline{|2}}f''(x)+\frac{81h^4}{\underline{|4}}f^{(4)}(x)+\frac{729h^6}{\underline{|6}}f^{(6)}(x)\right) \tag{I.37}$$

But $f(x+h)+f(x-h) = y_1 + y_{-1} = \Delta y_0 - \delta y_0$ (see Art. 8)

$$\tag{I.38}$$

$$f(x+2h)+f(x-2h) = y_2 + y_{-2} = 2(\Delta y_0 - \delta y_0) + (\Delta^2 y_0 + \delta^2 y_0) \quad . \tag{I.39}$$

and $f(x+3h)+f(x-3h) = y_3 + y_{-3} = 3(\Delta y_0 - \delta y_0)$

$$+ 3(\Delta^2 y_0 + \delta^2 y_0) + (\Delta^3 y_0 - \delta^3 y_0) \quad . \quad . \tag{I.40}$$

Equating the right-hand sides of (I.35) and (I.38), (I.36) and (I.39), (I.37) and (I.40) in order, we have

$$2\left(\frac{h^2}{\underline{|2}}f''(x)+\frac{h^4}{\underline{|4}}f^{(4)}(x)+\frac{h^6}{\underline{|6}}f^{(6)}(x)\right) = (\Delta y_0 - \delta y_0) \quad . \tag{I.41}$$

$$2\left(\frac{4h^2}{\underline{|2}}f''(x)+\frac{16h^4}{\underline{|4}}f^{(4)}(x)+\frac{64h^6}{\underline{|6}}f^{(6)}(x)\right)$$
$$= 2(\Delta y_0 - \delta y_0) + (\Delta^2 y_0 + \delta^2 y_0) \quad . \quad . \tag{I.42}$$

$$2\left(\frac{9h^2}{\underline{|2}}f''(x)+\frac{81h^4}{\underline{|4}}f^{(4)}(x)+\frac{729h^6}{\underline{|6}}f^{(6)}(x)\right)$$
$$= 3(\Delta y_0 - \delta y_0) + 3(\Delta^2 y_0 + \delta^2 y_0) + (\Delta^3 y_0 - \delta^3 y_0) \tag{I.43}$$

Multiply (I.41) by unity, (I.42) by a, and (I.43) by b and add, choosing a and b so that the terms involving $f^{(4)}(x)$ and $f^{(6)}(x)$ disappear. We have

$$h^2 f''(x)(1+4a+9b) = (\Delta y_0 - \delta y_0)(1+2a+3b)$$
$$+ (\Delta^2 y_0 + \delta^2 y_0)(a+3b) + b(\Delta^3 y_0 - \delta^3 y_0)$$

where $1+16a+81b=0$ and $1+64a+729b=0$, i.e. $a=-\dfrac{1}{10}$ and $b=\dfrac{1}{135}$

Substituting these in the above expression, we have

$$h^2 f''(x)\left(1-\frac{4}{10}+\frac{1}{15}\right) = (\Delta y_0 - \delta y_0)\left(1-\frac{2}{10}+\frac{1}{45}\right)$$
$$+ (\Delta^2 y_0 + \delta^2 y_0)\left(-\frac{1}{10}+\frac{1}{45}\right) + \frac{1}{135}(\Delta^3 y_0 - \delta^3 y_0)$$

$$\text{or } f''(x) = \frac{3}{2h^2}\left[\frac{37}{45}(\Delta y_0 - \delta y_0) - \frac{7}{90}(\Delta^2 y_0 + \delta^2 y_0)\right.$$
$$\left. + \frac{1}{135}(\Delta^3 y_0 - \delta^3 y_0)\right]$$

and finally

$$f''(x) = \frac{1}{h^2}\left[\frac{37}{30}(\Delta y_0 - \delta y_0) - \frac{7}{60}(\Delta^2 y_0 + \delta^2 y_0)\right.$$
$$\left. + \frac{1}{90}(\Delta^3 y_0 - \delta^3 y_0)\right] \quad . \qquad (I.44)$$

EXAMPLE

In a certain machine a slider moves along a fixed straight rod. Its distance x ft along the rod is given below for various values of the time t seconds. Find (1) the velocity of the slider, and (2) its acceleration, when $t = 0{\cdot}3$.

t = time in seconds . .	0	0·1	0·2	0·3	0·4	0·5	0·6
x = distance in feet . .	3·013	3·162	3·287	3·364	3·395	3·381	3·324

These values and the differences are tabulated below—

t (sec)	x (ft)	First Differences	Second Differences	Third Differences
0	3·013			
		0·149		
0·1	3·162		− 0·024	
		0·125		− 0·024
0·2	3·287		− 0·048	
		0·077		
0·3	3·364			
		0·031		
0·4	3·395		− 0·045	
		− 0·014		+ 0·002
0·5	3·381		− 0·043	
		− 0·057		
0·6	3·324			

By (I.34), $\dfrac{dx}{dt} = \dfrac{1}{0{\cdot}1}\left(\dfrac{0{\cdot}031 + 0{\cdot}077}{2} - \dfrac{-0{\cdot}045 + 0{\cdot}048}{10} + \dfrac{0{\cdot}002 - 0{\cdot}024}{60}\right)$

$$= 10(0{\cdot}054 - 0{\cdot}0003 - 0{\cdot}0004)$$

$$= 0{\cdot}533$$

or the velocity is 0·533 ft/sec.

By (I.44), $\dfrac{d^2x}{dt^2} = \dfrac{1}{(0{\cdot}1)^2}\left(\dfrac{37}{30}(0{\cdot}031 - 0{\cdot}077) - \dfrac{7}{60}(-0{\cdot}045 - 0{\cdot}048) + \dfrac{1}{90}(0{\cdot}002 + 0{\cdot}024)\right)$

$$= 100(-0{\cdot}05673 + 0{\cdot}01085 + 0{\cdot}00029)$$

$$= -4{\cdot}56 \text{ ft/sec}^2$$

which is the acceleration.

EXAMPLES I

Evaluate the following determinants—

(1) $\begin{vmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 8 & 9 \end{vmatrix}$ (2) $\begin{vmatrix} 3 & -4 & -3 \\ 2 & 7 & -31 \\ 5 & -9 & 2 \end{vmatrix}$ (3) $\begin{vmatrix} a+b & a & b \\ a & c+a & c \\ b & c & b+c \end{vmatrix}$

(4) $\begin{vmatrix} 5 & 7 & 4 \\ 6 & 25 & 11 \\ 7 & 52 & 21 \end{vmatrix}$ (5) $\begin{vmatrix} 14 & -9 & 25 \\ 15 & 16 & -11 \\ 13 & 10 & 6 \end{vmatrix}$ (6) $\begin{vmatrix} 11 & 71 & 27 \\ -6 & 9 & 10 \\ 1 & 16 & 17 \end{vmatrix}$

(7) $\begin{vmatrix} 3 & 5 & 7 \\ 11 & 9 & 13 \\ 15 & 17 & 19 \end{vmatrix}$ (8) $\begin{vmatrix} 10 & 10 & 21 & 18 \\ 4 & 5 & 8 & 6 \\ 2 & 1 & 3 & 7 \\ 3 & 4 & 10 & 5 \end{vmatrix}$ (9) $\begin{vmatrix} 2 & -5 & -7 & -5 \\ 8 & 3 & -11 & -27 \\ 3 & -9 & 4 & 2 \\ -5 & 2 & 3 & 13 \end{vmatrix}$

(10) Prove the identity $\begin{vmatrix} a^2 & a & bc \\ b^2 & b & ca \\ c^2 & c & ab \end{vmatrix} = \begin{vmatrix} a^3 & a^2 & 1 \\ b^3 & b^2 & 1 \\ c^3 & c^2 & 1 \end{vmatrix}$

Solve the following equations—

(11) $\begin{vmatrix} 3+x & 4+x & x \\ 2+x & x & 4+x \\ x & 2+x & 3+x \end{vmatrix} = 0$

(12) $\begin{vmatrix} x & 2 & 3 \\ 6 & x+4 & 4 \\ 7 & 8 & x+8 \end{vmatrix} = 0$ (13) $\begin{vmatrix} 2x^2 & -3 & -16 \\ x & 5 & 5 \\ 11 & 20 & -15 \end{vmatrix} = 0$

(14) Show that $x = 2$ is a root of the equation

$$\begin{vmatrix} 3x-1 & 4x & 6 \\ 2(1-x) & 1 & 3x \\ 11 & 5x & 4-3x \end{vmatrix} = 0$$

and find the other roots.

(15) Express the determinant $\begin{vmatrix} pa_1 + qa_2 & a_2 & a_3 \\ pb_1 + qb_2 & b_2 & b_3 \\ pc_1 + qc_2 & c_2 & c_3 \end{vmatrix}$

as the sum of two determinants of the third order, and prove that it is equal to

$$p\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Solve the equation $\begin{vmatrix} 4x+5 & 4x+7 & 4x+9 \\ 4x+9 & 4x+5 & 4x+7 \\ 4x+7 & 4x+9 & 4x+5 \end{vmatrix} = 0$

(U.L.)

(16) Show that $\begin{vmatrix} a_1 + A & a_2 \\ b_1 + B & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} A & a_2 \\ B & b_2 \end{vmatrix}$

Find the coefficient of the first power of X in the expansion of the determinant

$$\begin{vmatrix} 1+X & m & sk \\ m & m^2 + \frac{1}{2}(1-s)k^2 + X & \frac{1}{2}(1+s)mk \\ sk & \frac{1}{2}(1+s)mk & k^2 + \frac{1}{2}(1-s)m^2 \end{vmatrix}$$

(U.L.)

(17) Prove that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

and find the value of $\begin{vmatrix} 31 & 91 & 47 \\ 14 & 29 & 30 \\ 21 & 36 & 37 \end{vmatrix}$

(U.L.)

(18) Explain the method of solution of simultaneous linear equations by means of determinants. Find the equation of the circle which passes through the points $(-3, -5)$, $(1, -5)$, $(-3, 1)$. (U.L.)

(19) Solve by determinants the equations—

$$\begin{aligned} 7x + 5y - 13z + 4 &= 0 \\ 9x + 2y + 11z - 37 &= 0 \\ 3x - y + z - 2 &= 0 \end{aligned}$$

(U.L.)

(20) Solve by the aid of determinants, or otherwise—

$$\begin{aligned} 53x - 37y + 29z &= 32 \\ 22x + 31y - 99z &= 23 \\ 98x - 44y - 37z &= 0 \end{aligned}$$

(U.L.)

(21) In a determinant, prove that the same multiple of the constituents of any row (or column) may be added to the constituents of any other row (or column) without altering the value of the determinant.

$c_2r_2 - c_1r_1 = c_5r_5$; $c_1r_1 + c_3r_3 = v$; $c_2r_2 + c_4r_4 = v$; $c_1 - c_3 - c_5 = 0$; $c_4 - c_2 - c_5 = 0$, are five equations determining $c_1, c_2, \ldots c_5$ in terms of the other quantities. Write down the value of c_5 as the quotient of two determinants, and, hence, show that if $r_1r_4 = r_2r_3$, then $c_5 = 0$. (U.L.)

(22) In determining an electrical resistance, the following equations occur—

$$\begin{aligned} Gg + Pp - Qq &= e \\ -Gg + (P - G)r - (Q + G)s &= 0 \\ (P + Q)b + Qq + (Q + G)s &= E \end{aligned}$$

Find, as the ratio of two determinants, the value of G independent of P and Q, and determine the relation between p, q, r, and s if G be independent of E.

(U.L.)

(23) By means of a determinant, eliminate x, y, z, and w from the equations

$$\begin{aligned} tx + a(y + z + w) &= 0 \\ ty + b(z + w + x) &= 0 \\ tz + c(w + x + y) &= 0 \\ tw + d(x + y + z) &= 0 \end{aligned}$$

and show that the coefficient of t^2 in the result is

$$-(ab + ac + ad + bc + bd + cd)$$ (U.L.)

(24) Find all the values of t for which the equations

$$(t-1)x+(3t+1)y+2tz=0$$
$$(t-1)x+(4t-2)y+(t+3)z=0$$
$$2x+(3t+1)y+3(t-1)z=0$$

are compatible, and find the ratios of $x:y:z$ when t has the smallest of these values. What happens when t has the greatest of these values? (U.L.)

(25) Prove by determinants or otherwise that the equations

$$5x+3y+2z=12$$
$$2x+4y+5z=2$$
$$39x+43y+45z=c$$

are incompatible unless $c=74$; and that in that case the equations are satisfied by $x=2+t$, $y=2-3t$, $z=-2+2t$, where t is any arbitrary quantity. (U.L.)

(26) Obtain as a determinant the result of eliminating x from the equations

$$ax^3+bx^2+cx+d=0$$
$$a'x^2+b'x+c'=0$$

Solve the equations

$$x+2y+z=4$$
$$3x-4y-4z=10$$
$$5x+3y+7z=-9$$

(U.L.)

(27) Prove that $\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \delta_2 & \gamma_2 & \beta_2 & \alpha_2 \\ \delta_1 & \gamma_1 & \beta_1 & \alpha_1 \end{vmatrix} = \begin{vmatrix} \alpha_1+\delta_1 & \beta_1+\gamma_1 \\ \alpha_2+\delta_2 & \beta_2+\gamma_2 \end{vmatrix} \cdot \begin{vmatrix} \alpha_1-\delta_1 & \beta_1-\gamma_1 \\ \alpha_2-\delta_2 & \beta_2-\gamma_2 \end{vmatrix}$

(28) Express as a single determinant

$$\begin{vmatrix} 1 & 4 & -3 \\ -3 & -2 & 1 \\ 1 & 3 & 5 \end{vmatrix} \times \begin{vmatrix} 3 & 4 & 2 \\ 2 & 5 & 4 \\ -1 & 6 & 1 \end{vmatrix}$$

and find its value.

(29) Prove that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$

If a, b, c, have all different values and

$$\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = 0$$

prove that $abc=1$.

(U.L.)

(30) Show that $\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$

is the equation to the circle through the points (x_1, y_1), (x_2, y_2), (x_3, y_3).

(31) Find the value of λ for which the equations

$$(2-\lambda)x + 2y + 3 = 0$$
$$2x + (4-\lambda)y + 7 = 0$$
$$2x + 5y + 6 - \lambda = 0$$

are consistent, and find the values of x and y corresponding to each of these values of λ. (U.L.)

(32) If $y = f(x)$ and y_n denotes $f(x + nh)$, prove that, if powers of h above h^6 be neglected,

$$\frac{dy}{dx} = \frac{3}{4h}\left[y_1 - y_{-1} - \frac{1}{5}(y_2 - y_{-2}) + \frac{1}{45}(y_3 - y_{-3})\right]$$

The elevations above a datum line of seven points of a road, 100 yd apart, are 135, 149, 157, 183, 201, 205, 193 ft. Find the gradient of the road at the middle point. (U.L.)

(33) The following values of x and y are given. Using the method of Example 2, Art. 7, find the value of y when $x = 0{\cdot}7$ and also when $x = 0{\cdot}25$.

x	0·1	0·2	0·3	0·4	0·5	0·6
y	2·6841	3·0413	3·3753	3·6835	3·9636	4·2136

(34) From the following table of natural logarithms determine $\log_e 0{\cdot}0725742$.

x	$\log_e x$
0·071	$\bar{3}$·354925
0·072	$\bar{3}$·368911
0·073	$\bar{3}$·382704
0·074	$\bar{3}$·396301
0·075	$\bar{3}$·409733

(U.L.)

(35) The table gives the displacements x in. of a valve from its mean position for various crank angles. If t is the time in seconds, find the values of $\frac{dx}{dt}$ in./sec and $\frac{d^2x}{dt^2}$ in./sec² when the crank angle θ is 59°.

Values of θ degrees . .	50°	53°	56°	59°	62°	65°	68°
Values of x inches . .	2·72	2·78	2·84	2·88	2·92	2·95	2·98

The crank rotates at a uniform speed of 200 r.p.m.

[*Hint*. Since θ is in degrees $\frac{d\theta}{dt} = \frac{180}{\pi} \times$ angular velocity in radians per second. Also $\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt}$ and $\frac{dx}{d\theta}$ is found from (1.34).]

(36) Tabulate values of sin θ where $\theta = 22°, 23°, 24°, 25°, 26°, 27°$, and 28° in turn. Find from these, values of $\frac{d \sin \theta}{d\theta}$ and $\frac{d^2}{d\theta^2} \sin \theta$ when $\theta = 25°$, and compare these with the values obtained by putting $\theta = 25°$ in the differential coefficients, remembering that θ is in degrees and not radians.

(37) Given the following values of x and y, find y when $x = 3{\cdot}5$.

x	0	1	2	3	4	5	6
y	5·67921	5·91107	6·17940	6·48431	6·82610	7·20505	7·62143

(38) Taking the numbers from the last example, find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 3$.

(39) The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

h = height (feet) . .	100	150	200	250	300	350	400
d = distance (miles) . .	10·63	13·03	15·04	16·81	18·42	19·90	21·27

Find by the method indicated in the table in Example 2, Art. 7, the values of d when $h = 218$ ft and when $h = 265$ ft. Also find the rate at which the horizon is receding from a man in a balloon 250 ft high which is ascending at the rate of 6 nautical miles an hour.

(40) The areas of regular figures of 3, 5, 7, 9, 11 sides, each side of which is 1 in. long, are given in order by 0·4330, 1·7205, 3·6339, 6·1818, 9·3656 in.² Calling these y_0, y_1, y_2, y_3, y_4, find an expression for y_n in terms of y_0 and the differences. Putting $n = 1{\cdot}5$, calculate the area of a regular hexagon of 1 in. side. Compare this value with the value found by mensuration. Is the use of the method of finite differences justifiable in this case?

(41) Find an expression for the general term of the sequence of numbers 1, 3, 8, 21, 54, 126, 263, 498, and continue the sequence to include another term.

(42) Using the method of Example 2, Art. 7, calculate $\log_{10} \sin 30° \, 24'$ to five decimal places from the following table—

x	30°	31°	32°	33°	34°	35°
$y = \log_{10} \sin x$	$\bar{1}\cdot69897$	$\bar{1}\cdot71184$	$\bar{1}\cdot72421$	$\bar{1}\cdot73611$	$\bar{1}\cdot74756$	$\bar{1}\cdot75859$

(43) The following values of x and y are given. Find the best value of $\frac{dy}{dx}$ and also of $\frac{d^2y}{dx^2}$ when $x = 6\cdot5$—

x	5·0	5·5	6·0	6·5	7·0	7·5	8·0
y	3·2188	3·4096	3·5836	3·7436	3·8918	4·0298	4·1588

(44) A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of the time t seconds. Find (i) the angular velocity of the rod, and (ii) its angular acceleration, when $t = 0\cdot6$ sec.

t	0	0·2	0·4	0·6	0·8	1·0	1·2
θ	0	0·122	0·493	1·123	2·022	3·200	4·666

(45) From the following table of values of e^x estimate to 5 decimal places the value of e^x when $x = 1\cdot92635$. State, giving reasons, an upper limit to the error that may occur in your result, supposing your working to be as accurate as possible.

x	e^x
1·90	6·68589
1·91	6·75309
1·92	6·82096
1·93	6·88951
1·94	6·95875

(U.L.)

(46) From the following table of $\sec^{-1}x$ determine the value of $\sec^{-1} 2\cdot0136742$

x	$\sec^{-1}x$
2·00	1·0471975
2·01	1·0500675
2·02	1·0529045
2·03	1·0557090
2·04	1·0584816
2·05	1·0612229

(U.L.)

CHAPTER II

DOUBLE AND TRIPLE INTEGRALS AND THEIR APPLICATIONS

10. Notation. The symbol $\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y)dx\, dy$ is called a *double integral* and indicates that $f(x, y)$ is to be integrated with respect to x between the limits x_1 and x_2, and that the resulting expression is to be integrated with respect to y between the limits y_1 and y_2. Similarly, $\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y, z)dx\, dy\, dz$ denotes that $f(x, y, z)$ is to be integrated with respect to x between the limits x_1 and x_2, that the resulting function is to be integrated with respect to y between the limits y_1 and y_2, and that, finally, the result just obtained is to be integrated with respect to z between the limits z_1 and z_2.

EXAMPLE 1

Integrate $$\int_1^2\int_0^1\int_{-1}^1 (x^2 + y^2 + z^2)dx\, dy\, dz$$

Let I be the value of the integral, then

$$I = \int_1^2\int_0^1\int_{-1}^1 (x^2 + y^2 + z^2)dx\, dy\, dz$$

We first integrate with respect to x, assuming y and z to be constants. Then

$$I = \int_1^2\int_0^1 \left[\tfrac{1}{3}x^3 + (y^2 + z^2)x\right]_{-1}^1 dy\, dz$$

$$= \int_1^2\int_0^1 \{\tfrac{2}{3} + 2(y^2 + z^2)\}dy\, dz$$

Now, integrating with respect to y and assuming z to be a constant, we have

$$I = \int_1^2 \left[\tfrac{2}{3}y + 2(\tfrac{1}{3}y^3 + z^2y)\right]_0^1 dz = \int_1^2 \{\tfrac{2}{3} + 2(\tfrac{1}{3} + z^2)\}dz$$

$$= \int_1^2 (\tfrac{4}{3} + 2z^2)dz = \left[\tfrac{4}{3}z + \tfrac{2}{3}z^3\right]_1^2$$

$$= 6$$

The reader should notice that the order of integration is denoted by

$$I = \boxed{\int_{z_1}^{z_2} \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y, z)dx} \; dy} \; dz} \quad . \quad . \quad . \text{(II.1)}$$

the operations of integration being carried out in the rectangles shown in turn, starting in the innermost rectangle and working outwards to the outermost rectangle. Some writers alter the order of the symbols dx, dy, dz in (II.1), so that the first of the signs $\int$ is associated with the first of the differentials, thus, $\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y, z)dz\, dy\, dx$ being used to indicate what we mean by (II.1). We shall use the notation of (II.1). It is also important to notice that when integrating with respect to x in (II.1), y and z are treated as constants, and that when integrating with respect to y, z is treated as a constant.

EXAMPLE 2

Evaluate $I = \int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^{r} \rho^2 \sin\theta \, d\rho\, d\theta\, d\phi$, where r = constant.

Integrating with respect to ρ we have

$$I = \int_0^{\pi}\int_0^{\frac{\pi}{2}} \left[\tfrac{1}{3}\rho^3 \sin\theta\right]_{\rho=0}^{\rho=r} d\theta\, d\phi = \tfrac{1}{3}\int_0^{\pi}\int_0^{\frac{\pi}{2}} r^3 \sin\theta\, d\theta\, d\phi$$

$$= \tfrac{1}{3}\int_0^{\pi} r^3 \left[-\cos\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}} d\phi = \tfrac{1}{3}\int_0^{\pi} r^3\, d\phi = \tfrac{1}{3}r^3\left[\,\phi\,\right]_0^{\pi} = \frac{\pi r^3}{3}$$

In Ex. 1 and 2, we have dealt only with cases in which the limits are constants. The following examples illustrate the method of procedure when some of the limits are variables.

EXAMPLE 3

Evaluate $I = \int_0^{r}\int_0^{\sqrt{r^2 - x^2}} r\, dy\, dx$ in which r is constant.

$$I = \int_0^{r}\left[ry\right]_{y=0}^{y=\sqrt{r^2-x^2}} dx = \int_0^{r} r\sqrt{r^2 - x^2}\, dx = r\int_0^{r}\sqrt{r^2 - x^2}\, dx$$

$$= \frac{\pi r^3}{4}$$

EXAMPLE 4

Evaluate $I = \int_0^a \int_0^{\sqrt{a^2 - x^2}} x^2 y \, dy \, dx$ in which a is constant.

$$I = \int_0^a \left[\tfrac{1}{2} x^2 y^2 \right]_{y=0}^{y=\sqrt{a^2-x^2}} dx = \tfrac{1}{2} \int_0^a x^2 (a^2 - x^2) dx$$

$$= \tfrac{1}{2} \left[\frac{a^2 x^3}{3} - \frac{x^5}{5} \right]_0^a = \tfrac{1}{15} a^5$$

11. Volumes of Solids as Double Integrals. Let A (Fig. 1) be a portion of the surface $z = f(x, y)$ and let A' be its orthogonal projection on the xy-plane. We have to find an expression giving the volume of the solid included by A and A' and the surrounding cylindrical surface. Let a plane parallel to the zy-plane and at

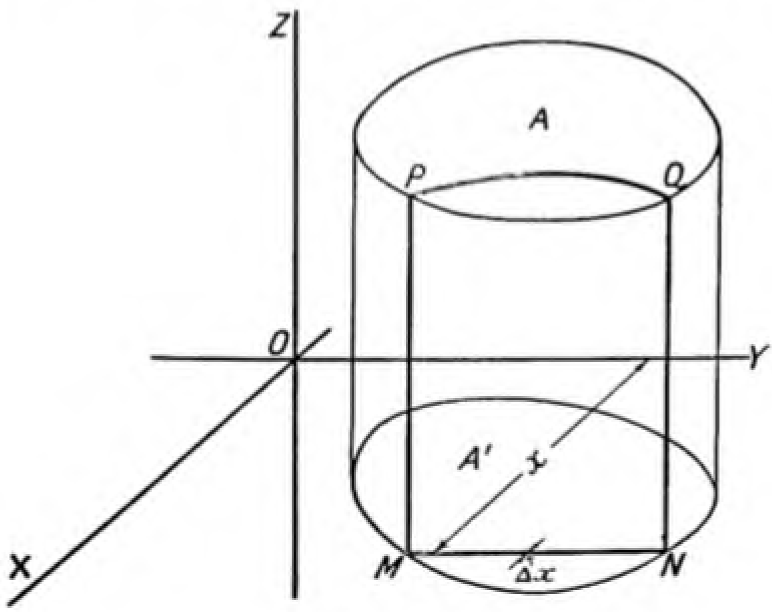


FIG. 1

distance x from it cut the solid in the section $PQNM$; then, in general, the area $PQNM$ will be a function of x, say $\phi(x)$, and the volume of the solid is $\int_{x_1}^{x_2} \phi(x) dx$, x_1 and x_2 being the extreme limits for x over the area A'.

Again, area $PQNM = \int_{y_1}^{y_2} z \, dy$, y_1 and y_2 being the extreme limits

for y over the section $PQNM$ and in general functions of x. Hence, if V is the volume of the solid,

$$V = \int_{x_1}^{x_2}\left[\int_{y_1}^{y_2} z\,dy\right]dx$$

i.e.

$$V = \int_{x_1}^{x_2}\int_{y_1}^{y_2} z\,dy\,dx \quad . \quad . \quad . \quad \text{(II.2)}$$

or

$$V = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)dy\,dx\,. \quad . \quad . \quad \text{(II.3)}$$

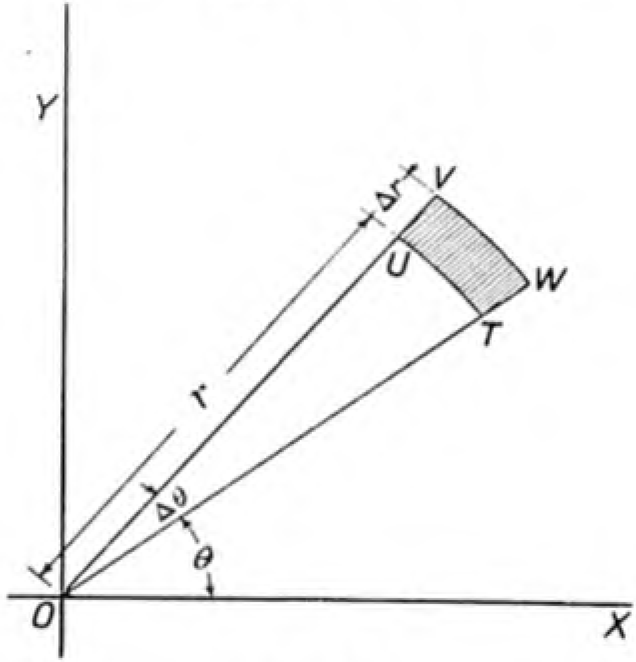


FIG. 2

We can obtain the same result from a different point of view. Let the area A' be divided up into elementary rectangular areas $\Delta x\ \Delta y$, and on each of these as base let a prism be constructed with its length parallel to the z-axis; then the volume of this prism between the surfaces A and A' is $z\ \Delta x\ \Delta y$ and the total volume is $\sum_{x=x_1}^{x=x_2}\ \sum_{y=y_1}^{y=y_2} z\ \Delta x\ \Delta y$, the limiting value of which is $\int_{x_1}^{x_2}\int_{y_1}^{y_2} z\,dy\,dx$, as above.

If we are using polar co-ordinates in the xy-plane, we may divide the area A' into elements such as $TUVW$ (Fig. 2) by means of radial lines through O and circles with their centres at O. Then, if T be the

point (r, θ) and V the point $(r + \Delta r, \theta + \Delta\theta)$. $TU = r\Delta\theta$ and $UV = \Delta r$, so that area $TUVW = r\Delta\theta\,\Delta r$.

Hence, for the volume of the solid above we replace $dy\,dx$ by $rd\theta\,dr$, and then we have

$$\text{Volume of solid} = \int\int zrd\theta\,dr \qquad (11.4)$$

z being assumed given as a function of r and θ.

EXAMPLE 1

A triangular prism is formed by the planes whose equations are $ay = bx$, $y = 0$, and $x = a$. Obtain the volume of this prism between the plane $z = 0$ and the surface $z = c + xy$. (U.L.)

The required volume $= \int_0^a \int_0^{\frac{bx}{a}} z\,dy\,dx$, for with x assumed constant y varies from 0 to $\dfrac{bx}{a}$

$$\therefore \qquad \text{Volume} = \int_0^a \int_0^{\frac{bx}{a}} (c + xy)dy\,dx = \int_0^a \left[cy + \frac{xy^2}{2}\right]_0^{\frac{bx}{a}} dx$$

$$= \int_0^a \left[\frac{bc}{a}x + \frac{b^2}{2a^2}\cdot x^3\right] dx$$

$$= \left(\frac{bc}{a}\cdot\frac{x^2}{2} + \frac{b^2}{2a^2}\cdot\frac{x^4}{4}\right)_0^a$$

$$= \frac{abc}{2} + \frac{a^2b^2}{8} = \frac{ab}{8}[4c + ab]$$

EXAMPLE 2

Show that the equation $(x^2 + y^2 + z^2 + c^2 - a^2)^2 = 4c^2(x^2 + y^2)$ represents an anchor ring. Sketch the surface, showing how it lies with respect to the axes of co-ordinates. Prove that the co-ordinates of any point on the surface may be expressed in the form $x = (c + a\cos\theta)\cos\phi$, $y = (c + a\cos\theta)\sin\phi$, $z = a\sin\theta$, and evaluate the surface area by calculating $\int\int a(c + a\cos\theta)d\theta\,d\phi$. (U.L.)

Put $z = 0$ in the given equation; then

$$(x^2 + y^2 + c^2 - a^2)^2 = 4c^2(x^2 + y^2)$$

i.e. $$(x^2 + y^2)^2 - 2(c^2 + a^2)(x^2 + y^2) + (c^2 - a^2)^2 = 0$$

i.e. $$[x^2 + y^2 - (c + a)^2][x^2 + y^2 - (c - a)^2] = 0$$

Hence, the xy-plane cuts the given surface in two concentric circles of radii $c + a$ and $c - a$, the origin being their common centre.

Again, the plane $y = mx$ cuts the given surface along two sections where

$$[(1 + m^2)x^2 + z^2 + c^2 - a^2]^2 = 4c^2(1 + m^2)x^2$$

$$(1 + m^2)x^2 + z^2 + c^2 - a^2 = \pm 2c\sqrt{1 + m^2} \,.\, x$$

or

$$[\sqrt{1 + m^2} \,.\, x \pm c]^2 + z^2 = a^2$$

which represents the orthogonal projections on the zx-plane of two circles of radius a in the plane $y = mx$.

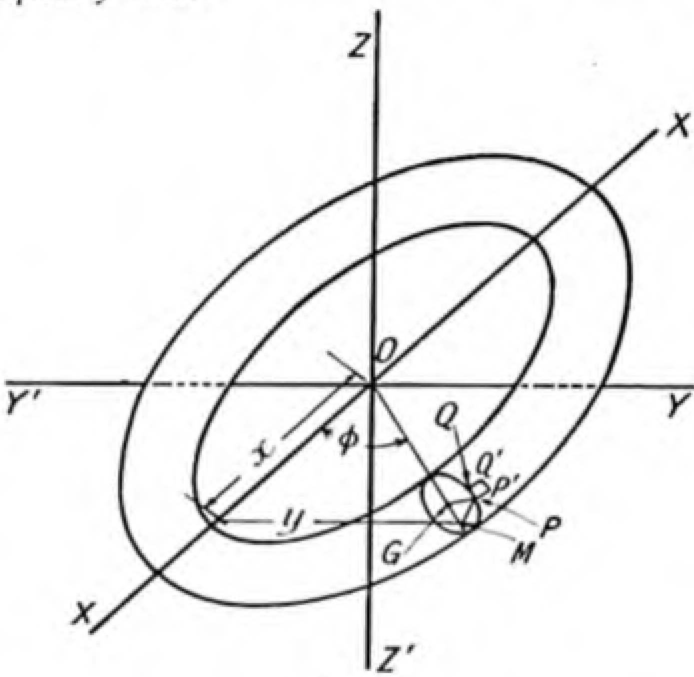


FIG. 3

The given surface is then that of an anchor ring generated by the rotation about the z-axis of a circle of radius a, the centre of the circle being in the xy-plane and at a distance c from the origin, and the plane of the circle containing the z-axis. Fig. 3 indicates how the surface lies with respect to the co-ordinate axes.

Let P be any point (x, y, z) on the surface, and let G be the centre of the generating circle through P. Join OG, PG, and draw $PM \perp OG$. Let $\widehat{XOG}$ be ϕ and $\widehat{PGM}$ be θ. Then $PG = a$, $OG = c$, $OM = c + a\cos\theta$; hence,

$$z = a\sin\theta$$

$$y = OM\sin\phi = (c + a\cos\theta)\sin\phi$$

$$x = OM\cos\phi = (c + a\cos\theta)\cos\phi$$

Let Q be a point on the generating circle through P such that $\widehat{PGQ} = \Delta\theta$, and let P move to P' and Q to Q', when ϕ increases by $\Delta\phi$; then the area $PQQ'P' = PQ \times PP'$ (very nearly) $= a\Delta\theta\,(c + a\cos\theta)\Delta\phi$.

Now, with ϕ constant, θ varies from $-\pi$ to $+\pi$ and then ϕ varies from $-\pi$ to $+\pi$.

$$\therefore \text{ Surface area } = \int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} a(c + a\cos\theta)\,d\phi\,d\theta$$

$$= a\int_{-\pi}^{+\pi}\left[\Big\{(c + a\cos\theta)\phi\Big\}_{-\pi}^{+\pi}\right]d\theta$$

$$= 2\pi a\int_{-\pi}^{+\pi}(c + a\cos\theta)\,d\theta$$

$$= 2\pi a\Big[c\theta + a\sin\theta\Big]_{-\pi}^{+\pi}$$

$$= 2\pi a\,.\,2c\pi$$

$$= 4\pi^2 ac$$

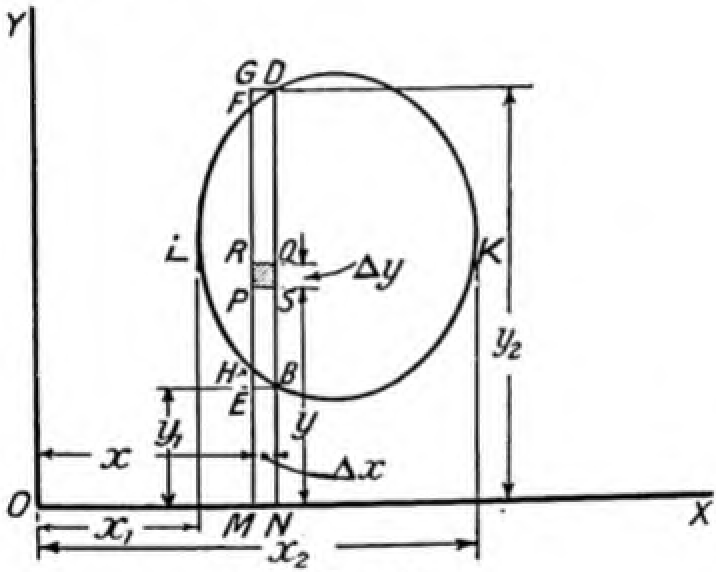


FIG. 4

12. **Area Enclosed by a Plane Closed Curve Expressed as a Double Integral.** Let $P(x, y)$ be any point in the area (Fig. 4) and let Q be the point $(x + \Delta x, y + \Delta y)$. Draw vertical lines through P and Q intersecting the curve in F, H and B, D as shown, and intersecting OX in the points M and N. Complete the rectangles $EGDB$ and $PRQS$. Then we have, since the area of $PRQS = \Delta y\,\Delta x$ and Δx is small

$$\text{area of } \quad EGDB = \underset{\Delta y\to 0}{Lt.}\sum_{y=y_1}^{y=y_2}\Delta y\,\Delta x$$

$$= (\underset{\Delta y\to 0}{Lt.}\sum_{y=y_1}^{y=y_2}\Delta y)\,\Delta x \quad . \quad . \quad . \quad (II.5)$$

Again, if A is the area of the figure,

$$A = \underset{\Delta x \to 0}{Lt.} \sum_{x=x_1}^{x=x_2} (\text{area of strip } EGDB)$$

$$= \underset{\Delta x \to 0}{Lt.} \sum_{x=x_1}^{x=x_2} \{\underset{\Delta y \to 0}{Lt.} \sum_{y=y_1}^{y=y_2} \Delta y\} \Delta x \quad . \quad . \quad \text{(II.6)}$$

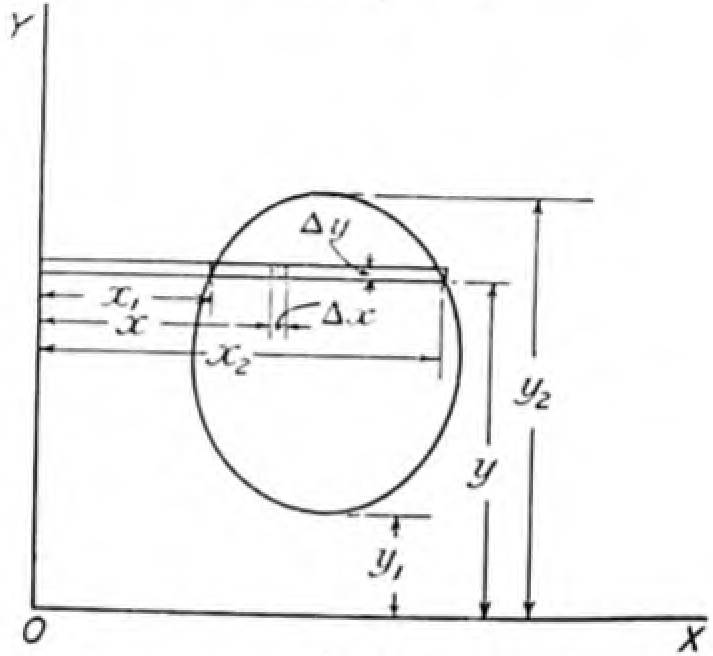


FIG. 5

and, as shown in Volume I, this becomes

$$A = \int_{x_1}^{x_2} \int_{y_1}^{y_2} dy\, dx \quad . \quad . \quad . \quad \text{(II.7)}$$

By similar reasoning with respect to Fig. 5, in which the area is supposed to be divided up into horizontal strips, we can show that

$$A = \int_{y_1}^{y_2} \int_{x_1}^{x_2} dx\, dy \quad . \quad . \quad . \quad \text{(II.8)}$$

Thus we see that the order of integration may be altered without altering the value of A if the limits x_1, x_2, y_1, y_2 are correctly determined. The reader should notice the difference between the values of x_1, x_2, y_1, y_2 in Fig. 4 and those in Fig. 5.

EXAMPLE 1

Find the area enclosed by the parabola $y = x(4 - x)$ and the axis of x.

The graph is shown in Fig. 6 (*a*). If we assume the area to be divided up into vertical strips like *PQNM*, we see that the limiting values of y are $y = 0$ to $y = PM = x(4 - x)$ and those of x are then $x = 0$ to $x = 4$. Hence, we have

$$\begin{aligned} A &= \text{required area} \\ &= \int_0^4 \int_0^{x(4-x)} dy\,dx \\ &= \int_0^4 x(4 - x)dx = \left[2x^2 - \frac{x^3}{3}\right]_0^4 = 10\tfrac{2}{3} \text{ units} \end{aligned}$$

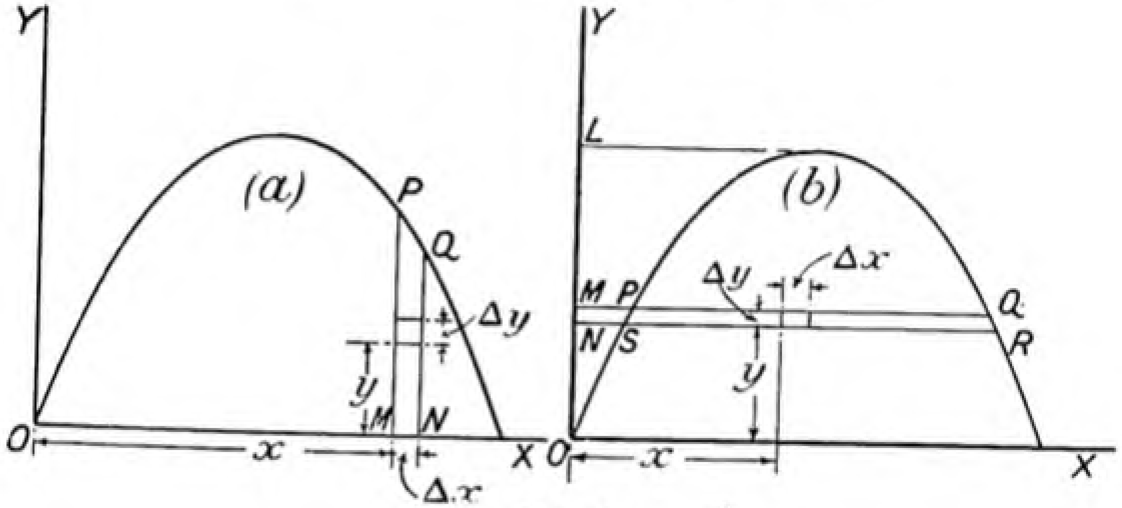


FIG. 6

If we assume the area to be divided into horizontal strips like *PQRS* in Fig. 6 (*b*), the limiting values of x are found from $y = x(4 - x)$ by solving for x; thus

$$x^2 - 4x + y = 0$$

or

$$x = \frac{4 \pm \sqrt{16 - 4y}}{2} = 2 \pm \sqrt{4 - y}$$

whence the limits for x are $NS = 2 - \sqrt{4 - y}$ and $NR = 2 + \sqrt{4 - y}$

Then the limits for y are $y = 0$ and $y = OL$, which is easily seen to be 4.

Thus

$$\begin{aligned} A &= \int_0^4 \int_{2-\sqrt{4-y}}^{2+\sqrt{4-y}} dx\,dy \\ &= \int_0^4 \Big[x\Big]_{2-\sqrt{4-y}}^{2+\sqrt{4-y}} dy \\ &= 2\int_0^4 \sqrt{4 - y}\,dy \\ &= \left[-\tfrac{4}{3}(4 - y)^{\frac{3}{2}}\right]_0^4 = \frac{32}{3} = 10\tfrac{2}{3} \text{ units, as before.} \end{aligned}$$

EXAMPLE 2

Prove that $\int\int f(x, y)dy\,dx$ taken over a closed area is equal to a single integral taken over the perimeter.

Taking the area given in Fig. 4, the value of the integral is

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)dy\,dx$$

Suppose we have a function $\phi(x, y)$ such that $\frac{\partial}{\partial y}\phi(x, y) = f(x, y)$, then

$$\begin{aligned}
\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)dy\,dx &= \int_{x_1}^{x_2}\Big[\phi(x, y)\Big]_{y\,=\,y_1}^{y\,=\,y_2} dx \\
&= \int_{x_1}^{x_2}\{\phi(x, y_2) - \phi(x, y_1)\}dx \\
&= \int_{x_1}^{x_2}\phi(x, y_2)dx - \int_{x_1}^{x_2}\phi(x, y_1)dx \\
&= \int_{x_1}^{x_2}\phi(x, y_2)dx + \int_{x_2}^{x_1}\phi(x, y_1)dx \\
&= \int \phi(x, y_2)dx \text{ over arc } LDK \\
&\qquad + \int \phi(x, y_1)dx \text{ over arc } KBL \\
&= \int \phi(x, y)dx \text{ over the perimeter of the curve.}
\end{aligned}$$

13. **Volume of a Solid as a Triple Integral.** We saw in Art. 11 that the volume V of the solid shown in Fig. 1 is given by

$$V = \int_{x_1}^{x_2}\int_{y_1}^{y_2} z\,dy\,dx \quad . \quad . \quad . \quad \text{(II.9)}$$

But $\int_{y_1}^{y_2} z\,dy$ represents the area $PQNM$ which, by the methods of Art. 12, may be written $\int_{y_1}^{y_2}\int_{z_1}^{z_2} dz\,dy$. The limit z_1 is zero in Fig. 1 where the xy-plane is a surface of the solid. In the general case, however, z_1 is not zero. Substituting the double integral for the single integral in (II.9), we have

$$V = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} dz\,dy\,dx \,. \quad . \quad . \quad \text{(II.10)}$$

It is usually more convenient to find volumes by double integrals as in Art. 11 than by using the relation just proved.

EXAMPLE

Find the volume of the paraboloid of revolution $x^2 + y^2 = 4z$ cut off by the plane $z = 4$.

We shall work this as an example in triple integration.

By (II.10) the volume is

$$V = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{\frac{x^2+y^2}{4}}^{4} dz\, dy\, dx$$

The limits are found thus—

(1) Assuming x and y fixed, z is found for the two surfaces of the solid. At the top surface $z = 4$, and at the lower surface $z = \dfrac{x^2 + y^2}{4}$

(2) Next, assuming x fixed, the least and greatest values of y are found by substituting $z = 4$ in $x^2 + y^2 = 4z$. Hence, $y = \pm\sqrt{16 - x^2}$ and the limits are as shown.

(3) The limiting values for x are clearly found by putting $z = 4$ and $y = 0$. Hence, the limits are $\pm\sqrt{16} = \pm 4$. Integrating with respect to z, we have

$$V = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \left(4 - \frac{x^2 + y^2}{4}\right) dy\, dx$$

$$= \int_{-4}^{4} \left[4y - \frac{x^2 y}{4} - \frac{y^3}{12}\right]_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} dx$$

$$= \int_{-4}^{4} \left(8\sqrt{16 - x^2} - \frac{x^2}{2}\sqrt{16 - x^2} - \frac{(16 - x^2)\sqrt{16 - x^2}}{6}\right) dx$$

$$= \tfrac{1}{3} \int_{-4}^{4} (16 - x^2)^{\frac{3}{2}}\, dx$$

$$= 32\pi$$

This result is most easily found by the method of Art. 11.

14. Centroids and Moments of Inertia. The following examples show how double integrals occur in problems involving the finding of centroids and moments of inertia.

EXAMPLE 1

A plate in the form of a quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is of small but varying thickness, the thickness at any point being proportional to the product of the distances of that point from the axes; show that the co-ordinates of the centroid are $\dfrac{8a}{15}$ and $\dfrac{8b}{15}$ (U.L.)

Consider the elementary area $\Delta x\ \Delta y$ at a point $P(x, y)$ in the quadrant; the density at this point is kxy, where k is a constant, so that the mass of the corresponding element of volume is $kxy\ \Delta x\ \Delta y$ and its moments about the x- and y-axes are $kxy^2\ \Delta x\ \Delta y$ and $kx^2y\ \Delta x\ \Delta y$ respectively.

Hence, if $(\bar{x}, \bar{y})$ be the co-ordinates of the centroid,

$$\bar{x} = \frac{\int_0^a \int_0^{y_1} x^2 y\, dy\, dx}{\int_0^a \int_0^{y_1} xy\, dy\, dx}; \ \bar{y} = \frac{\int_0^a \int_0^{y_1} xy^2\, dy\, dx}{\int_0^a \int_0^{y_1} xy\, dy\, dx}$$

where
$$y_1 = \frac{b}{a}\sqrt{a^2 - x^2}$$

Now
$$\int_0^a \int_0^{y_1} xy\, dy\, dx = \int_0^a \left[\left\{\frac{xy^2}{2}\right\}_0^{\frac{b}{a}\sqrt{a^2 - x^2}} \right] dx$$

$$= \frac{b^2}{2a^2}\int_0^a (a^2 x - x^3) dx$$

$$= \frac{b^2}{2a^2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{a^2b^2}{8}$$

$$\int_0^a \int_0^{y_1} x^2 y\, dy\, dx = \int_0^a \left[\left\{\frac{x^2y^2}{2}\right\}_0^{\frac{b}{a}\sqrt{a^2 - x^2}} \right] dx$$

$$= \frac{b^2}{2a^2}\int_0^a (a^2 x^2 - x^4) dx$$

$$= \frac{b^2}{2a^2}\left[\frac{a^2x^3}{3} - \frac{x^5}{5}\right]_0^a = \frac{a^3b^2}{15}$$

$$\int_0^a \int_0^{y_1} xy^2\, dy\, dx = \int_0^a \left[\left\{\frac{xy^3}{3}\right\}_0^{\frac{b}{a}\sqrt{a^2 - x^2}} \right] dx$$

$$= \frac{b^3}{3a^3}\int_0^a x(a^2 - x^2)^{\frac{3}{2}}\, dx$$

$$= \frac{b^3}{3a^3}\left[-\frac{(a^2 - x^2)^{\frac{5}{2}}}{5}\right]_0^a = \frac{b^3}{3a^3}\cdot\frac{a^5}{5} = \frac{a^2b^3}{15}$$

$\therefore$
$$\bar{x} = \frac{a^3b^2}{15} \Big/ \frac{a^2b^2}{8} = \frac{8a}{15}$$

and
$$\bar{y} = \frac{a^2b^3}{15} \Big/ \frac{a^2b^2}{8} = \frac{8b}{15}$$

EXAMPLE 2

The generators of a solid uniform cylinder are parallel to the axis of z. Its base is symmetrical about the axes of x and y. Show that the co-ordinates of the centre of gravity of the portion cut off from it by the plane $z = px + qy + h$ are given by $\bar{x} = pk_2^2/h$, $\bar{y} = qk_1^2/h$, $\bar{z} = \frac{1}{2}(p\bar{x} + q\bar{y} + h)$, where k_1 and k_2 are the radii of gyration of its base about the axes of x and y.

A solid cylinder whose base is a circle of radius a and whose length is $4a$ is cut into two equal portions by a plane making an angle of 45° with its axis. It is placed with its base on a horizontal plane which is gradually tilted about a horizontal straight line parallel to the minor axis of the ellipse which is the section of the cylinder by the dividing plane. If the cylinder does not slip on the plane and the upper portion does not slip on the lower, find the angle through which the plane can be tilted before the two parts separate. (U.L.)

When $x = 0$, $y = 0$, $z = h$ = mean height of portion, so that, if A = area of base of cylinder, volume of portion $= Ah$. If each element $\Delta x\ \Delta y$ of the area A be multiplied by the square of its distance y from OX, and the results all added together, the moment of inertia of the base about OX is obtained,

i.e.
$$Ak_1^2 = \int\int y^2 dx\, dy$$

Similarly,
$$Ak_2^2 = \int\int x^2 dx\, dy$$

Consider, now, an elementary prism parallel to OZ and with the area $\Delta x\ \Delta y$ as base; its volume $= z\Delta x\ \Delta y$ and its moments about the yz, zx, and xy planes are $zx\Delta x\ \Delta y$, $zy\Delta x\ \Delta y$, and $\frac{1}{2} z^2\ \Delta x\ \Delta y$ respectively.

Hence,
$$\bar{x} \,.\, Ah = \int\int zx\, dx\, dy$$
$$\bar{y} \,.\, Ah = \int\int zy\, dx\, dy$$
$$\bar{z} \,.\, Ah = \tfrac{1}{2} \int\int z^2\, dx\, dy$$

Now
$$\begin{aligned}\int\int zx\, dx\, dy &= \int\int (px + qy + h)x\, dx\, dy \\ &= p \int\int x^2\, dx\, dy + q \int\int xy\, dx\, dy + h \int\int x\, dx\, dy \\ &= p \,.\, Ak_2^2 + q(0) + h(0)\end{aligned}$$

for $\int\int xy\, dx\, dy$ = product of inertia of area A about a pair of principal axes = 0 (see Vol. I), and

$$\int\int x\, dx\, dy = A \times \text{distance of c.g. of base from } OY = A \times 0 = 0$$

Similarly, we prove
$$\int\int zy\, dx\, dy = q \,.\, Ak_1^2$$

Hence,
$$\bar{x} \,.\, Ah = pAk_2^2 \qquad \therefore\ \bar{x} = pk_2^2/h$$
$$\bar{y} \,.\, Ah = qAk_1^2 \qquad \therefore\ \bar{y} = qk_1^2/h$$

Again,
$$\begin{aligned}\tfrac{1}{2} \int\int z^2\, dx\, dy &= \tfrac{1}{2} \int\int (px + qy + h)^2 dx\, dy \\ &= \tfrac{1}{2}\, [p^2 \int\int x^2\, dx\, dy + q^2 \int\int y^2\, dx\, dy + h^2 \int\int dx\, dy \\ &\quad + 2pq \int\int xy\, dx\, dy + 2ph \int\int x\, dx\, dy + 2qh \int\int y\, dx\, dy] \\ &= \tfrac{1}{2}\, [p^2 \,.\, Ak_2^2 + q^2 \,.\, Ak_1^2 + h^2 \,.\, A]\end{aligned}$$

(the last three integrals vanishing)

$$\therefore \qquad \bar{z} \,.\, Ah = \tfrac{1}{2} A \left[p^2 \,.\, \frac{h\bar{x}}{p} + q^2 \,.\, \frac{h\bar{y}}{q} + h^2\right]$$

$$\therefore \qquad \bar{z} = \tfrac{1}{2}\, [p\bar{x} + q\bar{y} + h]$$

Let Fig. 7 represent the lower portion of the cylinder in the second part of the question, the minor axis of the elliptical section being assumed parallel to OX. Then, $h = 2a$, and if $z = px + qy + h$ be the equation of the dividing plane, $3a = p(0) + q(a) + 2a$, i.e. $q = 1$, since point $(0, a, 3a)$ lies on the plane; also $p = 0$, since the plane is parallel to OX. Therefore the equation of the dividing plane is $z = y + 2a$.

The radii of gyration of the base about OX and OY are each $\frac{a}{2}$. Hence, if $\bar{x}, \bar{y}, \bar{z}$ be the co-ordinates of the centre of gravity of the lower portion,

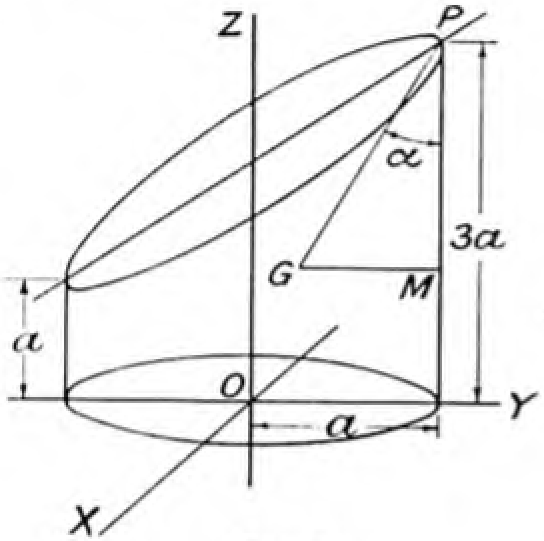


Fig. 7

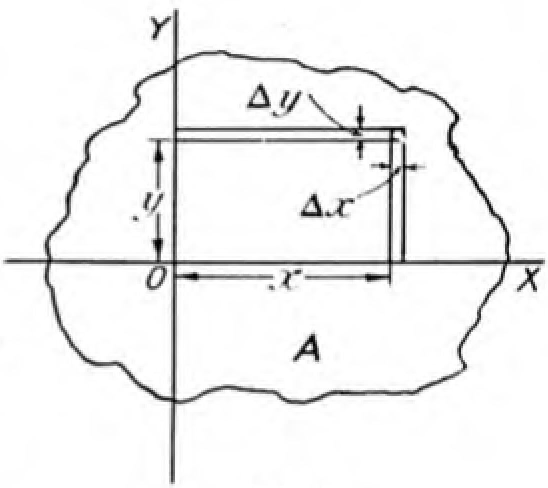


Fig. 8

$$\left.\begin{aligned} \bar{x} = 0,\ \bar{y} &= \frac{\frac{a^2}{4}}{2a} = \frac{a}{8} \\ \bar{z} &= \tfrac{1}{2}\left[0 + \frac{a}{8} + 2a\right] = \frac{17a}{16} \end{aligned}\right\} \text{from first part.}$$

Under the conditions stated, the two parts will be on the point of separating when the line joining the point $P(0, a, 3a)$ to the centre of gravity G of the lower portion is vertical.

Therefore, angle through which plane must be tilted is α where

$$\alpha = \widehat{GPM} = \tan^{-1}\frac{GM}{MP} = \tan^{-1}\frac{a - \frac{a}{8}}{3a - \frac{17a}{16}} = \tan^{-1}\frac{14}{31}$$

$$= 24^\circ\ 18'$$

15. **Centres of Pressure.** Let A (Fig. 8) be a plane area immersed in a homogeneous liquid. Taking rectangular axes OX, OY in its

plane, let A be divided up into an infinite number of elementary portions of area $\Delta x\ \Delta y$; the pressure on one of these $= p\ \Delta x\ \Delta y$, where p is the pressure at the point (x, y) in the area A. The moment of this pressure about OX

$$= py\ \Delta x\ \Delta y$$

and the moment about OY

$$= px\ \Delta x\ \Delta y$$

The resultant pressure on A

$$= \Sigma\ \Sigma p\ \Delta x\ \Delta y = \int\int p\ dx\ dy$$

and the sums of the moments of the pressures on all the elementary

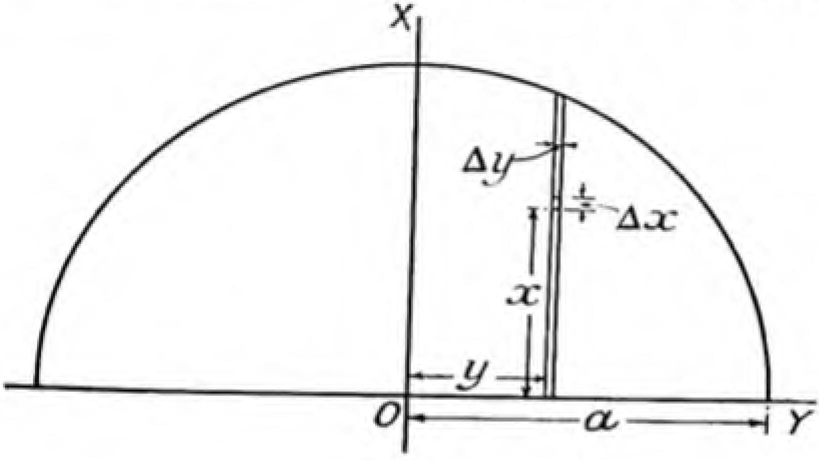


FIG. 9

areas about OX and OY are $\int\int py\ dx\ dy$ and $\int\int px\ dx\ dy$ respectively. Hence, if (ξ, η) be the co-ordinates of the centre of pressure,

$$\xi \cdot \int\int p\ dx\ dy = \int\int px\ dx\ dy \qquad \text{(II.11)}$$

and

$$\eta \cdot \int\int p\ dx\ dy = \int\int py\ dx\ dy \qquad \text{(II.12)}$$

and these equations give ξ, η, the integrals embracing the whole of the area A.

EXAMPLE 1

Find the value of $\int\int (a - x)^2 dx\ dy$ taken over half the circle $x^2 + y^2 = a^2$

A horizontal boiler has a flat bottom, and its ends are plane and semicircular. If it is just full of water, show that the depth of the centre of pressure of either end is $0{\cdot}7 \times$ total depth, very nearly. (U.L.)

To find the value of the double integral over the given area (Fig. 9), we imagine the area divided up into an infinite number of elements like $\Delta x\ \Delta y$, the product $(a - x)^2 \Delta x\ \Delta y$ calculated for each element and the results added together. For any assigned value of y, x varies from 0 to $\sqrt{a^2 - y^2}$ and y then varies from $- a$ to $+ a$. Hence, the integral is

$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}}(a-x)^2 dx\, dy = \int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}}(a^2-2ax+x^2)dx\, dy$$

$$= \int_{-a}^{a}\left[a^2x - ax^2 + \tfrac{1}{3}x^3\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_{-a}^{a}\{a^2\sqrt{a^2-y^2} - a(a^2-y^2) + \tfrac{1}{3}(a^2-y^2)^{\frac{3}{2}}\}dy$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{a^3\cos\theta - a^3\cos^2\theta + \tfrac{1}{3}a^3\cos^3\theta\}\, a\cos\theta\, d\theta$$

(with the substitution $y = a\sin\theta$)

$$= 2a^4\int_0^{\frac{\pi}{2}}(\cos^2\theta - \cos^3\theta + \tfrac{1}{3}\cos^4\theta)d\theta$$

$$= 2a^4\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{2}{3}\cdot 1 + \frac{1}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= 2a^4\left[\frac{5\pi}{16} - \frac{2}{3}\right]$$

$$= 0{\cdot}6302a^4$$

Let the semicircle in Fig. 9 represent an end of the boiler. The centre of pressure will lie on OX from symmetry. Then, if ξ be the height of the centre of pressure above OY,

$$\xi\iint p\, dx\, dy = \iint px\, dx\, dy$$

where p = pressure at point $(x, y) = w(a - x)$, w being the weight of unit volume of water. Also for any assigned value of y, x varies from 0 to $\sqrt{a^2 - y^2}$ and y then varies from $-a$ to $+a$.

Hence,
$$\xi = \frac{\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} w(a-x)x\, dx\, dy}{\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} w(a-x)dx\, dy}$$

$$= \frac{\int_{-a}^{a}\left[\left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^{\sqrt{a^2-y^2}}\right]dy}{\int_{-a}^{a}\left[\left[ax - \frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}}\right]dy}$$

$$= \frac{\int_{-a}^{a}\left[\frac{a}{2}(a^2-y^2) - \frac{1}{3}(a^2-y^2)^{\frac{3}{2}}\right]dy}{\int_{-a}^{a}\left[a(a^2-y^2)^{\frac{1}{2}} - \frac{1}{2}(a^2-y^2)\right]dy}$$

Let $y = a \sin \theta$, so that $dy = a \cos \theta \, d\theta$ and $\theta = -\frac{\pi}{2}, \frac{\pi}{2}$ when $y = -a, a$.

$$\therefore \qquad \bar{z} = \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{a^3}{2}\cos^2\theta - \frac{a^3}{3}\cos^3\theta\right] a \cos\theta \, d\theta}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[a^2 \cos\theta - \frac{a^2}{2}\cos^2\theta\right] a\cos\theta \, d\theta}$$

$$= \frac{2a^4 \int_0^{\frac{\pi}{2}} (\frac{1}{2}\cos^3\theta - \frac{1}{3}\cos^4\theta)d\theta}{2a^3 \int_0^{\frac{\pi}{2}} (\cos^2\theta - \frac{1}{2}\cos^3\theta)d\theta}$$

$$= a\left(\frac{\frac{1}{2}\cdot\frac{2}{3}\cdot 1 - \frac{1}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}}{\frac{1}{2}\cdot\frac{\pi}{2} \quad - \frac{1}{2}\cdot\frac{2}{3}\cdot 1}\right)$$

$$= a\left(\frac{16 - 3\pi}{12\pi - 16}\right) = 0{\cdot}3a, \text{ nearly.}$$

$\therefore$ Depth of centre of pressure $= 0{\cdot}7a$, nearly.

EXAMPLE 2

A rectangular lamina *ABCD* (Fig. 10) in a vertical plane is exposed to fluid pressure over the whole of one face, and its centroid *O* is at a depth *h* below the

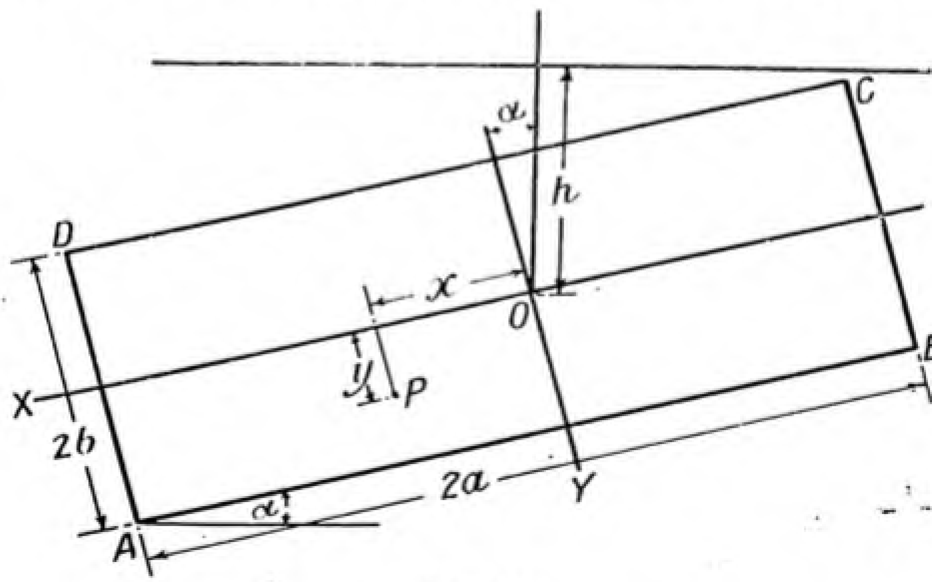


FIG. 10

free surface of the fluid. A is the lowest corner, AB and AD are of lengths $2a$ and $2b$, and AB makes an angle α with the horizontal. Rectangular axes OX, OY, with origin at O, are taken parallel to BA, DA. Express the co-ordinates of the centre of pressure by means of double integrals.

Evaluate the integrals, and prove that the co-ordinates of the centre of pressure are $\frac{1}{3}(a^2/h)\sin\alpha$ and $\frac{1}{3}(b^2/h)\cos\alpha$. (U.L.)

Consider an element of area $\Delta x\,\Delta y$ at the point $P(x,y)$.

$$\text{The depth of } P \text{ below } O = \text{projection of } OP \text{ on vertical}$$

$$= x\sin\alpha + y\cos\alpha$$

$$\therefore \quad \text{Pressure on area } \Delta x\,\Delta y = w(h + x\sin\alpha + y\cos\alpha)\,\Delta x\,\Delta y$$

where w = wt/unit vol of the fluid. The moments of this pressure about OY and OX are

$$w(h + x\sin\alpha + y\cos\alpha)x\,\Delta x\,\Delta y \text{ and } w(h + x\sin\alpha + y\cos\alpha)y\Delta x\,\Delta y$$

respectively.

The resultant pressure on the lamina $= 4ab\,wh$, so that if (ξ, η) be the co-ordinates of the centre of pressure,

$$\xi \,.\, 4ab\,wh = \int_{-a}^{a}\int_{-b}^{b} w(h + x\sin\alpha + y\cos\alpha)x\,dy\,dx$$

and

$$\eta \,.\, 4ab\,wh = \int_{-a}^{a}\int_{-b}^{b} w(h + x\sin\alpha + y\cos\alpha)y\,dy\,dx$$

Now

$$\int_{-a}^{a}\int_{-b}^{b} (hx + x^2\sin\alpha + xy\cos\alpha)dy\,dx$$

$$= \int_{-a}^{a}\left\{\left[(hx + x^2\sin\alpha)y + \frac{xy^2\cos\alpha}{2}\right]_{-b}^{b}\right\}dx$$

$$= 2b\int_{-a}^{a}(hx + x^2\sin\alpha)dx$$

$$= 2b\left[\frac{hx^2}{2} + \frac{x^3\sin\alpha}{3}\right]_{-a}^{a} = \tfrac{4}{3}\,a^3b\sin\alpha$$

Also

$$\int_{-a}^{a}\int_{-b}^{b}\{(h + x\sin\alpha)y + y^2\cos\alpha\}\,dy\,dx$$

$$= \int_{-a}^{a}\left[\left\{(h + x\sin\alpha)\frac{y^2}{2} + \frac{y^3\cos\alpha}{3}\right\}_{-b}^{b}\right]dx$$

$$= \int_{-a}^{a}\tfrac{2}{3}\,b^3\cos\alpha\,dx = \tfrac{2}{3}\,b^3\cos\alpha\,\Big(x\Big)_{-a}^{a} = \tfrac{4}{3}\,ab^3\cos\alpha$$

Hence,

$$\xi = \tfrac{4}{3}\,a^3b\sin\alpha/4abh = \tfrac{1}{3}\,(a^2/h)\sin\alpha$$

and

$$\eta = \tfrac{4}{3}\,ab^3\cos\alpha/4abh = \tfrac{1}{3}\,(b^2/h)\cos\alpha$$

EXAMPLE OF TRIPLE INTEGRATION

We work the following example as an illustration of the use of triple integrals.

A solid is cut out of the cylinder $x^2 + y^2 = a^2$ by the plane $z = 0$ and that part of the plane $z = mx$ (m constant) for which z is positive; the density of the solid at any point varies as the height of the point above the plane $z = 0$. Find the z-co-ordinate of the centre of mass of the solid.

Let the solid be divided up into elementary rectangular elements $\Delta x\ \Delta y\ \Delta z$ by planes parallel to the planes of reference. The mass of an element is $kz\ \Delta x\ \Delta y\ \Delta z$, and its moment about the plane $z = 0$ is $kz^2\Delta x\ \Delta y\ \Delta z$, k being some constant. Hence, if the z-co-ordinate of the mass-centre of the solid be $\bar{z}$ then

$$\bar{z} = \frac{\int\int\int kz^2 dz\, dy\, dx}{\int\int\int kz\, dz\, dy\, dx}$$

With x and y regarded as constant, z varies from 0 to mx; then with x regarded as constant, y varies from $-\sqrt{a^2 - x^2}$ to $+\sqrt{a^2 - x^2}$, and finally x varies from 0 to a.

Now
$$\int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{mx} z^2 dz\, dy\, dx$$

$$= \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\left[\frac{z^3}{3}\right]_0^{mx}\right] dy\, dx$$

$$= \frac{m^3}{3}\int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^3\, dy\, dx$$

$$= \frac{m^3}{3}\int_0^a \left[\left[x^3 y\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\right] dx$$

$$= \frac{2m^3}{3}\int_0^a x^3\sqrt{a^2 - x^2}\, dx$$

$$= \frac{2m^3a^5}{3}\int_0^{\frac{\pi}{2}} \sin^3\theta \cos^2\theta\, d\theta$$

(where $x = a \sin \theta$)

$$= \frac{2m^3a^5}{3}\left[\int_0^{\frac{\pi}{2}} \sin^3\theta\, d\theta - \int_0^{\frac{\pi}{2}} \sin^5\theta\, d\theta\right]$$

$$= \frac{2m^3a^5}{3}\left[\frac{2}{3}\cdot 1 - \frac{4}{5}\cdot\frac{2}{3}\cdot 1\right]$$

$$= \frac{4m^3a^5}{45}$$

$$\int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{mx} z\,dz\,dy\,dx$$

$$= \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\left[\frac{z^2}{2}\right]_0^{mx}\right] dy\,dx$$

$$= \frac{m^2}{2} \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2\,dy\,dx$$

$$= \frac{m^2}{2} \int_0^a \left[\left[x^2 y\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\right] dx$$

$$= m^2 \int_0^a x^2\sqrt{a^2-x^2}\,dx$$

$$= m^2a^4 \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta \text{ (where } x = a\sin\theta)$$

$$= m^2a^4 \left[\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta - \int_0^{\frac{\pi}{2}} \sin^4\theta\,d\theta\right]$$

$$= m^2a^4 \left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{m^2a^4\pi}{16}$$

Hence $$z = \frac{4m^3a^5}{45}\bigg/\frac{m^2a^4\pi}{16} = \frac{64ma}{45\pi}$$

THE PROBABILITY INTEGRAL. The integral $\int_a^b \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}\,dx$ is of importance in the theory of probability. It is not integrable in exact form but, as we saw in Volume I, it can be integrated in the form of an infinite series and approximately evaluated. In the particular case $a = -\infty$, $b = +\infty$, however, it can be evaluated exactly.

Let $$I = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}\,dx \quad . \quad . \quad (II.13)$$

Changing the independent variable to y does not alter the value of the integral, hence

$$I = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}}\,dy \quad . \quad . \quad (II.14)$$

and multiplying corresponding sides,

$$I^2 = \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}\,dx \times \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}\,dy$$

or

$$I^2 = \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2\sigma^2}}\,dx\,dy \qquad . \qquad . \qquad \text{(II.15)}$$

The area of integration in (II.15) just covers the whole xy-plane. Changing to polar co-ordinates (r, θ), with the same origin and the axis of x as initial line, the element of area $\Delta x\,\Delta y$ becomes $r\,\Delta\theta\,\Delta r$ and $x^2 + y^2 = r^2$. Since the changed integral must just cover the whole plane, the limits of integration are 0 to 2π for θ and 0 to ∞ for r. Thus (II.15) becomes

$$I^2 = \frac{1}{2\pi\sigma^2}\int_0^{\infty}\int_0^{2\pi} re^{-\frac{r^2}{2\sigma^2}}\,d\theta\,dr$$

$$I^2 = \frac{1}{\sigma^2}\int_0^{\infty} re^{-\frac{r^2}{2\sigma^2}}\,dr$$

$$= \frac{1}{\sigma^2} \times \sigma^2\left[-e^{-\frac{r^2}{2\sigma^2}}\right]_0^{\infty}$$

$$= 1$$

and

$$I = 1$$

a result we assumed in Volume I.

16. **Changing the Order of Integration.** In (II.7) and (II.8) we have two equivalent expressions for the area of a plane figure; each is a double integral, the order of integration in one being the opposite of that in the other. If we repeat the procedure of Art. 12 to find the sum of the products of each element of area $\Delta x\,\Delta y$ and $f(x, y)$, where the latter is any function of x and y, or of either alone, we obtain $\int\int f(x, y)\,dx\,dy$ and $\int\int f(x, y)\,dy\,dx$, each with appropriate limits for x and y, as equivalent expressions for the sum. The limits of x and y in each case depend upon the shape of the boundary of the area. Thus we see that with the necessary changes in the limits of integration the order of integration in a double integral may be changed. As the integral with changed order may be easier to evaluate than the original integral, it is convenient to be able to change the order. In Art. 15, Example 1, we found that the given double integral taken over the semicircle in Fig. 9 was equal to

$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}}(a-x)^2\,dx\,dy$$

Assuming the area to be divided into horizontal strips, we see that the limits of integration are 0 to a for x and $-\sqrt{a^2-x^2}$ to $+\sqrt{a^2-x^2}$ for y. The value of the integral over the given area is therefore given also by

$$\int_{0}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}(a-x)^2\,dy\,dx$$

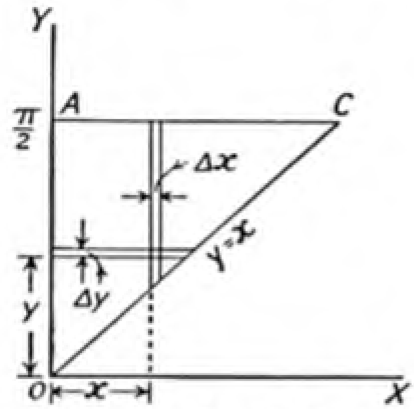


FIG. 11. CHANGING ORDER OF INTEGRATION

When the area of integration is not given, its boundary may be determined from the given limits of integration, as in the following example.

EXAMPLE

Show in a diagram the region over which the repeated integral

$$\int_{0}^{\frac{\pi}{2}} dy\int_{0}^{y}\cos 2y\sqrt{1-k^2\sin^2 x}\,dx$$

extends. Change the order of integration and evaluate the integral.

The limits for x are 0 to y which indicates that the horizontal strips into which the area is supposed to be divided extend from the axis of y to the line $y = x$. The limits for y are 0 to $\pi/2$, and the area of integration is that of the triangle bounded by the straight lines $x = 0$, $y = x$, and $y = \pi/2$, i.e. OAC, Fig. 11. Now to change the order we assume the area to be divided into vertical strips as shown. The limits for y are now x to $\pi/2$, and those for x are 0 to $\pi/2$. The changed integral is

$$I = \int_{0}^{\frac{\pi}{2}}\int_{x}^{\frac{\pi}{2}}\cos 2y\sqrt{1-k^2\sin^2 x}\,dy\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 x} \left[\int_x^{\frac{\pi}{2}} \cos 2y \, dy \right] dx$$

$$= -\tfrac{1}{2} \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 x} \sin 2x \, dx$$

$$= -\tfrac{1}{2} \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2 x)^{\frac{1}{2}} \, d(\sin^2 x), \text{ since } \frac{d}{dx}(\sin^2 x) = \sin 2x$$

$$\therefore \quad I = \frac{1}{2k^2} \times \frac{2}{3} \left[(1 - k^2 \sin^2 x)^{\frac{3}{2}} \right]_0^{\frac{\pi}{2}} = \frac{1}{3k^2} \left\{ (1 - k^2)^{\frac{3}{2}} - 1 \right\}$$

EXAMPLES II

Integrate the following, in which a, b, and R are constants—

(1) $\int_0^a \int_0^b (x + y) dx \, dy$ (2) $\int_0^a \int_0^b xy \, dx \, dy$

(3) $\int_0^a \int_0^b (x^2 + y^2) dx \, dy$ (4) $\int_0^{2\pi} \int_0^R r \, dr \, d\theta$

(5) $\int_0^{2\pi} \int_0^R r^3 \, dr \, d\theta$ (6) $\int_{-1}^{2} \int_{-3}^{3} (y^2 - 3xy) dx \, dy$

(7) Verify the truth of the relation $\int_a^b \int_c^d f(x, y) dx \, dy = \int_c^d \int_a^b f(x, y) dy \, dx$ in which a, b, c, and d are constants, by testing it for each of the integrals in Ex. (1) to (6) above.

Integrate the following—

(8) $\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} \cos (x + y) dx \, dy$ (9) $\int_1^2 \int_0^2 e^{(x+y)} \, dx \, dy$

(10) $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (a \cos 2\theta + b \sin 2\theta) d\theta \, d\phi$

(11) $\int_{-3}^{3} \int_0^1 \int_1^2 (x + y + z) dx \, dy \, dz$

Integrate the following and verify that when the limits of integration are not all constants, the relation of Ex. 7 is not true.

(12) $\int_0^r \int_0^{\sqrt{r^2 - x^2}} dy \, dx$ (13) $\int_0^r \int_0^{\sqrt{r^2 - y^2}} x^2 \, dx \, dy$

(14) $\int_0^r \int_0^{\sqrt{r^2 - x^2}} (x^2 + y^2) dy\, dx$ (15) $\int_0^{\pi} \int_0^{a(1 + \cos\theta)} x\, dx\, d\theta$

(16) Evaluate $\int_{-3}^{3} \int_1^2 \int_0^2 (x + y + z) dx\, dy\, dz$

(17) Evaluate $\int_0^a \int_0^b \int_0^c (x^2 + y^2) dx\, dy\, dz$

(18) Evaluate $\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2) dx\, dy\, dz$

(19) Evaluate $\int_0^r \int_0^\alpha \int_0^\beta (x^2 + y^2 + z^2) dx\, dy\, dz$ where r is constant, $\beta = \sqrt{r^2 - y^2 - z^2}$, $\alpha = \sqrt{r^2 - z^2}$

(20) Using double integrals, find (i) the area, (ii) the moment of inertia about OX of the area enclosed by the lines $x = 0$, $y = 0$, $\frac{x}{a} + \frac{y}{b} = 1$. Find also the position of the centroid of the area.

(21) Find, using double integrals, (i) the area, (ii) the moment of inertia about OX, of that portion of the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which lies above OX. Find the position of its centroid.

(22) Find the area enclosed by the cardioid $r = a(1 + \cos\theta)$ and find also the polar co-ordinates of its centroid.

(23) Evaluate $\iint x^2\, dx\, dy$ over the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. From this result, without further integration, evaluate $\iint r^2\, dx\, dy$ taken over the area, where $r^2 = x^2 + y^2$. What do these two integrals represent?

(24) Evaluate $\iint x\, dx\, dy$, $\iint x^2\, dx\, dy$, and $\iint xy\, dx\, dy$ over the area of the rectangle bounded by OX and OY and the lines $x = a$, $y = b$.

(25) Evaluate the integrals in the last example over the area of that half of the circle $x^2 + y^2 = a^2$ which lies above OX.

(26) Evaluate $\iint x^2\, dx\, dy$ over the area bounded by the parabola $y^2 = 4x$ and the line $x = 4$. Find the position of the centroid of this area.

(27) The density at any point (x, y) of a lamina is $\frac{\sigma}{a}(x + y)$ where σ and a are constants. The lamina is bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$. Find the position of the centre of gravity. If the total mass is M, find the moment of inertia of the mass (i) about OX, (ii) about O.

(28) If w lb is the load per foot run on a horizontal cantilever, the bending moment at a distance x from the free end is $\int_0^x \int_0^x w\, dx\, dx$. Find the bending moment in the cases (i) $w = w_0 \frac{x}{a}$ and (ii) $w = w_0 \frac{x^2}{a^2}$ where w_0 and a are constants.

(29) The normal stress, p lb/in.^{2}, at a distance y in. above the neutral axis of a beam section is given by $p = \frac{E}{R} y$ where E and R are constants. Show that the resultant normal force on the section is $\frac{E}{R} \iint y\, dx\, dy$ taken over the area, x being measured parallel to the neutral axis, and y perpendicular to it. Hence,

since the resultant normal force on the cross-section is zero in a horizontal beam with vertical loads, show that the neutral axis passes through the centroid of the section.

(30) With the notation of the last example, show that the effect of the normal stress is to produce a couple of moment $\frac{E}{R}\int\int y^2\,dx\,dy$ taken over the area.

(31) The elastic energy stored up in a piece of elastic material under uniform stress q lb/in.^{2} is $\frac{q^2}{2N}V$ where N is a constant and V the volume in cubic inches. Show that if a piece of material is under stress q which varies from point to point, the energy stored up is $\frac{1}{2N}\int\int\int q^2\,dx\,dy\,dz$, taken over the volume, x, y, and z being rectangular co-ordinates. Find the elastic energy in a cylindrical piece of material, radius R in., length L in., in which the stress q varies as the distance from the axis, being zero at the axis and q_0 at the outer surface.

(32) With the notation of Ex. 31, but using cylindrical co-ordinates ρ, θ, z, where $x = \rho\cos\theta$, $y = \rho\sin\theta$, $z = z$, show that the energy is

$$\frac{1}{2N}\int\int\int q^2\rho dz\,d\rho\,d\theta$$

taken over the volume, and that the energy in the cylinder is

$$\frac{L}{2N}\int_0^{2\pi}\int_0^R q^2\rho\,d\rho\,d\theta = \frac{q_0^2L}{2NR^2}\int_0^{2\pi}\int_0^R \rho^3\,d\rho\,d\theta$$

Find its value.

(33) Using polar co-ordinates and double integrals, find the position of the centroid of a sector of a circle of radius r and central angle 2α.

(34) Find the second moment of the sector of Ex. 33 about the centre of the circular arc.

(35) Integrate $\int\int r\cos\theta\,dr\,d\theta$ over the area of the circle $r = c\cos\theta$.

(36) Evaluate $\int\int(x^2+y^2)x\,dx\,dy$ over a quadrant of the circle $x^2+y^2=a^2$.

(37) Find the value of $\int\int(x^2+y^2)x\,dx\,dy$ for a quadrant of the ellipse $2x^2+y^2=1$. (U.L.)

(38) Find the values of the integrals $\int\int xy\,dx\,dy$ and $\int\int x^2y^2\,dx\,dy$ taken over the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. (U.L.)

(39) Find the value of $\int\int xy\,dx\,dy$ taken over the positive quadrant of the circle $x^2+y^2=a^2$. (U.L.)

(40) Work out the integrals $\int_0^5\int_0^{x^2} x(x^2+y^2)dx\,dy$; $\int_0^\infty\int_0^\infty e^{-x^2(1+t^2)}x\,dx\,dt$. (U.L.)

(NOTE. In the former integral the limits 0 to x^2 appear to refer to the integration with respect to y.)

(41) Find the centroid of a semicircular area using polar co-ordinates, (i) with the pole at one end of the diameter, (ii) with the pole at the centre of the diameter. Show that the results are consistent.

(42) Find the centroid of the area enclosed by the parabola $y^2 = kx$, the axis of x, and the latus-rectum of the parabola.

(43) Find the centroid of the area enclosed between the graphs of $y^2 = ax$ and $ay = x^2$.

(44) Find the position of the centroid of the area under $y = \sin x$ from $x = 0$ to $x = \pi$.

(45) The density of the material of a right circular cylinder of radius r varies as the distance from the axis and as the distance from one end. Find the radius of gyration about the axis.

(46) Evaluate $\int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} r\sqrt{a^2 - r^2}\, dr\, d\theta$ and $\int_0^{\pi} \int_0^{\pi} \sin(a\theta + b\phi)d\theta\, d\phi$

(47) Explain the use of double integrals in finding centres of pressures of areas under fluid thrust.

A closed rectangular tank is filled with water, the ends are vertical and of area 4 ft by 6 ft each, the 4 ft edge makes 30° with the upward vertical. Find the total fluid thrust on one end and the centre of pressure.

(NOTE. Assume the pressure at the highest point to be zero.) (U.L.)

(48) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta\, d\theta$ and $\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ gives the contour of the base of a right cylinder; the height is c. The cylinder is bevelled down so that the height z of any point (x, y) of the base in the first quadrant is given by $z = c\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)$. Express the volume left in this quadrant as a double integral and show that its value is $\frac{abc}{4}\left(\pi - \frac{13}{6}\right)$ (U.L.)

(49) Define the double integral $\iint (x, y)dx\, dy$ taken over a closed area S in the xy-plane, and show that it is equivalent to a repeated single integral.

The circular cylinder $x^2 + y^2 = a^2$ is cut by the plane $z = 0$ and by an oblique plane so that its greatest height is p and its least height q. Find by double integration the position of the centroid of the volume of the cylinder contained between these planes. (U.L.)

(50) The polar co-ordinates of any point in a sphere of radius r with its centre at the origin are ρ, θ, ϕ. Express its volume as a double integral and evaluate the integral.

(51) Find by using polar co-ordinates the volume and the centroid of the volume of a hemisphere.

(52) A rectangular block of wood floats in water, the water-line section being a square of side 10 in. If the block is immersed to a depth of 5 in., find the height of the metacentre above the base.

(53) A cylinder whose right section is an ellipse of semi-axes a and b $(a > b)$ floats in water with its axis just in the surface; find the metacentric height for small angular displacements about the axis when the longer axis of the right section is horizontal.

(54) Find the metacentric height in each of the following cases: (*a*) a right cone, height h, radius r, floating with its axis vertical and vertex downward; (*b*) a sphere of radius r; (*c*) a paraboloid of revolution formed by rotating about OX the parabola $y^2 = kx$ from $x = 0$ to $x = H$, floating so that its axis is vertical and vertex downward.

(55) Show that if $\bar{x}$, $\bar{y}$ are the co-ordinates of the centre of gravity of a plane area, then $\bar{x}\iint dy\, dx = \iint x\, dy\, dx$ taken over the area. Write down the corresponding relation involving $\bar{y}$. Use these relations to find the position of the c.g.

of the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the positive parts of the axes of x and y.

(56) Calculate the value of $\iint r^3\, dr\, d\theta$ for the area included between the two circles $r = 2 \sin \theta$ and $r = 4 \sin \theta$. Show that this represents the polar moment of inertia of the area about the origin.

(57) Show that if ρ represents the density of the material at any point (x, y, z) of a solid, the moment of inertia of the solid about the axis OX is $\iiint \rho(y^2 + z^2)dx\, dy\, dz$ taken over the solid, and that if ρ is constant, the average value of the square of the distance of a particle from OX is the above integral divided by the mass. Find the root mean square of the distance (i.e. the radius of gyration) in the case of the sphere $x^2 + y^2 + z^2 = a^2$.

(58) A rectangle 6 ft long and 4 ft high rests with its plane vertical and its upper 6 ft edge in the surface of a fluid at rest. Find the centre of pressure on the area of the rectangle.

(59) Find the centre of pressure on the area of a circle, radius r ft, lying in a plane inclined at an angle α to the vertical and whose centroid is at a depth h ft below the surface of the water.

(60) A cylinder floats with its axis just in the surface of the water. The radius is r ft and the length l ft, and the density σ is uniform. Find the position of the metacentre (i) for small angular displacements of the axis in a vertical plane, and (ii) for small angular displacements about the axis.

(61) Show that in evaluating $\iint r^2 \sin \theta\, dr\, d\theta$ over that half of the circle $r = 2a \cos \theta$ which lies above OX we must take the limits of integration as (i) 0 to $2a \cos \theta$ for r and 0 to $\frac{\pi}{2}$ for θ, if we integrate first with respect to r and then with respect to θ, and (ii) 0 to $\cos^{-1}\frac{r}{2a}$ for θ and 0 to $2a$ for r if we reverse the order of integration. Show that the value of the integral is $\frac{2a^3}{3}$

(62) Show that the product of inertia P of the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant is given by $\int_0^b \int_0^{a\sqrt{1-\frac{y^2}{b^2}}} xy\ dx\ dy$, or, if the order of integration is reversed, by $\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} xy\ dy\ dx$. Show that the value of P is $\frac{a^2b^2}{8}$

(63) Using polar co-ordinates, find the value of $\iint r^2 dr\ d\theta$ taken over the area of a quadrant of a circle of radius a, the centre of the circle being the pole and one of the bounding radii the initial line.

A circular disc of weight W lies on a rough horizontal plane and is capable of turning about a fixed vertical axis through a point O of its rim. Assuming the pressure of the disc on the plane to be equally distributed over the area, prove that the tangential force that must be applied at the other extremity of the diameter through O so as just to cause the disc to turn about O is $16\mu W/9\pi$, where μ is the coefficient of friction between the disc and the plane. (U.L.)

(64) Show that the equation

$$(x^2 + y^2 + z^2 + c^2 - a^2)^2 = 4c^2(x^2 + y^2)$$

represents an anchor ring. Sketch the surface, showing how it lies with respect to the axes of reference.

Prove that the co-ordinates of any point on the surface may be expressed in the form $[(c + a\cos\theta)\cos\phi,\ (c + a\cos\theta)\sin\phi,\ a\sin\theta]$, and evaluate the surface area by calculating $\int\int a(c + a\cos\theta)d\theta\, d\phi$ (U.L.)

(65) Indicate by a sketch the region over which the double integral

$$\int_0^a dx \int_x^a (x^2 + y^2)\,dy$$

is taken.

Write down the integral with the order of integration changed, and hence, or directly, evaluate it. (U.L.)

(66) Change the order of integration in

$$\int_0^a dx \int_0^x \frac{dy}{\{(a-x)(x-y)(a-y)(a+y)\}^{\frac{1}{2}}}$$

Hence, or otherwise, evaluate the integral. (U.L.)

CHAPTER III

PERIODIC FUNCTIONS—FOURIER'S SERIES—HARMONIC ANALYSIS

17. **Periodic Functions.** The functions $\sin x$, $\cos x$, $\tan x$, $\operatorname{cosec} x$, $\sec x$, and $\cot x$ have the functional relationship expressed by $f(x) = f(x + 2\pi)$. Similarly, $\sin(nx + \alpha)$, $\cos(nx + \alpha)$, etc., satisfy for all values of x the relation

$$f(x) = f\left(x + \frac{2\pi}{n}\right)$$

More generally, many functions of a single independent variable met with in mathematics applied to engineering satisfy the relation

$$f(x) = f(x + \alpha) \qquad . \qquad . \qquad . \qquad \text{(III.1)}$$

where x is the variable and α is constant.

Functions satisfying this relation are known as *periodic functions* and are such that as x increases through the series of ranges of values a to $a + \alpha$, $a + \alpha$ to $a + 2\alpha$, $a + 2\alpha$ to $a + 3\alpha$, etc., where a is any real number, $f(x)$ passes through the same sequence of values in each range. Considered graphically, the above statement means that if the graph of $f(x)$ is drawn (Fig. 12) and ordinates are drawn dividing the axes of x into equal parts of length α these ordinates will divide the graph into identical portions. Any one of the portions can be made to coincide with any other by sliding it parallel to the axis of x through a distance $n\alpha$, where n is any positive or negative integer. α is known as the *period* of the function, or as the *periodic time*, when x represents time. We assume here that α has the least value which will satisfy (III.1) for all values of x.

18. **Harmonic Analysis. Fourier's Series.** In addition to the ordinary trigonometrical functions, many periodic functions occur in connection with engineering problems. The displacement, velocity, and acceleration of the piston of the simple engine are periodic functions of the time. So also are the voltage and current in an alternating current circuit. The function representing the displacement of a particle in simple harmonic motion is very simple, that representing the displacement of the piston of the simple engine is more complicated. On p. 68, Volume I, we found it convenient to

substitute for the latter the sum of two terms representing respectively displacements of particles executing simple harmonic motions whose periods are in the ratio 1 to 2. By expanding

$$\left\{1 - \frac{r^2}{l^2}\sin^2(\omega t + \alpha)\right\}^{\frac{1}{2}}$$

and making use of the relation $2\sin^2 x = 1 - \cos 2x$ we can show (see p. 68, Vol. I) that if $\frac{r}{l} < 1$ the displacement x of the piston can

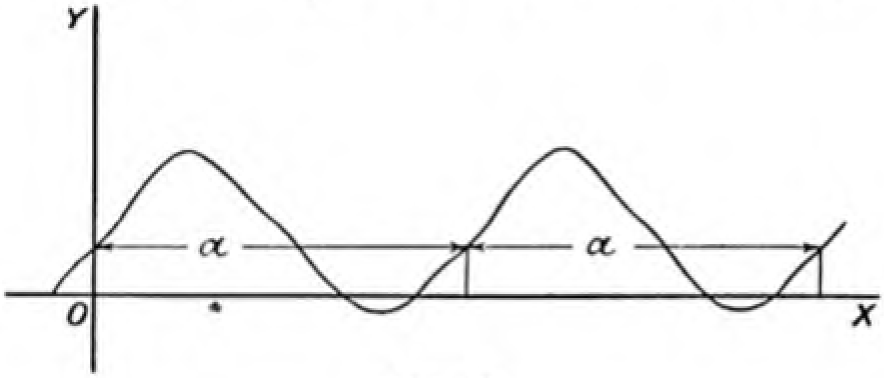


FIG. 12

be expressed to any required degree of accuracy by taking a sufficient number of terms of the infinite series

$$\begin{aligned} x = a_0 + a_1\cos(\omega t + \alpha) &+ a_2\cos(2\omega t + 2\alpha) \\ &+ a_3\cos(3\omega t + 3\alpha) + \ldots \\ &+ a_n\cos(n\omega t + n\alpha) + \ldots \qquad \text{(III.2)} \end{aligned}$$

where the quantities a_1, a_2, a_3, etc., are functions of $\frac{r}{l}$. If $l \nless 3r$, as is usually the case, the series converges rapidly and the first three terms of the series are usually sufficient to express x to the necessary degree of accuracy.

The method of splitting up a periodic function into a series such as that in (III.2) is known as *harmonic analysis*. The term $a_1\cos(\omega t + \alpha)$ is known as the *fundamental*, $a_2\cos(2\omega t + 2\alpha)$ is the *octave* or *second harmonic*, and the succeeding terms are the third, fourth, fifth, etc., harmonics. (III.2) is a particular series, the more general form of which is

$$\begin{aligned} f(x) = a_0 + a_1\cos(x + \alpha_1) &+ a_2\cos(2x + \alpha_2) \\ &+ a_3\cos(3x + \alpha_3) + \ldots \\ &+ a_n\cos(nx + \alpha_n) + \ldots \qquad \text{(III.3)} \end{aligned}$$

where $f(x)$ represents a periodic function of x of period 2π.

Since $\sin\left(A + \frac{\pi}{2}\right) = \cos A$ (III.3) may be written in the form

$$\begin{aligned} f(x) = a_0 &+ a_1 \sin(x + \alpha_1) + a_2 \sin(2x + \alpha_2) \\ &+ a_3 \sin(3x + \alpha_3) + \ldots \\ &+ a_n \sin(nx + \alpha_n) + \ldots \end{aligned} \qquad \text{(III.4)}$$

the values of α_1, α_2, α_3, etc., being different from those in (III.3).

The theorem which states that certain periodic functions can be expressed in either of the forms (III.3) or (III.4) is known as *Fourier's Theorem*, and the series is known as *Fourier's Series.* (We do not attempt any formal proof of this theorem.*)

Since each of the terms $a_n \sin(nx + \alpha_n)$ can be written in the form $a_n \cos\alpha_n \sin nx + a_n \sin\alpha_n \cos nx$ or $A \sin nx + B \cos nx$, where $A = a_n \cos\alpha_n$ and $B = a_n \sin\alpha_n$, we may write (III.4) in the form

$$\left.\begin{aligned} f(x) = B_0 &+ B_1 \cos x + B_2 \cos 2x + B_3 \cos 3x \\ &+ \ldots + B_n \cos nx + \ldots \\ &+ A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x \\ &+ \ldots + A_n \sin nx + \ldots \end{aligned}\right\} \quad \text{(III.5)}$$

We shall find the following a more convenient form

$$\left.\begin{aligned} f(x) = \tfrac{1}{2}b_0 &+ b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x \\ &+ \ldots + b_n \cos nx + \ldots \\ &+ a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x \\ &+ \ldots + a_n \sin nx + \ldots \end{aligned}\right\} \quad \text{(III.6)}$$

In the preceding we have assumed that the period of the function is 2π. We shall see later how to deal with functions whose period is any constant.

19. To Determine the Constants b_0, b_1, a_1, b_2, a_2, etc., in the Fourier Series. Integrating both sides of (III.6) between the limits $x = 0$ and $x = 2\pi$, we have, since

$$\int_0^{2\pi} \sin nx \, dx = \int_0^{2\pi} \cos nx \, dx = 0$$

* Readers should consult *A Practical Treatise on Fourier's Theorem and Harmonic Analysis*, by Eagle (Longmans, Green & Co.).

for all integral values of n,

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} \tfrac{1}{2}b_0\,dx$$

all the other integrals vanishing.

Hence, $$b_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = 2 \times \text{mean value of } f(x) \quad . \quad \text{(III.7)}$$

In what follows we make use of the fact that, if p and q are integers, the integrals from 0 to 2π of $\sin px \sin qx$, $\sin px \cos qx$, and $\cos px \cos qx$ are all zero.

Again, multiply both sides of (III.6) by $\cos nx$ and integrate as before.

Then $$\int_0^{2\pi} f(x) \cos nx\,dx = b_n \int_0^{2\pi} \cos^2 nx\,dx$$

all the other integrals vanishing.

Hence, $$\int_0^{2\pi} f(x) \cos nx\,dx = \pi b_n$$

and $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx = 2 \times \text{mean value of } f(x) \cos nx \quad . \quad . \quad . \quad \text{(III.8)}$$

Now multiply both sides of (III.6) by $\sin nx$ and integrate.

Then $$\int_0^{2\pi} f(x) \sin nx\,dx = a_n \int_0^{2\pi} \sin^2 nx\,dx$$

from which we see that

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\,dx = 2 \times \text{mean value of } f(x) \sin nx \quad . \quad . \quad . \quad \text{(III.9)}$$

By giving to n the values 1, 2, 3, etc., in (III.8) and (III.9) we evaluate a_1, b_1, a_2, b_2, a_3, b_3, etc., and (III.7) gives the value of b_0. In general we can also determine b_0 by putting $n = 0$ in the expression for b_n. It is to make this possible that we take $\frac{1}{2}b_0$ and not b_0 as the first term of the series.

EXAMPLE 1

Find the Fourier series which represents the periodic function $f(x)$ of period 2π such that between $x = 0$ and $x = 2\pi$, $f(x) = x$. The graph of $f(x)$ is given in Fig. 13.

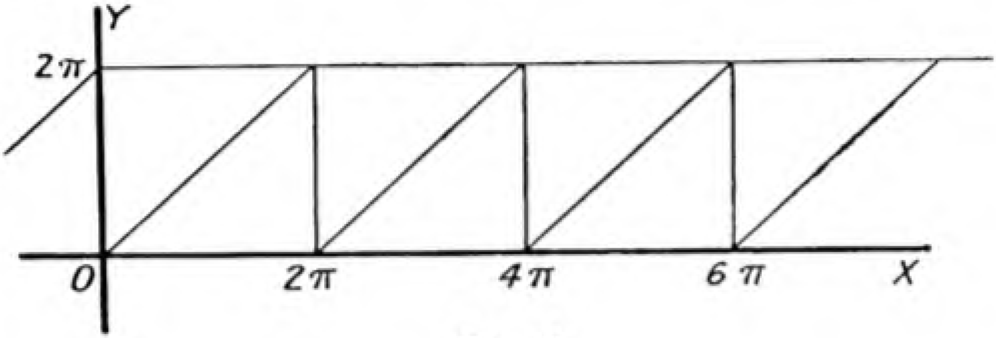


FIG. 13

Here $b_0 = \frac{1}{\pi}\int_0^{2\pi} x\,dx = 2\pi$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx = \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{2\pi} = 0 \text{ for all integral values of } n$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx = \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{2\pi} = -\frac{2}{n}$$

and substituting in (III.6)

$$f(x) = \pi - 2\left[\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \frac{\sin 5x}{5} + \ldots\right] \quad (1)$$

which is true for all values of x except those for which $f(x)$ is discontinuous (see below). This relation may be written

$$x = \pi - 2\left[\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \frac{\sin 5x}{5} + \ldots\right] \quad (2)$$

which is true only for values of x between $x = 0$ and $x = 2\pi$.

It should be noticed that when $x = 0$ the right-hand side of (1) apparently reduces to π whilst the left-hand side is zero. Thus the series fails to give a correct value of the function when $x = 0$. This will be found to be the case also when $x = 2\pi$. The reader will remember that we are not finding a series to represent the function x, but one to represent the function $f(x)$, which is the same as x only between the limits $x = 0$ and $x = 2\pi$. At each of the points $x = 0$ and $x = 2\pi$ the function is discontinuous (Fig. 13) and the function has two limiting values, i.e. $f(x) = 0$ and $f(x) = 2\pi$. The series (2) gives a value which is the arithmetic mean of these. It can be shown that at all points of finite discontinuity of such periodic functions as occur in engineering problems, the series gives a value which is the arithmetic mean of the two limiting values of the function.

EXAMPLE 2

Develop a Fourier series to represent a periodic function $f(x)$ of period 2π where $f(x) = 0$ from $x = 0$ to $x = \pi$ and $f(x) = 1$ from $x = \pi$ to $x = 2\pi$.

We have from (III.7)

$$b_0 = \frac{1}{\pi}\int_0^{\pi} 0\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 1\,dx = 1$$

the two integrals occurring because of the discontinuity in the function.

From (III.8)

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} 0\,dx + \int_{\pi}^{2\pi}\cos nx\,dx\right\}$$

$$\therefore \qquad b_n = \frac{1}{n\pi}\left[\sin nx\right]_{\pi}^{2\pi} = 0$$

Notice that we cannot evaluate b_0 directly by putting $n = 0$ in this last result, for this would give $\frac{0}{0}$, which is indeterminate. We have, however, already determined the value of b_0.

Again, from (III.9),

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} 0\,dx + \int_{\pi}^{2\pi}\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{\pi}^{2\pi} = \frac{1}{\pi}\left\{-\frac{\cos 2n\pi}{n} + \frac{\cos n\pi}{n}\right\}$$

$$= \frac{1}{n\pi}\{-1 + (-1)^n\}$$

$$\therefore \qquad a_n = -\frac{2}{n\pi} \text{ if } n \text{ is odd}$$

and $\qquad a_n = 0$ if n is even

Hence, the series is

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \ldots\}$$

In finding the values of b_0, b_n, and a_n in the above we have dealt with the function over the range $x = 0$ to $x = 2\pi$. We could equally well have selected the range $x = -\pi$ to $x = \pi$, in which case we

should have obtained in place of (III.7), (III.8) and (III.9) the relations

$$b_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad . \quad . \quad . \quad \text{(III.10)}$$

With these values of the coefficients, the series (III.6) gives values of the periodic function between the limits $x = -\pi$ and $x = \pi$.

EXAMPLE 3

Find a Fourier series to represent a periodic function $f(x)$ of period 2π such that $f(x) = x$ from $x = -\pi$ to $x = \pi$. (Compare with Ex. 1 above.)

Using (III.10) we have

$$b_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = 0$$

since $x \cos nx$ is an odd function of x, and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$x \sin nx$ being an even function

$$\therefore \qquad a_n = \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = -\frac{2\cos n\pi}{n} = \frac{2}{n}(-1)^{n+1}$$

The series is therefore

$$f(x) = 2\,\{\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \tfrac{1}{5}\sin 5x\ .\ .\ .\}$$

for all values of x except those at points of discontinuity, or

$$x = 2\,\{\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \tfrac{1}{5}\sin 5x\ .\ .\ .\}$$

for values of x between, but not including, $x = -\pi$ and $x = \pi$.

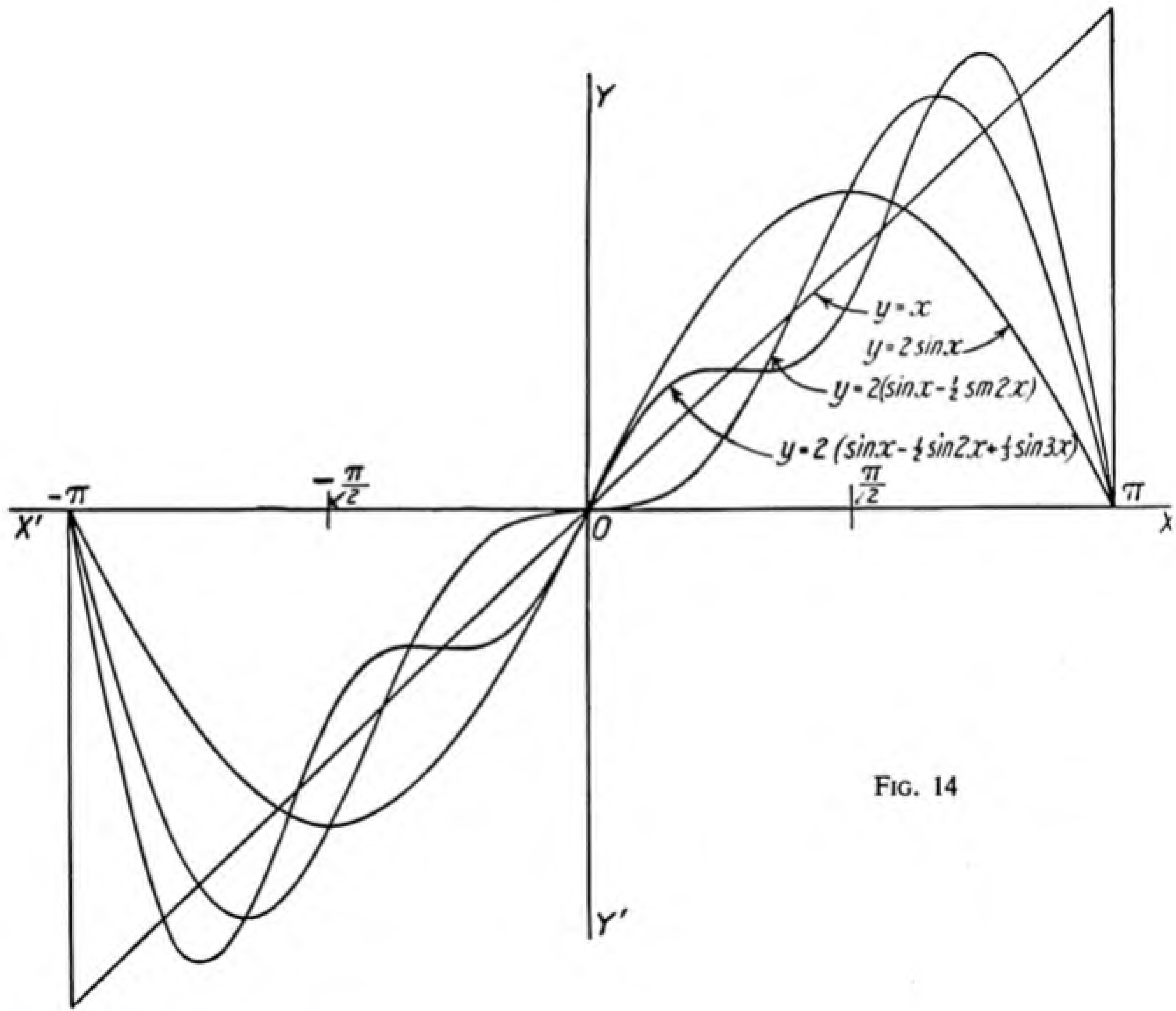
Fig. 14 shows the graphs of $y = x$, $y = 2\sin x$, $y = 2\left(\sin x - \frac{\sin 2x}{2}\right)$ and $y = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3}\right)$ and illustrates the manner in which successive approximations to the series approach more and more closely to the value of the function.

EXAMPLE 4

Find a Fourier's series to represent e^x from $x = -\pi$ to $x = \pi$.

We have

$$b_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{e^{\pi} - e^{-\pi}}{\pi} = \frac{2\sinh\pi}{\pi}$$



Y
Y′
X′
X
O
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$y = x$
$y = 2\sin x$
$y = 2(\sin x - \frac{1}{2}\sin 2x)$
$y = 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x)$

FIG. 14

Also $\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx = \frac{1}{\pi}\left[\frac{e^x(\cos nx + n\sin nx)}{1+n^2}\right]_{-\pi}^{\pi}$ (see Vol. I)

$$= \frac{1}{\pi(1+n^2)}\cos n\pi(e^{\pi} - e^{-\pi}) = \frac{2}{\pi(1+n^2)}\sinh \pi(-1)^n$$

and $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\,dx = \frac{1}{\pi}\left[\frac{e^x(\sin nx - n\cos nx)}{1+n^2}\right]_{-\pi}^{\pi}$

$$= \frac{1}{\pi(1+n^2)}\{-n\cos n\pi\,(e^{\pi} - e^{-\pi})\} = \frac{2n}{\pi(1+n^2)}\sinh \pi(-1)^{n+1}$$

The series is therefore

$$e^x = \frac{2\sinh \pi}{\pi}\{(\tfrac{1}{2} - \tfrac{1}{2}\cos x + \tfrac{1}{5}\cos 2x - \tfrac{1}{10}\cos 3x + \tfrac{1}{17}\cos 4x - \ldots)$$
$$+ (\tfrac{1}{2}\sin x - \tfrac{2}{5}\sin 2x + \tfrac{3}{10}\sin 3x - \tfrac{4}{17}\sin 4x + \ldots)\}$$

which holds only between $x = -\pi$ and $x = \pi$.

20. **Periodic Function of Period 2*c*.** If we write $\frac{\pi}{c}x$ for x in (III.6) and (III.10) but retain $f(x)$ to represent $f\left(\frac{\pi}{c}x\right)$ we have

$$\left.\begin{aligned} f(x) = \tfrac{1}{2}b_0 &+ b_1\cos\frac{\pi}{c}x + b_2\cos\frac{2\pi}{c}x + b_3\cos\frac{3\pi}{c}x \\ &+ \ldots + b_n\cos\frac{n\pi}{c}x + \ldots \\ &+ a_1\sin\frac{\pi}{c}x + a_2\sin\frac{2\pi}{c}x + a_3\sin\frac{3\pi}{c}x \\ &+ \ldots + a_n\sin\frac{n\pi}{c}x + \ldots \end{aligned}\right\} \quad \text{(III.11)}$$

$$\left.\begin{aligned} \text{where } b_0 &= 2 \times \text{mean value of } f(x) \\ &= \frac{1}{c}\int_{-c}^{c} f(x)\,dx \\ b_n &= 2 \times \text{mean value of } f(x)\cos\frac{n\pi}{c}x \\ &= \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi}{c}x\,dx \\ a_n &= 2 \times \text{mean value of } f(x)\sin\frac{n\pi}{c}x \\ &= \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi}{c}x\,dx \end{aligned}\right\} \quad \text{(III.12)}$$

With these values of the coefficients the series and the function are equivalent throughout the range $x = -c$ to $x = c$.

EXAMPLE 1

Find a Fourier series to represent x^2 from $x = -1$ to $x = 1$.

Here $b_0 = \frac{1}{1}\int_{-1}^{1} x^2\,dx = \frac{2}{3}$

$$b_n = \int_{-1}^{1} x^2 \cos n\pi x\,dx = \left[\frac{1}{n\pi}x^2 \sin n\pi x + \frac{2}{n^2\pi^2}x\cos n\pi x - \frac{2}{n^3\pi^3}\sin n\pi x\right]_{-1}^{1}$$

$$= \frac{1}{n^2\pi^2}(2\cos n\pi + 2\cos(-n\pi)) = \frac{4}{n^2\pi^2}\cos n\pi = \frac{4}{n^2\pi^2}(-1)^n$$

$\therefore\ b_n = -\frac{4}{n^2\pi^2}$ if n is odd

and $b_n = \frac{4}{n^2\pi^2}$ if n is even

Also $a_n = \int_{-1}^{1} x^2 \sin n\pi x\,dx$

and since the integrand is an odd function of x

$a_n = 0$

and $x^2 = \frac{1}{3} - \frac{4}{\pi^2}\left(\cos\pi x - \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x - \frac{1}{4^2}\cos 4\pi x + \ldots\right)$

between $x = -1$ and $x = 1$.

EXAMPLE 2

Find a Fourier series to represent $-a$ from $x = -c$ to $x = 0$ and a from $x = 0$ to $x = c$.

We have

$b_0 = 2 \times$ mean value of $f(x) = 0$

$b_n = 2 \times$ mean value of $f(x)\cos\frac{n\pi x}{c}$

$$= \frac{1}{c}\left\{\int_{-c}^{0} -a\cos\frac{n\pi x}{c}\,dx + \int_0^c a\cos\frac{n\pi x}{c}\,dx\right\} = 0$$

$a_n = 2 \times$ mean value of $f(x)\sin\frac{n\pi x}{c}$

$$= \frac{1}{c}\left\{\int_{-c}^{0} -a\sin\frac{n\pi x}{c}\,dx + \int_0^c a\sin\frac{n\pi x}{c}\,dx\right\}$$

$$= \frac{1}{c}\left\{2\int_0^c a\sin\frac{n\pi}{c}x\,dx\right\} = \frac{2a}{n\pi}\left[-\cos\frac{n\pi}{c}x\right]_0^c = \frac{2a}{n\pi}(1 - \cos n\pi)$$

$\therefore\ a_n = 0$ if n is even

and $a_n = \frac{4a}{n\pi}$ if n is odd

Hence, the series is

$$f(x) = \frac{4a}{\pi}\left\{\sin\frac{\pi}{c}x + \frac{1}{3}\sin\frac{3\pi}{c}x + \frac{1}{5}\sin\frac{5\pi}{c}x + \frac{1}{7}\sin\frac{7\pi}{c}x + \ldots\right\}$$

21. Expansion of a Periodic Function in a Series of Sines Alone or of Cosines Alone. If $f(x)$ in (III.11) is an odd function of x, then $f(x)\cos\frac{n\pi}{c}x$ is also an odd function of x, and $f(x)\sin\frac{n\pi}{c}x$ is an even function. Then, by the properties of odd and even functions,

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi}{c}x\,dx = 0$$

and $$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi}{c}x\,dx = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi}{c}x\,dx \quad \text{(III.13)}$$

Thus, when we are expanding an odd function of x the cosine terms disappear from (III.11) and the term $\frac{1}{2}b_0$ disappears with them. The resulting series is

$$\left.\begin{aligned} f(x) &= a_1\sin\frac{\pi}{c}x + a_2\sin\frac{2\pi}{c}x + a_3\sin\frac{3\pi}{c}x \\ &\quad + \ldots + a_n\sin\frac{n\pi}{c}x + \ldots \\ \text{where } a_n &= \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi}{c}x\,dx \\ &= 2\times\text{mean value of } f(x)\sin\frac{n\pi}{c}x \\ &\qquad \text{from } x=0 \text{ to } x=c \end{aligned}\right\} \quad \text{(III.14)}$$

Similarly, if $f(x)$ is an even function of x the sine terms disappear from the series and we obtain

$$\left.\begin{aligned} f(x) &= \tfrac{1}{2}b_0 + b_1\cos\frac{\pi}{c}x + b_2\cos\frac{2\pi}{c}x \\ &\quad + b_3\cos\frac{3\pi}{c}x + \ldots + b_n\cos\frac{n\pi}{c}x + \ldots \\ \text{where } b_n &= \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi}{c}x\,dx \\ &= 2\times\text{mean value of } f(x)\cos\frac{n\pi}{c}x \\ &\qquad \text{from } x=0 \text{ to } x=c \end{aligned}\right\} \quad \text{(III.15)}$$

We are now able to represent $f(x)$ between $x=0$ and $x=c$ either as a series of sines, as in (III.14), or as a series of cosines, as in (III.15). In the first case we assume that $f(x)$ is an odd function

of x, and in the second case that it is an even function. Thus, consider the following example—

EXAMPLE 1

Express $f(x) = x$ as the sum of a series of sines for values of x between 0 and π. Then express it as a series of cosines over the same range.

From (III.13)

$$a_n = \frac{2}{\pi}\int_0^\pi x \sin nx\, dx$$

$$= \frac{2}{\pi}\left\{\left[-\frac{x}{n}\cos nx\right]_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\, dx\right\}$$

$$= -\frac{2}{n}\cos n\pi + \frac{2}{\pi n^2}\Big[\sin nx\Big]_0^\pi = -\frac{2}{n}\cos n\pi$$

$\therefore$ $$a_n = +\frac{2}{n} \text{ if } n \text{ is odd}$$

and $$a_n = -\frac{2}{n} \text{ if } n \text{ is even}$$

Hence, the series of (III.14) becomes

$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\right) \quad . \quad . \quad (1)$$

Now, using (III.15) we have

$$b_n = \frac{2}{\pi}\int_0^\pi x \cos nx\, dx = \frac{2}{\pi}\left\{\left[\frac{x}{n}\sin nx\right]_0^\pi - \frac{1}{n}\int_0^\pi \sin nx\, dx\right\}$$

$$= \frac{2}{\pi n^2}(\cos n\pi - 1)$$

$\therefore$ $$b_n = -\frac{4}{\pi n^2} \text{ if } n \text{ is odd}$$

and $$b_n = 0 \text{ if } n \text{ is even}$$

Also $$b_0 = \frac{2}{\pi}\int_0^\pi x\, dx = \pi$$

Hence, the series of (III.15) becomes

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \ldots\right) \quad . \quad (2)$$

It is important to notice that though either of the series in (1) and (2) is equivalent to x over the range $x = 0$ to $x = \pi$, they differ very much at some points outside of that range. Fig. 15 shows the graph of $y = x$ from $x = -\pi$ to $x = \pi$.

The series in (1) represents a periodic function whose period is 2π; the graph of the function over the period $x = -\pi$ to $x = \pi$ is the straight line POQ.

The series in (2) represents a periodic function whose period is 2π, the graph of which over the period $x = -\pi$ to $x = \pi$ is ROQ, RO being part of the graph of $y = -x$.

It must be clearly understood that the function which we are expanding between $x = 0$ and $x = \pi$ (or $x = c$ in the general case) may be either an odd or an even function, or neither. We expand it as a series of sines or cosines in

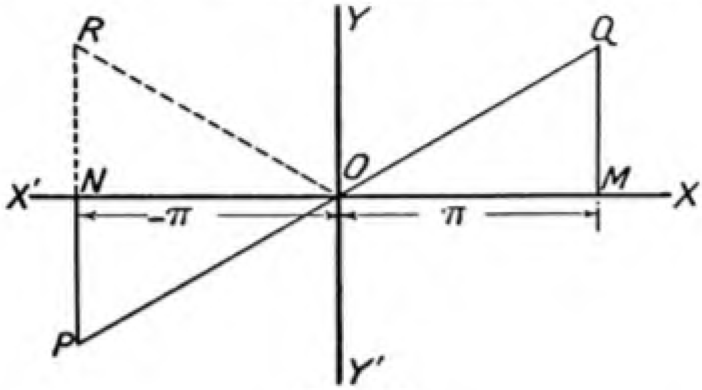


FIG. 15

any case, merely looking upon it as an odd or an even function of period 2π, whichever is convenient for our purpose.

EXAMPLE 2

Expand $f(x)$ in a series of sines if $f(x) = \dfrac{cx}{b}$ from $x = 0$ to $x = b$ and $f(x) = c\dfrac{l-x}{l-b}$ from $x = b$ to $x = l$.

From (III.13)

$$a_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

Hence,
$$\frac{l}{2}a_n = \int_0^b \frac{cx}{b}\sin\frac{n\pi x}{l}\,dx + \int_b^l c\frac{l-x}{l-b}\sin\frac{n\pi x}{l}\,dx$$
$$= -\frac{cl}{n\pi b}\left[x\cos\frac{n\pi x}{l}\right]_0^b + \frac{cl}{n\pi b}\int_0^b \cos\frac{n\pi x}{l}\,dx$$
$$-\frac{cl}{n\pi(l-b)}\left[(l-x)\cos\frac{n\pi x}{l}\right]_b^l - \frac{cl}{n\pi(l-b)}\int_b^l \cos\frac{n\pi x}{l}\,dx$$

Since $\dfrac{cx}{b}$ and $c\dfrac{l-x}{l-b}$ have the same values when $x = b$ this becomes

$$\frac{l}{2}a_n = \frac{cl}{n\pi b}\int_0^b \cos\frac{n\pi x}{l}\,dx - \frac{cl}{n\pi(l-b)}\int_b^l \cos\frac{n\pi x}{l}\,dx$$

$\therefore$
$$\frac{l}{2}a_n = \sin\frac{n\pi b}{l}\left\{\frac{cl^2}{n^2\pi^2 b} + \frac{cl^2}{n^2\pi^2(l-b)}\right\}$$

and
$$a_n = \frac{2cl^2}{n^2\pi^2 b(l-b)}\sin\frac{n\pi b}{l}$$

Hence, writing α for $\frac{\pi b}{l}$ and substituting in (III.14) for a_1, a_2, etc.,

$$f(x) = \frac{2cl^2}{\pi^2 b(l-b)}\left\{\frac{1}{1^2}\sin\alpha\sin\frac{\pi x}{l} + \frac{1}{2^2}\sin 2\alpha\sin\frac{2\pi x}{l} + \frac{1}{3^2}\sin 3\alpha\sin\frac{3\pi x}{l} + \frac{1}{4^2}\sin 4\alpha\sin\frac{4\pi x}{l} + \ldots\right\}$$

which holds for all values of x between $x = 0$ and $x = l$.

EXAMPLE 3

Find a cosine series for $f(x)$ if $f(x) = x$ from $x = 0$ to $x = \frac{\pi}{2}$ and $f(x) = \pi - x$ from $x = \frac{\pi}{2}$ to $x = \pi$.

From (III.15)

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

Hence,

$$\frac{\pi}{2}b_n = \int_0^{\frac{\pi}{2}} x\cos nx\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\cos nx\,dx$$

$$= \left[\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\frac{\pi}{2}} + \left[\frac{1}{n}(\pi - x)\sin nx - \frac{1}{n^2}\cos nx\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{n^2}\left(\cos\frac{n\pi}{2} - \cos 0 - \cos n\pi + \cos\frac{n\pi}{2}\right)$$

$$= -\frac{1}{n^2}\left(1 + \cos n\pi - 2\cos\frac{n\pi}{2}\right)$$

and

$$b_n = -\frac{2}{\pi n^2}\left(1 + \cos n\pi - 2\cos\frac{n\pi}{2}\right)$$

If n is odd

$$b_n = 0. \text{ Hence, } b_1 = b_3 = b_5 = \ldots = 0$$

If n is even and $\frac{n}{2}$ is also even, then

$$b_n = 0. \text{ Hence, } b_4 = b_8 = b_{12} = \ldots = 0$$

If n is even and $\frac{n}{2}$ is odd

$$b_n = -\frac{2}{\pi n^2}(1 + 1 + 2) = -\frac{2\times 4}{\pi n^2} = -\frac{2}{\pi\left(\frac{n}{2}\right)^2}$$

Hence, $b_2 = -\dfrac{2}{\pi \times 1^2}$, $b_6 = -\dfrac{2}{\pi \times 3^2}$, $b_{10} = -\dfrac{2}{\pi \times 5^2}$, $b_{14} = -\dfrac{2}{\pi \times 7^2}$, etc.

Also
$$b_0 = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\,dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\,dx\right]$$
$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \frac{\pi^2}{8}\right] = \frac{\pi}{2}$$

The series is
$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left\{\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \frac{\cos 14x}{7^2} + \ldots\right\}$$

22. **Function Represented by a Graph or by a Series of Pairs of Values.** When dealing with a periodic function which is not known as an explicit function of the independent variable, it is necessary to know a sufficient number of pairs of values of the function and the independent variable. These values may be found from a graph or may be given in the form of a table. The method of solution in such cases is shown in the following example. The values of the function should be known for a sufficient number of values of the independent variable, and the intervals between successive values of the latter should be equal.

EXAMPLE 1

A machine completes its cycle of operations every time a certain pulley completes a revolution. The displacement y in. of a point on a certain portion of the machine is given in the following table for twelve positions of the pulley, θ being the angle in degrees turned through by the pulley. Find a Fourier series to represent y for all values of θ.

Number of Position	1	2	3	4	5	6	7	8	9	10	11	12
θ	30	60	90	120	150	180	210	240	270	300	330	360
y	7·976	8·026	7·204	5·676	3·674	1·764	0·552	0·262	0·904	2·492	4·736	6·824

The sum of the values of $y = 50{\cdot}090$, i.e. $\Sigma y = 50{\cdot}090$.

By (III.7), $\quad b_0 = 2 \times$ mean value of y
$$= \tfrac{1}{6}\,\Sigma y = \frac{50{\cdot}090}{6} = 8{\cdot}348$$

Again, by (III.8), $\quad b_1 = 2 \times$ mean value of $y \cos\theta$
$$= \tfrac{1}{6}\,\Sigma y \cos\theta$$

and by (III.9), $\quad a = \tfrac{1}{6}\,\Sigma y \sin\theta$

In order to facilitate the calculation of these coefficients, we make use of Fig. 16, which shows a circle divided into twelve equal parts with radii drawn so as to make with the horizontal, angles whose values are those in the preceding table. If any value of y in the table is set off from the centre of the circle along the radius bearing the same position number, we see that the vertical projection of y is equal to $y \sin \theta$ and the horizontal projection is $y \cos \theta$. By proceeding in this manner we may find the values of $\Sigma y \sin \theta$ and $\Sigma y \cos \theta$, and, hence, those

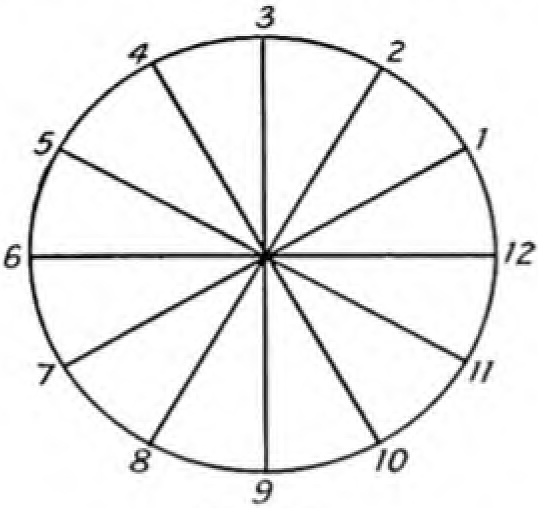


FIG. 16

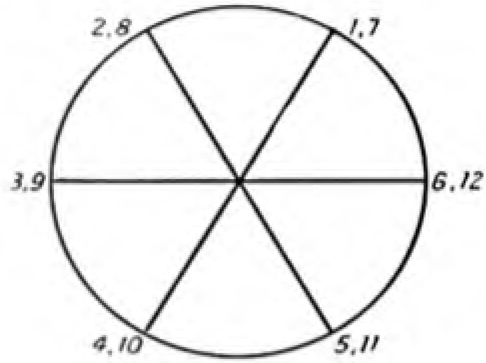


FIG. 17

of b_1 and a_1. It is not necessary, however, to find the projections graphically. The figure enables us to see readily that $\sin 150° = \sin 30°$, $\sin 120° = \sin 60°$, $\sin 180° = \sin 0°$, $\sin 210° = \sin 330° = -\sin 30°$, $\sin 240° = \sin 300° = -\sin 60°$, and $\sin 270° = -\sin 90°$.

$$\text{Hence,}\quad \Sigma y \sin \theta = \sin 30°(y_1 + y_5 - y_7 - y_{11}) + \sin 60°(y_2 + y_4 - y_8 - y_{10}) + (y_3 - y_9)\sin 90°$$

where $\quad y_n =$ value of y for position n in table

$$\begin{aligned} \text{Thus,}\quad \Sigma y \sin \theta &= 0{\cdot}5(7{\cdot}976 + 3{\cdot}674 - 0{\cdot}552 - 4{\cdot}736) \\ &\quad + 0{\cdot}8660(8{\cdot}026 + 5{\cdot}676 - 0{\cdot}262 - 2{\cdot}492) \\ &\quad + 7{\cdot}204 - 0{\cdot}904 \\ &= 18{\cdot}963 \end{aligned}$$

$$\text{and}\quad a_1 = \frac{\Sigma y \sin \theta}{6} = 3{\cdot}161$$

$$\begin{aligned} \text{Similarly,}\ \Sigma y \cos \theta &= (y_{12} - y_6) + \cos 30°(y_1 + y_{11} - y_5 - y_7) \\ &\quad + \cos 60°(y_2 + y_{10} - y_4 - y_8) \\ &= (6{\cdot}824 - 1{\cdot}764) + 0{\cdot}8660(7{\cdot}976 + 4{\cdot}736 - 3{\cdot}674 - 0{\cdot}552) \\ &\quad + 0{\cdot}5(8{\cdot}026 + 2{\cdot}492 - 5{\cdot}676 - 0{\cdot}262) \\ &= 14{\cdot}699 \end{aligned}$$

$$\text{and}\quad b_1 = \tfrac{1}{6}\,\Sigma y \cos \theta = 2{\cdot}450$$

In Fig. 17 we have made the angles between successive radii twice as large as those in Fig. 16, and so we are able to calculate $\Sigma y \sin 2\theta$ and $\Sigma y \cos 2\theta$.

As before

$$\begin{aligned}\Sigma y \sin 2\theta &= \sin 60°(y_1 + y_7 + y_2 + y_8 - y_4 - y_{10} - y_5 - y_{11}) \\ &= 0{\cdot}8660(7{\cdot}976 + 0{\cdot}552 + 8{\cdot}026 + 0{\cdot}262 - 5{\cdot}676 - 2{\cdot}492 \\ &\qquad - 3{\cdot}674 - 4{\cdot}736) \\ &= 0{\cdot}2061\end{aligned}$$

and $\qquad a_2 = \frac{1}{6}\,\Sigma y \sin 2\theta = 0{\cdot}034$

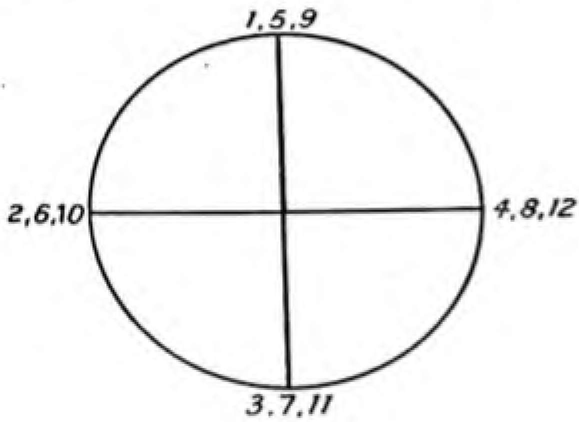


FIG. 18

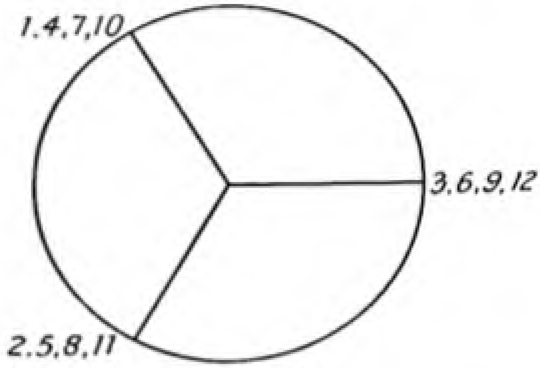


FIG. 19

$$\begin{aligned}\text{Again } \Sigma y \cos 2\theta &= \cos 60°(y_1 + y_7 + y_5 + y_{11} - y_2 - y_8 - y_4 - y_{10}) \\ &\qquad + (y_{12} + y_6 - y_9 - y_3) \\ &= 0{\cdot}5(7{\cdot}976 + 0{\cdot}552 + 3{\cdot}674 + 4{\cdot}736 - 8{\cdot}026 - 0{\cdot}262 \\ &\qquad - 5{\cdot}676 - 2{\cdot}492) + (6{\cdot}824 + 1{\cdot}764 - 0{\cdot}904 - 7{\cdot}204) \\ &= 0{\cdot}721\end{aligned}$$

and $\qquad b_2 = \frac{1}{6}\,\Sigma y \cos 2\theta = 0{\cdot}120$

To determine a_3 and b_3, we make use of Fig. 18, in which the angles between successive radii are $3 \times 30° = 90°$.

We have

$$\begin{aligned}\Sigma y \sin 3\theta &= \sin 90°(y_1 + y_5 + y_9 - y_3 - y_7 - y_{11}) \\ &= 7{\cdot}976 + 3{\cdot}674 + 0{\cdot}904 - 7{\cdot}204 - 0{\cdot}552 - 4{\cdot}736 \\ &= 0{\cdot}062\end{aligned}$$

and $\qquad a_3 = \frac{1}{6}\,\Sigma y \sin 3\theta = 0{\cdot}010$

$$\begin{aligned}\text{Again, } \Sigma y \cos 3\theta &= \cos 0°(y_4 + y_8 + y_{12} - y_2 - y_6 - y_{10}) \\ &= 5{\cdot}676 + 0{\cdot}262 + 6{\cdot}824 - 8{\cdot}026 - 1{\cdot}764 - 2{\cdot}492 \\ &= 0{\cdot}480\end{aligned}$$

$\therefore \qquad b_3 = \frac{1}{6}\,\Sigma y \cos 3\theta = 0{\cdot}080$

Continuing, we have from Fig. 19,

$$\begin{aligned}\Sigma y \sin 4\theta &= \sin 60°(y_1 + y_4 + y_7 + y_{10} - y_2 - y_5 - y_8 - y_{11}) \\ &= -\,0{\cdot}0017\end{aligned}$$

(We leave the calculation to the reader.)

$\therefore \qquad a_4 = 0$ to three places of decimals.

Similarly,

$$\begin{aligned}\Sigma y \cos 4\theta &= \cos 0^\circ(y_3 + y_6 + y_9 + y_{12}) \\ &\qquad - \cos 60^\circ(y_1 + y_2 + y_4 + y_5 + y_7 + y_8 + y_{10} + y_{11}) \\ &= -0{\cdot}001\end{aligned}$$

$\therefore \qquad b_4 = 0$ to three places of decimals.

The series is therefore up to seven terms

$$\begin{aligned}y = 4{\cdot}174 &+ 2{\cdot}450 \cos \theta + 0{\cdot}120 \cos 2\theta + 0{\cdot}080 \cos 3\theta \\ &+ 3{\cdot}161 \sin \theta + 0{\cdot}034 \sin 2\theta + 0{\cdot}010 \sin 3\theta\end{aligned}$$

EXAMPLE 2

Prove that if $\alpha = 2\pi p/n$, $\beta = 2\pi q/n$ where p, q, n are integers

$$\sum_{r=0}^{r=n-1} \cos r\alpha \ \cos r\beta = 0 \text{ unless } p = q$$

Assuming that for all values of x from 0 to 2π,

$$y = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x$$

and that y has the values $y_0, y_1, y_2, y_3, y_4, y_5$, when $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$ respectively, find the values of these coefficients a and b in terms of the six values of y. (U.L.)

$$\begin{aligned}\sum_{r=0}^{r=n-1} \cos r\alpha \,.\, \cos r\beta &= \cos 0^\circ \cos 0^\circ + \cos \alpha \cos \beta + \cos 2\alpha \cos 2\beta \\ &\qquad + \cos 3\alpha \cos 3\beta + \ldots \text{ to } n \text{ terms}\end{aligned}$$

But $\qquad \cos A \cos B = \frac{1}{2}[\cos (A + B) + \cos (A - B)]$

Hence,

$$\begin{aligned}\sum_{r=0}^{r=n-1} \cos r\alpha \,.\, \cos r\beta &= \tfrac{1}{2}[(\cos 0^\circ + \cos (\alpha + \beta) + \cos 2(\alpha + \beta) \\ &\qquad + \cos 3(\alpha + \beta) + \ldots \text{ to } n \text{ terms}) \\ &\qquad + \cos 0^\circ + \cos (\alpha - \beta) + \cos 2(\alpha - \beta) \\ &\qquad + \cos 3(\alpha - \beta) + \ldots \text{ to } n \text{ terms}]\end{aligned}$$

$$= \frac{1}{2}\left[\frac{\cos \frac{n-1}{2}(\alpha + \beta) \sin \frac{1}{2}n(\alpha + \beta)}{\sin \frac{1}{2}(\alpha + \beta)} + \frac{\cos \frac{n-1}{2}(\alpha - \beta) \sin \frac{1}{2}n(\alpha - \beta)}{\sin \frac{1}{2}(\alpha - \beta)}\right]$$

(See Art. 19, Vol. I.)

$$= \frac{1}{2}\left[\frac{\cos \frac{n-1}{n}\pi(p + q) \sin \pi(p + q)}{\sin \frac{\pi}{n}(p + q)} + \frac{\cos \frac{n-1}{n}\pi(p - q) \sin \pi(p - q)}{\sin \frac{\pi}{n}(p - q)}\right]$$

Since p and q are integers, the numerator of each fraction is zero for all integral values of p and q, but when $p = q$, the denominator of the second fraction is also zero and the fraction takes the form $\frac{0}{0}$. But $\underset{\theta \to 0}{Lt.} \dfrac{\sin \theta}{\sin \frac{\theta}{n}} = n$, and when $p = q$ the value of the expression is therefore $\frac{n}{2}$. Hence, $\sum_{r=0}^{r=n-1} \cos rx \cos r\beta = 0$, except when $p = q$, and has the value $\frac{n}{2}$ when $p = q$.

Fig. 20 shows how the values of y would be arranged round the circles if we applied the method of the previous example to this case. All the circles are drawn in one figure; a_1 and b_1 of (III.6) would be determined with the aid of the inner circle, a_2 and b_2 with the second circle, and so on. The reader will notice, however, that this method is not applicable in the present case as we are asked to determine, not the coefficients of some of the terms of an infinite series like (III.6), but the values of the coefficients of the terms of a finite series of six terms, so that the given pairs of values of x and y shall satisfy exactly the given relation. The method of doing this is explained fully on pages 76 to 78 of the treatise referred to in the footnote, page 69. The method is to substitute in turn in the given relation the pairs of corresponding values of x and y. Thus we obtain the relations

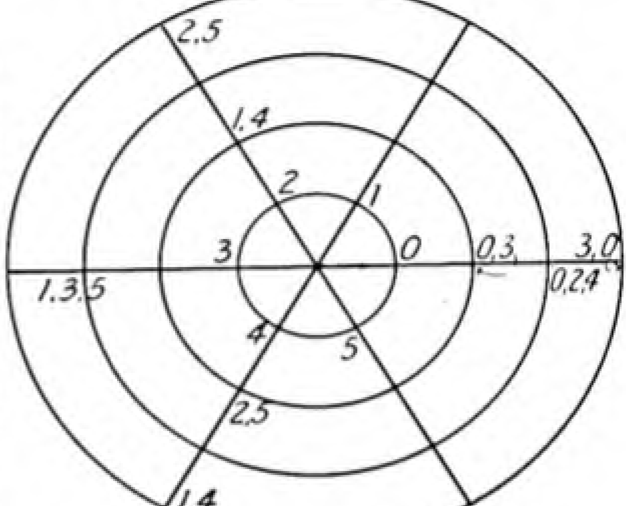


FIG. 20

$$y_0 = a_0 + a_1 + a_2 + a_3$$

$$y_1 = a_0 + \tfrac{1}{2}a_1 - \tfrac{1}{2}a_2 - a_3 + \frac{\sqrt{3}}{2}b_1 + \frac{\sqrt{3}}{2}b_2$$

$$y_2 = a_0 - \tfrac{1}{2}a_1 - \tfrac{1}{2}a_2 + a_3 + \frac{\sqrt{3}}{2}b_1 - \frac{\sqrt{3}}{2}b_2$$

$$y_3 = a_0 - a_1 + a_2 - a_3$$

$$y_4 = a_0 - \tfrac{1}{2}a_1 - \tfrac{1}{2}a_2 + a_3 - \frac{\sqrt{3}}{2}b_1 + \frac{\sqrt{3}}{2}b_2$$

$$y_5 = a_0 + \tfrac{1}{2}a_1 - \tfrac{1}{2}a_2 - a_3 - \frac{\sqrt{3}}{2}b_1 - \frac{\sqrt{3}}{2}b_2$$

On solving these simultaneous equations in six unknowns, we obtain

$$\begin{aligned}
a_0 &= \tfrac{1}{6}(y_0 + y_1 + y_2 + y_3 + y_4 + y_5)\\
a_1 &= \tfrac{1}{3}\{y_0 - y_3 + 0{\cdot}5(y_1 - y_2 - y_4 + y_5)\}\\
a_2 &= \tfrac{1}{3}\{y_0 + y_3 - 0{\cdot}5(y_1 + y_2 + y_4 + y_5)\}\\
a_3 &= \tfrac{1}{6}\{y_0 + y_2 + y_4 - (y_1 + y_3 + y_5)\}\\
b_1 &= \tfrac{1}{3} \times 0{\cdot}8660(y_1 + y_2 - y_4 - y_5)\\
&= 0{\cdot}2887(y_1 + y_2 - y_4 - y_5)\\
b_2 &= \tfrac{1}{3} \times 0{\cdot}8660(y_1 + y_4 - y_2 - y_5)\\
&= 0{\cdot}2887(y_1 + y_4 - y_2 - y_5)
\end{aligned}$$

EXAMPLE 3

(1) If E is a function of t such that E has a constant value E_0 when $0 < t < \tau$ and $E = 0$ when $\tau < t < 2\tau$, express E in the form

$$a_0 + a_1 \cos\frac{\pi t}{\tau} + a_2 \cos\frac{2\pi t}{\tau} + \ldots + b_1 \sin\frac{\pi t}{\tau} + b_2 \sin\frac{2\pi t}{\tau} + \ldots$$

(2) Evaluate the integral $\int e^{\frac{Rt}{L}} \sin\frac{n\pi t}{\tau}\,dt$.

The current j in an electric circuit satisfies the equation $L\dfrac{dj}{dt} + Rj = E$, where E is the electromotive force. If E is constant and equal to E_0 when t lies between 0 and τ, 2τ and 3τ, 4τ and 5τ, etc., and is zero at other times, find the current in the circuit when the steady state has been reached. (U.L.)

(1) Let $E = f(t)$, then

$$f(t) = a_0 + a_1 \cos\frac{\pi t}{\tau} + a_2 \cos\frac{2\pi t}{\tau} + \ldots + b_1 \sin\frac{\pi t}{\tau} + b_2 \sin\frac{2\pi t}{\tau} + \ldots$$

With the necessary changes in the symbols the relations (III.12) become

$$a_0 = \frac{1}{2\tau}\int_0^{2\tau} f(t)\,dt$$

$$b_n = \frac{1}{\tau}\int_0^{2\tau} f(t)\sin\frac{n\pi t}{\tau}\,dt$$

$$a_n = \frac{1}{\tau}\int_0^{2\tau} f(t)\cos\frac{n\pi t}{\tau}\,dt$$

As $f(t)$ is discontinuous, we split the definite integrals into two parts. Thus

$$a_0 = \frac{1}{2\tau}\left\{\int_0^{\tau} E_0\,dt + \int_{\tau}^{2\tau} 0\,dt\right\} = \tfrac{1}{2}E_0$$

$$\begin{aligned}
a_n &= \frac{1}{\tau}\left\{\int_0^{\tau} E_0 \cos\frac{n\pi t}{\tau}\,dt + \int_{\tau}^{2\tau}(0)\cos\frac{n\pi t}{\tau}\,dt\right\}\\
&= \frac{E_0}{\tau}\left[\frac{\tau}{n\pi}\sin\frac{n\pi t}{\tau}\right]_0^{\tau} = 0
\end{aligned}$$

$$b_n = \frac{1}{\tau}\left\{\int_0^{\tau} E_0 \sin\frac{n\pi t}{\tau}\,dt + \int_{\tau}^{2\tau}(0)\sin\frac{n\pi t}{\tau}\,dt\right\}$$

$$= \frac{E_0}{\tau}\left[-\frac{\tau}{n\pi}\cos\frac{n\pi t}{\tau}\right]_0^{\tau} = -\frac{E_0}{n\pi}(\cos n\pi - 1)$$

$$= \frac{2E_0}{n\pi} \text{ if } n \text{ is odd and 0 if } n \text{ is even.}$$

We have then as the required series

$$E = \frac{E_0}{2}\left\{1 + \frac{4}{\pi}\left(\sin\frac{\pi t}{\tau} + \frac{1}{3}\sin\frac{3\pi t}{\tau} + \frac{1}{5}\sin\frac{5\pi t}{\tau} + \ldots\right)\right\}$$

(2) The given integral is evaluated thus—

Let $$A = \int e^{\frac{Rt}{L}} \sin\frac{n\pi t}{\tau}\,dt \text{ and } B = \int e^{\frac{Rt}{L}}\cos\frac{n\pi t}{\tau}\,dt$$

Integrating by parts $$A = \frac{L}{R}e^{\frac{Rt}{L}}\sin\frac{n\pi t}{\tau} - \frac{n\pi L}{R\tau}B$$

and $$B = \frac{L}{R}e^{\frac{Rt}{L}}\cos\frac{n\pi t}{\tau} + \frac{n\pi L}{R\tau}A$$

Now, eliminating B, we have

$$A = e^{\frac{Rt}{L}}\left(\frac{L}{R}\sin\frac{n\pi t}{\tau} - \frac{n\pi L^2}{R^2\tau}\cos\frac{n\pi t}{\tau}\right) - \frac{n^2\pi^2L^2}{R^2\tau^2}A$$

or $$A = \frac{R^2\tau^2}{n^2\pi^2L^2 + R^2\tau^2}\cdot e^{\frac{Rt}{L}}\left(\frac{L}{R}\sin\frac{n\pi t}{\tau} - \frac{n\pi L^2}{R^2\tau}\cos\frac{n\pi t}{\tau}\right)$$

$$L\frac{dj}{dt} + Rj = E \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where E has the value

$$E = \frac{E_0}{2}\left\{1 + \frac{4}{\pi}\left(\sin\frac{\pi t}{\tau} + \frac{1}{3}\sin\frac{3\pi t}{\tau} + \frac{1}{5}\sin\frac{5\pi t}{\tau} + \ldots\right)\right\}. \quad . \quad (2)$$

In order to obtain the complete solution of (1) we add the solutions obtained by substituting for E in (1) each of the separate terms on the right-hand side of (2). Putting $E = \frac{E_0}{2}$ in (1), we see from (5), page 368, Vol. I, that the solution of (1) is

$$j = \frac{E_0}{2R} + Ae^{-\frac{R}{L}t} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Replacing E_0 and ω in (8), page 368, Vol. I, by $\frac{2E_0}{\pi}$ and $\frac{\pi}{\tau}$ respectively, and then by $\frac{2}{3}\frac{E_0}{\pi}$, and $\frac{3\pi}{\tau}$, $\frac{2}{5}\frac{E_0}{\pi}$ and $\frac{5\pi}{\tau}$, $\frac{2}{7}\frac{E_0}{\pi}$ and $\frac{7\pi}{\tau}$, in order we obtain the part solutions of (1) due to the other terms on the right-hand side of (2). These solutions are

$$j = \frac{2E_0}{\pi\sqrt{R^2 + \frac{\pi^2 L^2}{\tau^2}}} \sin\left(\frac{\pi t}{\tau} - \alpha_1\right), \text{ where } \alpha_1 = \tan^{-1}\frac{\pi L}{R\tau} \quad . \quad . \quad . \quad (4)$$

$$j = \frac{2E_0}{3\pi\sqrt{R^2 + \frac{9\pi^2 L^2}{\tau^2}}} \sin\left(\frac{3\pi t}{\tau} - \alpha_2\right), \text{ where } \alpha_2 = \tan^{-1}\frac{3\pi L}{R\tau} \quad . \quad . \quad (5)$$

$$j = \frac{2E_0}{5\pi\sqrt{R^2 + \frac{25\pi^2 L^2}{\tau^2}}} \sin\left(\frac{5\pi t}{\tau} - \alpha_3\right), \text{ where } \alpha_3 = \tan^{-1}\frac{5\pi L}{R\tau} \quad . \quad . \quad (6)$$

etc., etc., etc.

We have omitted the term $Ae^{-\frac{Rt}{L}}$ from (4), (5), (6), etc., as this already appears in the partial solution (3). The complete solution of (1) is therefore

$$j = \frac{E_0}{2R} + Ae^{-\frac{Rt}{L}} + \frac{2E_0}{\pi}\left[\frac{\sin\left(\frac{\pi t}{\tau} - \alpha_1\right)}{\sqrt{R^2 + \frac{\pi^2 L^2}{\tau^2}}} + \frac{\sin\left(\frac{3\pi t}{\tau} - \alpha_2\right)}{3\sqrt{R^2 + \frac{9\pi^2 L^2}{\tau^2}}} + \frac{\sin\left(\frac{5\pi t}{\tau} - \alpha_3\right)}{5\sqrt{R^2 + \frac{25\pi^2 L^2}{\tau^2}}} + \ldots\right] \quad . \quad . \quad . \quad . \quad (7)$$

As t increases, the second term on the right becomes negligible, and when the steady state is reached, (7) with this term omitted gives the value of j.

23. **Fourier Series containing Even Harmonics only or Odd Harmonics only.** In Art. 21 we saw that an even function of x can be expanded in a series of cosines only and an odd function in a series of sines only. In both these cases the coefficients of the terms can be determined by integration over half the range as in (III.14) and (III.15) thus leading to a reduction in the labour of integration when there is a point of discontinuity of $f(x)$ in the half-range or where the function is given by a graph or a series of pairs of values. There are two other types of functions for which the trouble of determining the coefficients may be reduced. These are defined by

$$f(x) = f(x - c) \quad . \quad . \quad . \quad (\text{III.16})$$

and

$$f(x) = -f(x - c) \quad . \quad . \quad . \quad (\text{III.17})$$

Typical graphs of these are shown in Fig. 21. In the upper figure the two portions, AB and CD, of the graph are such that each may be superposed on the other by sliding it horizontally through a distance c. In the lower figure the portions $A'B'$ and $C'D'$ are such that each may be superposed on the other by rotating it through 180° about

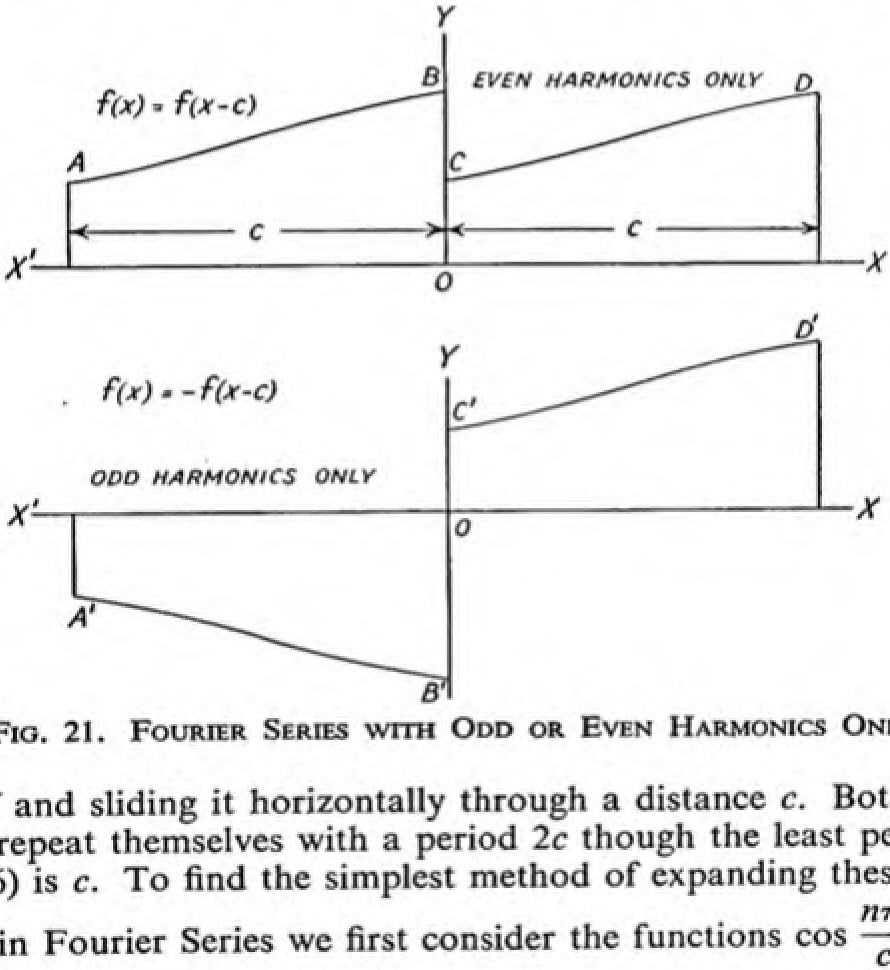


FIG. 21. FOURIER SERIES WITH ODD OR EVEN HARMONICS ONLY

$X'OX$ and sliding it horizontally through a distance c. Both functions repeat themselves with a period $2c$ though the least period in (III.16) is c. To find the simplest method of expanding these functions in Fourier Series we first consider the functions $\cos \frac{n\pi}{c} x$ and $\sin \frac{n\pi}{c} x$, where n is a positive integer.

We have $$\cos \frac{n\pi}{c}(x-c) = \cos\left(\frac{n\pi}{c}x - n\pi\right)$$

and $$\sin \frac{n\pi}{c}(x-c) = \sin\left(\frac{n\pi}{c}x - n\pi\right)$$

If n is even, $\sin \frac{n\pi}{c}(x-c) = \sin \frac{n\pi x}{c}$ and $\cos \frac{n\pi}{c}(x-c) = \cos \frac{n\pi}{c} x$, as in (III.16).

If n is odd, $\sin \frac{n\pi}{c}(x-c) = -\sin \frac{n\pi x}{c}$ and $\cos \frac{n\pi}{c}(x-c) = -\cos \frac{n\pi x}{c}$, as in (III.17).

From (III.12) $$a_n = \frac{1}{c}\int_{-c}^{c} f(x) \sin \frac{n\pi}{c} x \, dx$$

Taking first the relation $f(x) = f(x-c)$, we see that $f(x) \sin \frac{n\pi}{c} x$ passes through the same range of numerical values as x increases from $-c$ to 0 as it does while x increases from 0 to c, so that the integral of $f(x) \sin \frac{n\pi}{c} x$ has the same numerical value over each half-range. If n is even, these numerical values have the same algebraic sign, whilst, if n is odd, they have opposite signs. Thus,

$$\left.\begin{aligned} &\text{if } n \text{ is even,} && a_n = \frac{2}{c}\int_0^c f(x) \sin \frac{n\pi}{c} x \, dx \\ &\text{and if } n \text{ is odd,} && a_n = 0 \end{aligned}\right\} \quad . \quad . \quad \text{(III.18)}$$

$$\left.\begin{aligned} &\text{Similarly, if } n \text{ is even,} && b_n = \frac{2}{c}\int_0^c f(x) \cos \frac{n\pi}{c} x \, dx \\ &\text{and if } n \text{ is odd,} && b_n = 0 \end{aligned}\right\} \quad . \quad . \quad \text{(III.19)}$$

Taking next the relation $f(x) = -f(x-c)$ and repeating the above reasoning, we see that—

$$\left.\begin{aligned} &\text{if } n \text{ is even,} && a_n = b_n = 0 \\ &\text{and if } n \text{ is odd,} && a_n = \frac{2}{c}\int_0^c f(x) \sin \frac{n\pi}{c} x \, dx \\ & && b_n = \frac{2}{c}\int_0^c f(x) \cos \frac{n\pi}{c} x \, dx \end{aligned}\right\} \quad . \quad . \quad \text{(III.20)}$$

Summarizing these results, we see then that, if $f(x) = f(x-c)$, the Fourier expansion of $f(x)$ contains only even harmonics and, if $f(x) = -f(x-c)$, the expansion contains only odd harmonics. In each case the coefficients of the terms which appear in the series may be found by integration over the half-range. The combinations

of these two cases with the cases of odd and even functions are shown in Table I.

TABLE I

TABLE SHOWING THE KINDS OF TERMS IN FOURIER SERIES

	$f(x) = f(-x)$ Even function. Cosine terms only.	$f(x) = -f(-x)$ Odd function. Sine terms only.
$f(x) = f(x-c)$ Even harmonics only.	Even cosines only.	Even sines only.
$f(x) = -f(x-c)$ Odd harmonics only.	Odd cosines only.	Odd sines only.

It is easy to show that in each of the four cases in the table the graph of $f(x)$ against x is symmetrical about the line $x = \frac{c}{2}$. For example, consider the case $f(x) = f(-x) = f(x-c)$. If $x = \frac{c}{2} + a$, where a is any constant, then, since $f(x) = f(x-c)$, $f\left(\frac{c}{2} + a\right)$ $= f\left(-\frac{c}{2} + a\right)$, and, since $f(x) = f(-x)$, $f\left(-\frac{c}{2} + a\right) =$ $f\left(\frac{c}{2} - a\right)$, so that $f\left(\frac{c}{2} + a\right) = f\left(\frac{c}{2} - a\right)$, which shows the symmetry about the line $x = \frac{c}{2}$

In evaluating a_n or b_n, therefore, we need integrate only over the quarter period 0 to $\frac{c}{2}$ with a consequent reduction in the working. In these cases

$$a_n = \frac{4}{c}\int_0^{\frac{c}{2}} f(x) \sin \frac{n\pi}{c} x \, dx \text{ and } b_n = \frac{4}{c}\int_0^{\frac{c}{2}} f(x) \cos \frac{n\pi}{c} x \, dx \quad \text{(III.21)}$$

These formulae will give correct results only for the terms indicated in Table I, and will generally give incorrect results if evaluated for terms which the table shows to be missing and for all of which $a_n = b_n = 0$.

EXAMPLE 1

The graph of $f(x)$ against x is made up of the two straight lines joining the pairs of prints $(-c, 0)$, $(0, a)$ and $(0, 0)$, (c, a). Find a Fourier series to represent $f(x)$ over the range $x = -c$ to $x = c$. Here $f(x)$ satisfies the relation $f(x) = f(x - c)$, so that the series contains even harmonics only. From $x = 0$ to $x = c, f(x) = \frac{a}{c}x$.

From (III.18), $$a_n = \frac{2}{c}\int_0^c \frac{a}{c}\, x \sin\frac{n\pi}{c}x\, dx,\ n \text{ even}$$

$$= -\frac{2a}{n\pi c}\left\{\left[x\cos\frac{n\pi}{c}x\right]_0^c - \int_0^c \cos\frac{n\pi}{c}x\, dx\right\},\ n \text{ even}$$

Hence, $$a_n = -\frac{2a}{n\pi},\ n \text{ even}$$

By inspection, $b_0 = 2\times$ mean value of $f(x) = a$

From (III.19), $$b_n = \frac{2}{c}\int_0^c \frac{a}{c}x\cos\frac{n\pi}{c}x\, dx,\ n \text{ even}$$

$$= \frac{2a}{n\pi c}\left\{\left[x\sin\frac{n\pi}{c}x\right]_0^c - \int_0^c \sin\frac{n\pi}{c}x\, dx\right\},\ n \text{ even}$$

Hence, $$b_n = 0$$

Thus, the series is

$$f(x) = \frac{a}{2} - \frac{a}{\pi}\left(\sin\frac{2\pi}{c}x + \frac{1}{2}\sin\frac{4\pi}{c}x + \frac{1}{3}\sin\frac{6\pi}{c}x + \ldots\right)$$

We could have seen without integration that $b_n = 0$ (where $n \neq 0$), for $f(x) - \frac{a}{2}$ is an odd function and can be expressed as a series of sines only over the given range.

EXAMPLE 2

If $f(x) = -f(-x)$ in the range $-\pi < x < \pi$, show that the Fourier series for $f(x)$ in this range contains no cosine terms.

If the above function is defined by $f(x) = x$ when $0 < x < \frac{\pi}{2}$, and $f(x) = \pi - x$ when $\frac{\pi}{2} < x < \pi$, show that in the range $-\pi < x < \pi$,

$$f(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\sin(2n+1)x}{(2n+1)^2} \qquad \text{(U.L.)}$$

The first part is explained above. With the given conditions the graph of $f(x)$ between $-\pi$ and π consists of the three straight lines joining the points $(-\pi, 0)$ and $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$, $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $(\pi, 0)$. The reader should sketch the graph. It is seen that $f(x) = -f(x - \pi)$, and, as also $f(x) = -f(-x)$, the Fourier series contains no cosine terms and only odd sine terms. The coefficient of the general term can be denoted by a_{2n+1}, where $n = 0, 1, 2, \ldots$ up to ∞.

Then from (III.21), since $f(x) = x$ from 0 to $\frac{\pi}{2}$,

$$a_{2n+1} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} x \sin(2n+1)x\,dx$$

$$= -\frac{4}{\pi(2n+1)}\left\{\Big[x\cos(2n+1)x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos(2n+1)x\,dx\right\}$$

$$= \frac{4}{\pi(2n+1)^2}\Big[\sin(2n+1)x\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{4}{\pi}\cdot\frac{\sin(2n+1)\frac{\pi}{2}}{(2n+1)^2}$$

Hence,
$$a_{2n+1} = \frac{4}{\pi}\cdot\frac{(-1)^n}{(2n+1)^2}$$

and
$$f(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\sin(2n+1)x}{(2n+1)^2}$$

The reader should compare this example with Example 3 of Art. 21 and should note the difference in the graphs in the two cases.

EXAMPLE 3

Show that, in the range $0 < x < \pi$, $\sin x$ can be represented by the cosine Fourier series

$$\frac{4}{\pi}\left[\frac{1}{2} - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \frac{\cos 6x}{35} - \ldots - \frac{\cos 2px}{4p^2-1} - \ldots\right]$$

Show by a sketch the function represented by the sum of the series for values of x from $-\pi$ to $+3\pi$. (U.L.)

The given series contains no sine terms and only even cosine terms, so that $f(x) = f(-x) = f(x-\pi)$, and the graph of the function is the rectified sine curve.

Denoting the coefficient of the general term by b_{2n}, we have from (III.21),

$$b_{2n} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \sin x \cos 2n x\,dx$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \{\sin(2n+1)x - \sin(2n-1)x\}\,dx$$

$$= \frac{2}{\pi}\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right]$$

Hence,
$$b_{2n} = -\frac{4}{\pi(4n^2-1)}$$

Giving n the values 0, 1, 2, 3, . . ., p, . . . in turn, we find $b_0 = \frac{4}{\pi}$, $b_2 = -\frac{4}{3\pi}$, $b_4 = -\frac{4}{15\pi}$, $b_6 = -\frac{4}{35\pi}$, . . ., $b_{2p} = -\frac{4}{(4p^2-1)\pi}$, . . .

The first term is $\frac{1}{2}b_0$, and the series is

$$\frac{4}{\pi}\left[\frac{1}{2} - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \frac{\cos 6x}{35} - \ldots - \frac{\cos 2px}{4p^2-1} - \ldots\right]$$

We leave the reader to sketch the graph of the function represented by the sum of the series for values of x from $-\pi$ to $+3\pi$.

EXAMPLES III

(1) Expand $(1 - a^2 \sin^2 t)^{\frac{1}{2}}$ as far as the term involving a^6 and express the result in the form $A + B\cos 2t + C\cos 4t + D\cos 6t$.

Find Fourier series to represent the following—

(2) $f(x) = a$ from $x = 0$ to $x = \pi$ and $f(x) = 0$ from $x = \pi$ to $x = 2\pi$.
(3) $f(x) = x$ from $x = 0$ to $x = \pi$ and $f(x) = 2\pi - x$ from $x = \pi$ to $x = 2\pi$.
(4) $f(x) = a$ from $x = 0$ to $x = \pi$ and $f(x) = -a$ from $x = \pi$ to $x = 2\pi$.
(5) $f(x) = x^2$ from $x = 0$ to $x = 2\pi$.
(6) $f(x) = e^{-x}$ from $x = 0$ to $x = 2\pi$.
(7) $f(x) = x^2$ from $x = -a$ to $x = a$.
(8) $f(x) = e^{-x}$ from $x = -a$ to $x = a$.
(9) Find a sine series to represent e^x from $x = 0$ to $x = 1$.
(10) Find a cosine series to represent e^{-x} from $x = 0$ to $x = a$.
(11) Find a sine series to represent $f(x)$ from $x = 0$ to $x = \pi$, where $f(x) = c$ from $x = 0$ to $x = \frac{\pi}{2}$ and $f(x) = 0$ from $x = \frac{\pi}{2}$ to $x = \pi$.

(12) Find (i) a sine series, and (ii) a cosine series, to represent $f(x)$ between $x = 0$ and $x = l$ where $f(x) = x$ from $x = 0$ to $x = \frac{l}{3}$ and $f(x) = \frac{1}{2}(l - x)$ from $x = \frac{l}{3}$ to $x = l$.

(13) Calculate values of x^2 from $x = 1$ to $x = 12$, taking the values $x = 1$, 2, 3, etc. Use the method of Art. 22 to find the equivalent Fourier series.

(14) Find a Fourier series to represent the function $f(x) = x^3$ from $x = 0$ to $x = 2\pi$.

(15) Find (i) a sine series, and (ii) a cosine series, to represent $f(x)$ where $f(x) = c$ from $x = 0$ to $x = \frac{\pi}{2}$ and $f(x) = 0$ from $x = \frac{\pi}{2}$ to $x = \pi$. Sketch graphs showing what the series represents outside of the range $x = 0$ to $x = \pi$.

(16) The displacement y of any point of a stretched string of length l in a state of transverse vibration is connected with the distance x from a point in the string and the time in seconds by the relation $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 x}{\partial t^2}$ where a is a constant. The solution of this equation is

$$y = a_1 \sin\frac{\pi x}{l}\cos\frac{\pi at}{l} + a_2 \sin\frac{2\pi x}{l}\cos\frac{2\pi at}{l} + a_3 \sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l} + \ldots$$

where a_1, a_2, a_3, etc. are constants depending on the original shape of the string. By putting $t = 0$, convert the above into a simple series of sines, and determine a_1, a_2, a_3, etc., so that the original shape of the string shall be that of an isosceles triangle of height c.

(17) Find a series of sines to represent $f(x)$ from $x = 0$ to $x = \pi$ if $f(x) = 0$ from $x = 0$ to $x = \frac{\pi}{3}$ and from $x = \frac{2\pi}{3}$ to π and $f(x) = c$ from $x = \frac{\pi}{3}$ to $\frac{2\pi}{3}$

(18) The turning moment T lb-ft on the crankshaft of a steam-engine is given for a series of values of the crank angle θ degrees.

θ	0	15	30	45	60	75	90	105	120	135	150	165	180
T	0	2 782	5 224	7 051	8 097	8 332	7 850	6 832	5 499	4 051	2 626	1 202	0

Expand T in a series of sines.

(19) The following values of y and x are given; expand y in the form of a Fourier series.

x	0	1	2	3	4	5	6	7	8	9	10	11
y	90	138	182	218	244	261	278	279	275	258	220	159

(20) Repeat Ex. 3, using values of x calculated from the given particulars. Use the values of $x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$, etc., up to $x = 2\pi$.

(21) Draw to scale a diagram showing the position of the crank and connecting-rod of a simple engine mechanism for each of the crank angles 0, 30°, 60°, 90°, 120°, 150°, etc. . . . 330°. Assume the crank to be 1 ft long, and the connecting-rod 4 ft long; obtain a Fourier series for the distance x of the crosshead from the crankshaft.

Express each of the following empirical functions as a Fourier series, taking the period to be 2π in each case, and assuming that the first value given is measured at one-twelfth of the period from the beginning, and the others after equal intervals of one-twelfth of the period.

(22) 25, 40, 50·5, 57·5, 61·5, 63·3, 63·2, 59·2, 52·5, 44·2, 35·8, 28·7.

(23) 3·5, 6·09, 7·82, 8·58, 8·43, 7·73, 6·98, 6·19, 6·04, 5·55, 5·01, 3·35.

(24) 2·8, – 2·2, – 6·6, – 8·5, – 8·4, – 7·2, – 4·4, – 0·6, 2·2, 4·6, 5·9, 4·7.

(25) Find the coefficients A_0, A_r, B_r, if

$$f(x) = A_0 + A_1 \cos x + \ldots + A_r \cos rx + \ldots$$
$$+ B_1 \sin x + \ldots + B_r \sin rx + \ldots$$

between the values $x = 0$ and $x = 2\pi$; and determine their values in the case in which $f(x) = x$ between these limits. (U.L.)

(26) Give concisely with proofs a graphical method for determining the coefficients of a Fourier series valid between $x = 0$ and $x = 2\pi$ which is approximately the equation to a given graph. (U.L.)

(27) A function $f(x)$ us equal to sin $(\pi x/c)$ when sin $(\pi x/c)$ is positive, and to – sin $(\pi x/c)$ when sin $(\pi x/c)$ is negative. Prove that, when $f(x)$ is expanded in a Fourier series of sines and cosines of multiples of $(\pi x/c)$, the coefficients of the sines are all zero, and obtain the expansion. (U.L.)

(28) Express $F(x)$ in a Fourier series of sines and cosines, given that

$$F(x) = l - x \text{ when } 0 < x < l$$

$$F(x) = 0 \text{ when } l < x < 2l$$

and that $F(x)$ is periodic with a period $2l$. (U.L.)

(29) Assuming it possible for the series

$$a_1 \sin \theta + a_2 \sin 2\theta + a_2 \sin 3\theta + \ldots$$

to have the same values as a given function $f(\theta)$ for all values of θ between 0 and π, prove that

$$a_n = \frac{2}{\pi}\int_0^\pi f(\theta) \sin n\theta \, d\theta$$

Suppose
$$f(\theta) = 0 \text{ from } \theta = 0 \text{ to } \theta = \frac{\pi}{4}$$

$$f(\theta) = h \text{ from } \theta = \frac{\pi}{4} \text{ to } \theta = \frac{3\pi}{4}$$

$$f(\theta) = 0 \text{ from } \theta = \frac{3\pi}{4} \text{ to } \theta = \pi$$

Express $f(\theta)$ in a series of sines. (U.L.)

(30) Assuming that $f(x)$ can be expanded in a series of the form

$$A_1 \sin \frac{\pi x}{l} + A_2 \sin \frac{2\pi x}{l} + \ldots + A_n \sin \frac{n\pi x}{l} + \ldots$$

for all values of x between 0 and l, find the form of the coefficients.

If $f(x) = \frac{x}{a}$ from 0 to a, and $\frac{l-x}{l-a}$ from a to l, prove that, for all values of x between 0 and l,

$$f(x) = \frac{2l^2}{a(l-a)\pi^2}\left\{ \sin\frac{\pi a}{l} \sin\frac{\pi x}{l} + \frac{1}{2^2}\sin\frac{2\pi a}{l}\sin\frac{2\pi x}{l} + \ldots \right.$$
$$\left. \ldots + \frac{1}{n^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l} + \ldots \right\}$$ (U.L.)

(31) In each of the following cases sketch the graph from $x = -c$ to $x = c$ of any function which satisfies the given condition—

(*a*) The Fourier series contains sine terms only.
(*b*) ,, ,, ,, ,, cosine terms only.
(*c*) ,, ,, ,, ,, odd sine and cosine terms only.
(*d*) ,, ,, ,, ,, even sine and cosine terms only.
(*e*) ,, ,, ,, ,, odd sine terms only.
(*f*) ,, ,, ,, ,, odd cosine terms only.
(*g*) ,, ,, ,, ,, even sine terms only.
(*h*) ,, ,, ,, ,, even cosine terms only.

(32) If $f(x) = -a$ from $x = -c$ to $x = 0$ and $f(x) = a$ from $x = 0$ to $x = c$, and the period is $2c$, show that in finding the Fourier series we only need to look for the odd sine terms. Find the series.

(33) Repeat the working of Ex. 3, Art. 21, shortening the analysis as much as possible.

(34) The graph of $f(x)$ over the range $x = -c$ to $x = c$ consists of four straight lines joining in succession the points $(-c, 0)$, $\left(-\frac{c}{2}, a\right)$, $(0, 0)$, $\left(\frac{c}{2}, a\right)$, and $(c, 0)$. Find in the simplest manner a Fourier series of period $2c$ which is equivalent to $f(x)$ over the given range.

(35) Find a Fourier series for $f(x)$ of period 6 on the x scale, where $f(x) = x(3 + x)$ from $x = -3$ to $x = 0$ and $f(x) = x(3 - x)$ from $x = 0$ to $x = 3$.

(36) Express as a Fourier series of period $2c$ the function of x which is equal to $-a$ from $x = 0$ to $x = c$ and to a from $x = c$ to $x = 2c$.

(37) What conditions must be satisfied by a periodic function $f(x)$ of period 2π if its Fourier series is to contain only sines of odd multiples of x? Show that, if these conditions are satisfied, the coefficient of $\sin(2n + 1)x$ is

$$b_{2n+1} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x) \sin(2n + 1)x\, dx$$

If such a function is zero for $0 < x < \frac{\pi}{3}$, and unity for $\frac{\pi}{3} < x < \frac{\pi}{2}$, determine the first four terms of its Fourier series. (U.L.)

(38) A function $f(x) = \frac{1}{4}x^2$ in the interval $-\pi < x < \pi$, and is repeated with this period. Prove that

$$f(x) = \frac{\pi^2}{12} - \cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \ldots$$

and find the value of the term containing $\cos 2px$ when $x = \frac{\pi}{2}$ (U.L.)

(39) The function $f(x)$ is defined in the range $0 < x < \pi$ by the formulae—

$$f(x) = \frac{\pi}{3}, 0 < x < \frac{\pi}{3};\ f(x) = 0, \frac{\pi}{3} < x < \frac{2\pi}{3};\ f(x) = -\frac{\pi}{3}, \frac{2\pi}{3} < x < \pi$$

Obtain the expansion in a series of cosines of multiples of x. Show that the terms $\cos nx$ are absent when the remainder on dividing n by 6 is 0, 2, 3 or 4. (U.L.)

(40) Obtain the series of sines of integer multiples of x which represents the function $\pi x - x^2$ from $x = 0$ to $x = \pi$. Sketch the graph of the function, and also of the sum of the series, from $x = -\pi$ to $x = 2\pi$. Sketch also, for the same range, the sum of the cosine series which would represent the original function from $x = 0$ to $x = \pi$. (U.L.)

(41) Show how to obtain the coefficients in the Fourier expansion of a function $f(x)$ which is given in the range $(-\pi, \pi)$. Obtain the harmonic analysis of the function defined by $f(x) = c$ in the range $(0, \alpha)$, $f(x) = 0$ in the range (α, π), and $f(x) = f(-x)$. Describe briefly the effect of (i) integrating; (ii) differentiating a Fourier series. (U.L.)

(42) If the function $f(x)$ can be expanded in the Fourier series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

show that
$$a_n + i\,b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)e^{inx}\,dx$$

Hence, or otherwise, expand the function $x(x-1)(x-2)$ in a full range Fourier series, valid in the interval $(0, 2)$, giving the general term. Illustrate your result by drawing a graph of the function represented by the series for values of x outside the range. (U.L.)

CHAPTER IV

THE COMPLEX VARIABLE—CONFORMAL TRANSFORMATION

BEFORE studying the subject-matter of this chapter the reader should revise Arts. 13 to 19 of Vol. I, which treat of complex numbers.

24. **Functions of a Complex Variable.** If x and y are real variables, then $z = x + iy$, where $i = \sqrt{-1}$, is a *complex variable*. Let z represent any point in some region of the xy-plane, and let Z be a second complex variable which assumes a definite value or definite values for each value of z in the region. Then Z is said to be a *function of the complex variable* z, and the relationship is denoted by

$$Z = f(z)$$

Z is a *single-valued* function of z when it assumes a unique value for each value of z in the region. $Z = \frac{1}{z}$ is an example of a single-valued function, and $Z = \sqrt{z}$ is an example of a multi-valued function of z. The former is defined at all points in the xy-plane except that at $z = 0$, and the latter assumes two values for each value of z except $z = 0$.

By the methods of the above-mentioned Arts. of Vol. I, the value of $f(z)$ can be readily obtained in the form $u + iv$, where, in general, both u and v will be functions of x and y, provided that the processes involved are only those of addition, subtraction, multiplication, division, and extraction of root.

For example,

$$f(z) = z^2 - 4z + 3 = (x + iy)^2 - 4(x + iy) + 3$$
$$= (x^2 - y^2 - 4x + 3) + i(2xy - 4y);$$

$$f(z) = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2};$$

$$f(z) = \sqrt{z} = (x + iy)^{\frac{1}{2}} = r^{\frac{1}{2}}\left[\cos\frac{\theta + 2n\pi}{2} + i\sin\frac{\theta + 2n\pi}{2}\right],$$

where $n = 0, 1$, $r = \sqrt{x^2 + y^2}$,

and $\quad \theta = \tan^{-1}\frac{y}{x};$

$$f(z) = \frac{1}{\sqrt[3]{1 - z^2}} = \frac{1}{[1 - (x + iy)^2]^{\frac{1}{3}}} = \frac{1}{[(1 - x^2 + y^2) - 2ixy]^{\frac{1}{3}}}$$

$$= \frac{1}{r^{\frac{1}{3}}\left[\cos\dfrac{\theta + 2n\pi}{3} - i\sin\dfrac{\theta + 2n\pi}{3}\right]}$$

$$= r^{-\frac{1}{3}}\left[\cos\frac{\theta + 2n\pi}{3} + i\sin\frac{\theta + 2n\pi}{3}\right],$$

where $n = 0, 1, 2$, $r = \sqrt{(1 - x^2 + y^2)^2 + 4x^2y^2}$,

and $\quad \theta = \tan^{-1}\dfrac{2xy}{1 - x^2 + y^2}$

25. **Exponential and Circular Functions.** Transcendental functions, such as e^z, $\sin z$, etc., of the complex variable z require special definition, and it is obviously desirable that this definition be so framed that the corresponding functions of the real variable x are included as particular cases.

Consider the infinite series

$$1 + z + \frac{z^2}{\underline{|2}} + \frac{z^3}{\underline{|3}} + \frac{z^4}{\underline{|4}} + \ldots \qquad . \qquad . \qquad (IV.1)$$

$$z - \frac{z^3}{\underline{|3}} + \frac{z^5}{\underline{|5}} - \frac{z^7}{\underline{|7}} + \ldots \qquad . \qquad . \qquad . \qquad (IV.2)$$

$$1 - \frac{z^2}{\underline{|2}} + \frac{z^4}{\underline{|4}} - \frac{z^6}{\underline{|6}} + \ldots \qquad . \qquad . \qquad . \qquad (IV.3)$$

where $z = x + iy$.

In (IV.1), $z^n = (x + iy)^n = r^n(\cos n\theta + i\sin\theta)$, where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\dfrac{y}{x}$, so that (IV.1) can be written in the form

$$\left(1 + r\cos\theta + \frac{r^2}{\underline{|2}}\cos 2\theta + \frac{r^3}{\underline{|3}}\cos 3\theta + \ldots\right)$$

$$+\, i\left(r\sin\theta + \frac{r^2}{\underline{|2}}\sin 2\theta + \frac{r^3}{\underline{|3}}\sin 3\theta + \ldots\right)$$

Now r is always positive, and by Example, Art. 6, Vol. I, the infinite series $1 + r + \frac{r^2}{\underline{|2}} + \frac{r^3}{\underline{|3}} + \ldots$ is convergent for all real values of r. It follows from Test 1, Art. 6, Vol. I, that the series $1 + r\cos\theta + \frac{r^2}{\underline{|2}}\cos 2\theta + \ldots$ and $r\sin\theta + \frac{r^2}{\underline{|2}}\sin 2\theta + \frac{r^3}{\underline{|3}}\sin 3\theta + \ldots$ are also convergent. If these two latter series converge to the values C and S respectively, then the series (IV.1) converges to the value $C + iS$, and is thus convergent for all complex values of z which have finite moduli. The series (IV.2) and (IV.3) are similarly convergent.

If in $z = x + iy$, y is assumed zero, the series (IV.1), (IV.2), and (IV.3) take the forms of the expansions in ascending powers of x of the real functions e^x, $\sin x$, and $\cos x$ respectively, that is

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ldots \qquad \text{(IV.4)}$$

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \ldots \qquad \text{(IV.5)}$$

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \ldots \qquad \text{(IV.6)}$$

These three functions are, therefore, particular cases of the functions represented by (IV.1), (IV.2), and (IV.3).

The function represented by (IV.1) is often denoted by the symbol exp. z.

The two infinite series obtained by substituting $z = z_1$ and $z = z_2$ in turn in (IV.1) are absolutely convergent and thus satisfy the condition under which the result of multiplying the two series together is the equivalent of the product of the functions which the two series represent. Now it is easy to verify that the sum of the terms of the nth degree in z_1 and z_2 in the product of the two series is

$$\frac{{z_1}^n}{\underline{|n}} + \frac{{z_1}^{n-1}z_2}{\underline{|n-1}} + \frac{{z_1}^{n-2}{z_2}^2}{\underline{|2}\,\underline{|n-2}} + \ldots + \frac{z_1{z_2}^{n-1}}{\underline{|n-1}} + \frac{{z_2}^n}{\underline{|n}}, \text{ i.e. } \frac{(z_1+z_2)^n}{\underline{|n}},$$

so that the resulting series is

$$1 + (z_1 + z_2) + \frac{(z_1+z_2)^2}{\underline{|2}} + \frac{(z_1+z_2)^3}{\underline{|3}} + \ldots + \frac{(z_1+z_2)^n}{\underline{|n}} + \ldots$$

Thus, $$\exp. z_1 \times \exp. z_2 = \exp. (z_1 + z_2) \quad . \quad . \quad \text{(IV.7)}$$

Further, (IV.7) includes the case when $z_1 = x_1$ and $z_2 = x_2$ are real, and, since from (IV.4) $\exp. x_1 = e^{x_1}$ and $\exp. x_2 = e^{x_2}$, (IV.7) becomes

$$e^{x_1} \times e^{x_2} = e^{x_1 + x_2} \quad . \quad . \quad . \quad \text{(IV.8)}$$

It is convenient, therefore, to write (IV.7) as

$$e^{z_1} \times e^{z_2} = e^{z_1 + z_2} \quad . \quad . \quad . \quad \text{(IV.9)}$$

so that the real number e with an exponent, whether complex or real, obeys the fundamental index law of elementary algebra.

Thus, e^z, where z is complex, is defined by the relation

$$e^z = 1 + z + \frac{z^2}{\underline{|2}} + \frac{z^3}{\underline{|3}} + \frac{z^4}{\underline{|4}} + \ldots \quad . \quad \text{(IV.10)}$$

The functions $\sin z$ and $\cos z$ are defined similarly as given by the relations

$$\sin z = z - \frac{z^3}{\underline{|3}} + \frac{z^5}{\underline{|5}} - \frac{z^7}{\underline{|7}} + \ldots \quad . \quad \text{(IV.11)}$$

and $$\cos z = 1 - \frac{z^2}{\underline{|2}} + \frac{z^4}{\underline{|4}} - \frac{z^6}{\underline{|6}} + \ldots \quad . \quad \text{(IV.12)}$$

From (IV.12) and (IV.11),

$$\cos z + i \sin z = 1 + iz - \frac{z^2}{\underline{|2}} - i\frac{z^3}{\underline{|3}} + \frac{z^4}{\underline{|4}} + i\frac{z^5}{\underline{|5}} - \frac{z^6}{\underline{|6}} - i\frac{z^7}{\underline{|7}} + \ldots$$

$$= 1 + iz + \frac{(iz)^2}{\underline{|2}} + \frac{(iz)^3}{\underline{|3}} + \frac{(iz)^4}{\underline{|4}} + \frac{(iz)^5}{\underline{|5}}$$

$$+ \frac{(iz)^6}{\underline{|6}} + \frac{(iz)^7}{\underline{|7}} + \ldots$$

i.e.

$$\cos z + i \sin z = e^{iz} \quad . \quad . \quad . \quad \text{(IV.13)}$$

It is established similarly that

$$\cos z - i \sin z = e^{-iz} \quad . \quad . \quad . \quad \text{(IV.14)}$$

If corresponding sides of (IV.13) and (IV.14) be subtracted and added and the result in each case be divided by 2, then

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) \quad . \quad . \quad . \quad (\text{IV}.15)$$

and

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}) \quad . \quad . \quad . \quad (\text{IV}.16)$$

These relations hold for all values of z, real or complex, and might have been taken as defining $\sin z$ and $\cos z$ respectively.

The other circular functions of the complex variable z are defined as follows—

$$\operatorname{cosec} z = \frac{1}{\sin z},\ \sec z = \frac{1}{\cos z},\ \tan z = \frac{\sin z}{\cos z},\ \text{and } \cot z = \frac{\cos z}{\sin z}$$

just as for a real variable.

From (IV.15), $\sin(-z) = \frac{1}{2i}(e^{-iz} - e^{iz}) = -\frac{1}{2i}(e^{iz} - e^{-iz})$

i.e.

$$\sin(-z) = -\sin z \quad . \quad . \quad . \quad . \quad (\text{IV}.17)$$

It is proved similarly that

$$\cos(-z) = \cos z \quad . \quad . \quad . \quad (\text{IV}.18)$$

and

$$\tan(-z) = -\tan z \quad . \quad . \quad . \quad (\text{IV}.19)$$

again just as in the case of a real variable.

The usual formulae connecting the circular functions of a real variable hold good for those of a complex variable.

For example, from (IV.15) and (IV.16),

$$\sin^2 z + \cos^2 z = \frac{1}{4i^2}(e^{iz} - e^{-iz})^2 + \tfrac{1}{4}(e^{iz} + e^{-iz})^2$$

$$= -\tfrac{1}{4}(e^{2iz} - 2 + e^{-2iz}) + \tfrac{1}{4}(e^{2iz} + 2 + e^{-2iz})$$

i.e.

$$\sin^2 z + \cos^2 z = 1 \quad . \quad . \quad . \quad (\text{IV}.20)$$

It is easy to show similarly that

$$1 + \tan^2 z = \sec^2 z \quad . \quad . \quad . \quad (\text{IV}.21)$$

and

$$1 + \cot^2 z = \operatorname{cosec}^2 z \quad . \quad . \quad (\text{IV}.22)$$

Also $\sin z_1 \cos z_2 = \frac{1}{2i}(e^{iz_1} - e^{-iz_1}) \times \frac{1}{2}(e^{iz_2} + e^{-iz_2})$

$$= \frac{1}{4i}\left[e^{i(z_1 + z_2)} - e^{-i(z_1 - z_2)} + e^{i(z_1 - z_2)} - e^{-i(z_1 + z_2)}\right]$$

and similarly,

$$\cos z_1 \sin z_2 = \frac{1}{4i}[e^{i(z_1 + z_2)} + e^{-i(z_1 - z_2)} - e^{i(z_1 - z_2)} - e^{-i(z_1 + z_2)}]$$

By addition,

$$\sin z_1 \cos z_2 + \cos z_1 \sin z_2 = \frac{1}{2i}[e^{i(z_1 + z_2)} - e^{-i(z_1 + z_2)}]$$

i.e. $$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \quad \text{(IV.23)}$$

By subtraction,

$$\sin z_1 \cos z_2 - \cos z_1 \sin z_2 = \frac{1}{2i}[e^{i(z_1 - z_2)} - e^{-i(z_1 - z_2)}]$$

i.e. $$\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2 \quad \text{(IV.24)}$$

It may be proved in a similar manner that

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \quad . \quad \text{(IV.25)}$$

and $$\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2 \quad . \quad \text{(IV.26)}$$

Other formulae based on these addition and subtraction formulae can be readily established.

If in (IV.15) and (IV.16), z is purely imaginary, say iy, then

$$\sin iy = \frac{1}{2i}(e^{i^2 y} - e^{-i^2 y}) = \frac{1}{2i}(e^{-y} - e^{y}) = \frac{(-1)}{2i}(e^{y} - e^{-y})$$

i.e. $$\sin iy = \frac{i}{2}(e^{y} - e^{-y}) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV.27)}$$

and $$\cos iy = \tfrac{1}{2}(e^{i^2 y} + e^{-i^2 y}) = \tfrac{1}{2}(e^{-y} + e^{y})$$

i.e. $$\cos iy = \tfrac{1}{2}(e^{y} + e^{-y}) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV.28)}$$

If n is any integer, positive or negative, then

$$\begin{aligned}\sin(z + 2n\pi) &= \sin z \cos 2n\pi + \cos z \sin 2n\pi \\ &= \sin z\end{aligned}$$

and $$\begin{aligned}\cos(z + 2n\pi) &= \cos z \cos 2n\pi - \sin z \sin 2n\pi \\ &= \cos z\end{aligned}$$

Thus, $\sin z$ and $\cos z$ remain unaltered when z is increased or decreased by 2π or 4π or 6π or etc.; hence $\sin z$ and $\cos z$ are periodic functions with period 2π.

Further, $$\begin{aligned}e^{z + 2n\pi i} &= e^{z}(\cos 2n\pi + i \sin 2n\pi) \\ &= e^{z}\end{aligned}$$

so that e^z is periodic with period $2\pi i$.

26. **Hyperbolic Functions.** The hyperbolic functions of a complex variable z are defined as

$$\left.\begin{aligned} \sinh z &= \tfrac{1}{2}(e^{z} - e^{-z}) \\ \cosh z &= \tfrac{1}{2}(e^{z} + e^{-z}) \\ \tanh z &= \frac{\sinh z}{\cosh z} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}} \text{ or } \frac{e^{2z} - 1}{e^{2z} + 1} \\ \operatorname{cosech} z &= \frac{1}{\sinh z} \\ \operatorname{sech} z &= \frac{1}{\cosh z} \\ \text{and} \qquad \coth z &= \frac{1}{\tanh z} \end{aligned}\right\} \quad (\text{IV}.29)$$

just as in the case of hyperbolic functions of a real variable, and the formulae connecting the latter hold good for the former.

The relations (IV.15) and (IV.16) may then be expressed as

$$\sinh iz = i \sin z \quad . \quad . \quad . \quad (\text{IV}.30)$$

and

$$\cosh iz = \cos z \quad . \quad . \quad . \quad (\text{IV}.31)$$

By division,

$$\tanh iz = i \tan z \quad . \quad . \quad . \quad (\text{IV}.32)$$

Also, the relations (IV.27) and (IV.28) may be expressed as

$$\sin iy = i \sinh y \quad . \quad . \quad . \quad (\text{IV}.33)$$

and

$$\cos iy = \cosh y \quad . \quad . \quad . \quad (\text{IV}.34)$$

By division,

$$\tan iy = i \tanh y \quad . \quad . \quad . \quad (\text{IV}.35)$$

(IV.33) to (IV.35) can be deduced from (IV.30) to (IV.32) by substituting $z = iy$ and noting that $\sinh(-y) = -\sinh y$, $\cosh(-y) = \cosh y$, and $\tanh(-y) = -\tanh y$.

From the definitions of $\sinh z$ and $\cosh z$ it is easy to deduce the expansions of these functions in infinite series, thus,

$$\sinh z = z + \frac{z^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \frac{z^7}{\underline{|7}} + \ldots \quad . \quad (\text{IV}.36)$$

and

$$\cosh z = 1 + \frac{z^2}{\underline{|2}} + \frac{z^4}{\underline{|4}} + \frac{z^6}{\underline{|6}} + \ldots \quad . \quad (\text{IV}.37)$$

From (I.34), Vol. I,

$$\sinh (x + iy) = \sinh x \cosh iy + \cosh x \sinh iy$$
$$= \sinh x \cos y + i \cosh x \sin y$$

since from the above $\cosh iy = \cos y$ and $\sinh iy = i \sin y$

Hence, if n is an integer, positive or negative,

$$\sinh (x + iy + 2n\pi i) = \sinh [x + i(y + 2n\pi)]$$
$$= \sinh x \cos (y + 2n\pi) + i \cosh x \sin (y + 2n\pi)$$
$$= \sinh x \cos y + i \cosh x \sin y$$

Thus, when $z = x + iy$, $\sinh z$ and also $\cosh z$, as can be shown similarly, remain unaltered in value when z is increased or decreased by $2\pi i$ or $4\pi i$ or $6\pi i$ or etc., so that $\sinh z$ and $\cosh z$ are periodic, the period being imaginary and equal to $2\pi i$. It is left as an exercise for the reader to show that the period of $\tanh z$ is πi.

EXAMPLE 1

If $x + iy = \cos (u + iv)$, express x and y in terms of u and v.

Show that $\cos^2 u$ and $\cosh^2 v$ are the roots of the equation

$$\lambda^2 - (x^2 + y^2 + 1)\lambda + x^2 = 0$$

If $\cos (u + iv) = (5 + 4i\sqrt{3})/6$, find u and v. (U.L.)

We have $x + iy = \cos (u + iv) = \cos u \cos iv - \sin u \sin iv$

$$= \cos u \cosh v - i \sin u \sinh v \quad . \quad . \quad (1)$$

Equating real and imaginary parts, we obtain

$$x = \cos u \cosh v \qquad \text{and} \qquad y = - \sin u \sinh v$$

$$\therefore \quad x^2 = \cos^2 u \cosh^2 v \text{ and } y^2 = \sin^2 u \sinh^2 v$$
$$= (1 - \cos^2 u) (\cosh^2 v - 1)$$
$$= - 1 + \cos^2 u + \cosh^2 v - \cos^2 u \cosh^2 v$$
$$= - 1 + \cos^2 u + \cosh^2 v - x^2$$

so that
$$x^2 + y^2 + 1 = \cos^2 u + \cosh^2 v$$

The equation becomes

$$\lambda^2 - (\cos^2 u + \cosh^2 v)\lambda + \cos^2 u \cosh^2 v = 0$$

i.e.
$$(\lambda - \cos^2 u) (\lambda - \cosh^2 v) = 0$$

so that the roots of the equation are $\lambda = \cos^2 u$ and $\lambda = \cosh^2 v$.

Since $\cos (u + iv) = (5 + 4i\sqrt{3})/6$, then $x = \dfrac{5}{6}$ and $y = \dfrac{4\sqrt{3}}{6}$ so that

$$x^2 + y^2 + 1 = \frac{25 + 48 + 36}{36} = \frac{109}{36} \text{ and } x^2 = \frac{25}{36}$$

Hence $\cos^2 u$ and $\cosh^2 v$ are the roots of the equation

$$\lambda^2 - \frac{109}{36}\lambda + \frac{25}{36} = 0, \text{ i.e. } \left(\lambda - \frac{25}{9}\right)\left(\lambda - \frac{1}{4}\right) = 0$$

It follows that $\cos^2 u = \dfrac{1}{4}$ and $\cosh^2 v = \dfrac{25}{9}$

Since $\cosh v$ is always positive and $\cos u \cosh v = x = \dfrac{5}{6}$, then $\cos u$ must be positive, and, if the principal value of u is taken, $\sinh v$ and, therefore, v, must be negative. Thus, $\cos u = \dfrac{1}{2}$ and $u = \dfrac{\pi}{3}$; also $\cosh v = \dfrac{5}{3}$ and

$$v = \log_e\left(\frac{5}{3} - \sqrt{\frac{25}{9} - 1}\right) = \log_e \frac{1}{3} \quad \text{[see (I.38), Vol. I]}$$

$$\therefore \qquad u = \frac{\pi}{3} \text{ and } v = \log_e \frac{1}{3}$$

EXAMPLE 2

(i) If $x + iy = \cosh(u + iv)$, express x and y in terms of u and v, and find the equations of the curves u = constant, v = constant.

(ii) By means of the substitution $\phi = \frac{1}{2}\pi - 2t$, or otherwise, prove that

$$\left(\frac{1 + \sin\phi + i\cos\phi}{1 + \sin\phi - i\cos\phi}\right)^n = \cos(\tfrac{1}{2}n\pi - n\phi) + i\sin(\tfrac{1}{2}n\pi - n\phi) \quad \text{(U.L.)}$$

(i) By (I.34), Vol. I, $\cosh(u + iv) = \cosh u \cosh iv + \sinh u \sinh iv$

Now, from (IV.31) and (IV.30),

$$\cosh iv = \cos v \text{ and } \sinh iv = i\sin v$$

Hence $\qquad x + iy = \cosh u \cos v + i \sinh u \sin v$

whence $\qquad x = \cosh u \cos v$ and $y = \sinh u \sin v$

From these relations, $\dfrac{x}{\cosh u} = \cos v$ and $\dfrac{y}{\sinh u} = \sin v$ and, by squaring and adding, noting that $\cos^2 v + \sin^2 v = 1$,

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1$$

which, if u is constant, is the equation of an ellipse of semi-axis $\cosh u$ and $\sinh u$.

Also, $\qquad \dfrac{x}{\cos v} = \cosh u$ and $\dfrac{y}{\sin v} = \sinh u$

and, by squaring and subtracting, noting that $\cosh^2 u - \sinh^2 u = 1$,

$$\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1$$

which, if v is constant, is the equation of a hyperbola with semi-transverse axis $= \cos v$ and semi-conjugate axis $= \sin v$.

(ii) $$\frac{1+\sin\phi+i\cos\phi}{1+\sin\phi-i\cos\phi}=\frac{(1+\sin\phi+i\cos\phi)^2}{(1+\sin\phi-i\cos\phi)(1+\sin\phi+i\cos\phi)}$$
$$=\frac{(1+\sin\phi)^2+2i(1+\sin\phi)\cos\phi-\cos^2\phi}{(1+\sin\phi)^2+\cos^2\phi}$$
$$=\frac{(1+\sin\phi)^2+2i(1+\sin\phi)\cos\phi-(1-\sin^2\phi)}{2(1+\sin\phi)}$$
$$=\tfrac{1}{2}[1+\sin\phi+2i\cos\phi-1+\sin\phi]$$
$$=\sin\phi+i\cos\phi$$
$$=\cos\left(\frac{\pi}{2}-\phi\right)+i\sin\left(\frac{\pi}{2}-\phi\right)$$

$$\therefore\quad\left(\frac{1+\sin\phi+i\cos\phi}{1+\sin\phi-i\cos\phi}\right)^n=\left[\cos\left(\frac{\pi}{2}-\phi\right)+i\sin\left(\frac{\pi}{2}-\phi\right)\right]^n$$
$$=\cos(\tfrac{1}{2}n\pi-n\phi)+i\sin(\tfrac{1}{2}n\pi-n\phi)$$

Or, if $\phi=\frac{1}{2}\pi-2t$, $\cos\phi=\sin 2t$ and $\sin\phi=\cos 2t$, so that

$$\frac{1+\sin\phi+i\cos\phi}{1+\sin\phi-i\cos\phi}=\frac{1+\cos 2t+i\sin 2t}{1+\cos 2t-i\sin 2t}=\frac{2\cos^2 t+2i\sin t\cos t}{2\cos^2 t-2i\sin t\cos t}$$
$$=\frac{\cos t+i\sin t}{\cos t-i\sin t}=\frac{(\cos t+i\sin t)^2}{\cos^2 t+\sin^2 t}$$
$$=(\cos^2 t-\sin^2 t)+2i\sin t\cos t$$
$$=\cos 2t+i\sin 2t$$

$$\therefore\ \text{Given expression}=(\cos 2t+i\sin 2t)^n$$
$$=\cos 2nt+i\sin 2nt$$
$$=\cos(\tfrac{1}{2}n\pi-n\phi)+i\sin(\tfrac{1}{2}n\pi-n\phi)$$

27. **Logarithmic Functions.** If $e^u=x$, where x is real and positive, then $u=\log x$, and there is but one real solution of this equation. The logarithm of a complex quantity z is defined in similar manner, i.e. if $z=e^w$, then $w=\log z$, but it will be shown that in this case $\log z$ has infinitely many values.

If $z=x+iy$ and $w=u+iv$, the relation $z=e^w$ becomes

$$x+iy=e^{u+iv}=e^u\,.\,e^{iv}=e^u(\cos v+i\sin v)$$

Now $x+iy=r[\cos(\theta+2n\pi)+i\sin(\theta+2n\pi)]$, where n has any integral value, $r=+\sqrt{x^2+y^2}$, and $\theta=\tan^{-1}\dfrac{y}{x}$ is the principal value of the argument.

Hence $r \cos(\theta + 2n\pi) + ir \sin(\theta + 2n\pi) = e^u \cos v + ie^u \sin v$

Equating real and imaginary parts,

$$e^u \cos v = r \cos(\theta + 2n\pi)$$

and $$e^u \sin v = r \sin(\theta + 2n\pi)$$

whence $e^u = r$ and $v = \theta + 2n\pi$

Since u and r are both real, u is identical with the logarithm of r,

i.e. $$u = \log r = \log \sqrt{x^2 + y^2}$$

Thus, $w = u + iv = \log \sqrt{x^2 + y^2} + i(\theta + 2n\pi)$

or $$\log z = \log \sqrt{x^2 + y^2} + i\left(2n\pi + \tan^{-1}\frac{y}{x}\right) \quad . \quad . \quad \text{(IV.38)}$$

The principal value of $\log z$ is that obtained by putting $n = 0$, i.e. the one in which the argument has its principal value. Frequently the symbol $\text{Log}\, z$ is used to denote the many valued form of the logarithm of z, and $\log z$ to denote the principal value, thus,

$$\text{Log}\, z = \log \sqrt{x^2 + y^2} + i\left(2n\pi + \tan^{-1}\frac{y}{x}\right) \quad . \quad \text{(IV.39)}$$

and $$\log z = \log \sqrt{x^2 + y^2} + i \tan^{-1}\frac{y}{x} \quad . \quad . \quad \text{(IV.40)}$$

If in (IV.39), y is assumed to be zero, and x positive, then

$$\text{Log}\, x = \log x + 2n\pi i \quad . \quad . \quad \text{(IV.41)}$$

which shows that any positive quantity has a real logarithm $\log x$ and infinitely many imaginary logarithms which are given by adding an integral multiple of $2\pi i$ to the real logarithm.

Again, with $y = 0$ and x negative, say $-x_1$, where x_1 is positive, then (IV.39) gives

$$\text{Log}(-x_1) = \log x_1 + i(2n\pi + \pi) \quad . \quad \text{(IV.42)}$$

since here the principal value of the argument of z is such that its cosine is $\dfrac{-x_1}{x_1} = -1$ and its sine is 0, so that this principal value is π.

If $n = 0$, (IV.42) gives

$$\log(-x_1) = \log x_1 + \pi i \quad . \quad . \quad \text{(IV.43)}$$

which shows that the principal value of the logarithm of a negative quantity is the sum of the ordinary logarithm of the absolute value of the quantity and πi.

Now assume $x = 0$; then (IV.39) becomes

$$\operatorname{Log}(iy) = \log y + i\left(2n\pi + \frac{\pi}{2}\right) \quad . \quad . \quad \text{(IV.44)}$$

since here $\tan^{-1}\dfrac{y}{x} = \tan^{-1}\infty = \dfrac{\pi}{2}$

Thus, the logarithm of a purely imaginary quantity is the sum of a real logarithm and a many-valued imaginary quantity.

EXAMPLE

(*a*) Show that (i) $\log\dfrac{x+iy}{x-iy} = 2i\tan^{-1}\dfrac{y}{x}$

(ii) $\operatorname{Log}(1+i) = \frac{1}{2}\log 2 + i\left(2n\pi + \dfrac{\pi}{4}\right)$

(*b*) Show that the principal value of $\operatorname{Log}\sin(x+iy)$ is

$$\tfrac{1}{2}\log\frac{\cosh 2y - \cos 2x}{2} + i\tan^{-1}(\cot x\tanh y)$$

(*a*) (i) Let $\quad \log\dfrac{x+iy}{x-iy} = u + iv$; then $\dfrac{x+iy}{x-iy} = e^{u+iv}$

Now $\quad \dfrac{x+iy}{x-iy} = \dfrac{(x+iy)^2}{x^2+y^2} = \dfrac{x^2-y^2}{x^2+y^2} + i\dfrac{2xy}{x^2+y^2}$

and $\quad e^{u+iv} = e^u \,.\, e^{iv} = e^u(\cos v + i\sin v)$

Hence $\quad e^u \,.\, \cos v + ie^u \,.\, \sin v = \dfrac{x^2-y^2}{x^2+y^2} + i\dfrac{2xy}{x^2+y^2}$

Equating real and imaginary parts,

$$e^u\cos v = \frac{x^2-y^2}{x^2+y^2}$$

and $\quad e^u\sin v = \dfrac{2xy}{x^2+y^2}$

By division, $\quad \tan v = \dfrac{2xy}{x^2-y^2} = \dfrac{2\dfrac{y}{x}}{1-\dfrac{y^2}{x^2}} = \dfrac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta$

so that $v = 2\theta$, where $\theta = \tan^{-1}\dfrac{y}{x}$

By squaring and adding,

$$e^{2u} = \frac{(x^2 - y^2)^2 + 4x^2y^2}{(x^2 + y^2)^2} = \frac{(x^2 + y^2)^2}{(x^2 + y^2)^2} = 1$$

so that $\qquad 2u = 0, \quad \text{i.e. } u = 0$

Hence $\qquad \log \dfrac{x + iy}{x - iy} = i2\theta = 2i \tan^{-1} \dfrac{y}{x}$

(ii) Letting $\operatorname{Log}(1 + i) = u + iv$, and proceeding as above, we find

$$e^u \cos v = 1 \text{ and } e^u \sin v = 1$$

so that $\qquad \tan v = 1 \text{ and } e^{2u} = 2$

Hence $\qquad v = 2n\pi + \dfrac{\pi}{4},\ u = \tfrac{1}{2} \log 2,$

and $\qquad \operatorname{Log}(1 + i) = \tfrac{1}{2} \log 2 + i\left(2n\pi + \dfrac{\pi}{4}\right)$

(*b*) If $\operatorname{Log} \sin (x + iy) = u + iv$, then

$$\sin (x + iy) = e^{u + iv} = e^u(\cos v + i \sin v)$$

Now $\quad \sin (x + iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$

Hence, $\qquad \sin x \cosh y + i \cos x \sinh y = e^u \cos v + i \,.\, e^u \sin v$

It follows that $\qquad e^u \cos v = \sin x \cosh y$

and $\qquad e^u \sin v = \cos x \sinh y$

which give $\tan v = \dfrac{\cos x \sinh y}{\sin x \cosh y} = \cot x \tanh y$, i.e. $v = \tan^{-1} (\cot x \tanh y)$ and

$$\begin{aligned} e^{2u} &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\ &= (1 - \cos^2 x) \cosh^2 y + \cos^2 x(\cosh^2 y - 1) \\ &= \cosh^2 y - \cos^2 x \\ &= \tfrac{1}{2}(1 + \cosh 2y) - \tfrac{1}{2}(1 + \cos 2x) = \tfrac{1}{2}(\cosh 2y - \cos 2x) \end{aligned}$$

so that $\qquad u = \tfrac{1}{2} \log \dfrac{\cosh 2y - \cos 2x}{2}$

Thus, the principal value of $\operatorname{Log} \sin (x + iy)$ is

$$\tfrac{1}{2} \log \frac{\cosh 2y - \cos 2x}{2} + i \tan^{-1} (\cot x \tanh y)$$

28. **General Power w^z.** The function w^z is defined by the relation

$$w^z = e^{z \operatorname{Log} w} \qquad . \quad . \quad . \quad \text{(IV.45)}$$

whether w and z be real or complex.

The principal value of w^z is given by

$$w^z = e^{z \log w} \qquad . \quad . \quad . \quad \text{(IV.46)}$$

where $\log w$ is the principal value of $\operatorname{Log} w$.

EXAMPLE

(i) Show that $2^i = e^{-2n\pi}[\cos(\log 2) + i\sin(\log 2)]$, where n has any integral value.

(ii) Find the principal value of i^i

(i) By (IV.45), $2^i = e^{i\,\mathrm{Log}\,2}$, and by (IV.41), $\mathrm{Log}\,2 = \log 2 + 2n\pi i$

Hence
$$\begin{aligned} 2^i &= e^{i(\log 2 + 2n\pi i)} \\ &= e^{i\log 2 - 2n\pi} \\ &= e^{-2n\pi} \times e^{i\log 2} \end{aligned}$$

i.e.
$$2^i = e^{-2n\pi}[\cos(\log 2) + i\sin(\log 2)]$$

(ii)
$$i^i = e^{i\,\mathrm{Log}\,i}, \text{ and by (IV.44), } \mathrm{Log}\,i = i\left(2n\pi + \frac{\pi}{2}\right)$$

Hence
$$\begin{aligned} i^i &= e^{i^2\left(2n\pi + \frac{\pi}{2}\right)} \\ &= e^{-\left(2n\pi + \frac{\pi}{2}\right)} \end{aligned}$$

The principal value $e^{-\frac{\pi}{2}}$ is obtained by putting $n = 0$.

29. **Continuity.** If $z = x + iy$ and $Z = f(z) = f(x + iy) = u + iv$, where u and v are real functions of the variables x and y, z and Z can be represented on separate Argand diagrams or complex planes,

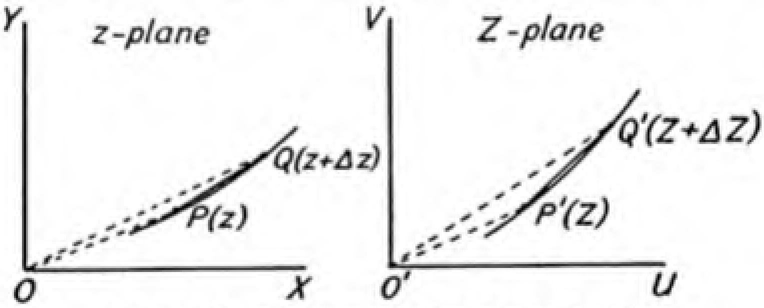


FIG. 22. GRAPHICAL REPRESENTATION

which, for convenience, may be termed the z-plane and the Z-plane respectively.

In this chapter we generally use the symbol Z to denote $f(z)$, but the reader should note that in textbooks and in examination papers the symbol w is often used instead of Z. x and y are respectively the horizontal and vertical co-ordinates of the point in the z-plane, and u and v are those in the Z-plane. Fig. 22 shows these two planes. P' and Q' in the Z-plane correspond respectively to P and Q in the z-plane. As P describes the curve PQ, P' describes the curve $P'Q'$. If, for example, $u = x^2$ and $v = y^2$ and P describes the

circle $x^2 + y^2 = a^2$, then P' moves to and fro along that portion of the straight line $u + v = a^2$ which is cut off by the axes $O'U$ and $O'V$.

If, as shown in the figure, Q is the point $z + \Delta z$, then, since $z = \overrightarrow{OP}$, $z + \Delta z = \overrightarrow{OQ}$, then $\Delta z = \overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{PO} = \overrightarrow{PQ}$. Similarly, $\Delta Z = \overrightarrow{P'Q'}$.

Let $Z = f(z)$ be a single-valued function of z, and let $z = z_1$ represent some point in the z-plane. Then, provided that $f(z_1)$ and $\underset{z \to z_1}{Lt.} f(z)$ exist, $f(z)$ is said to be continuous at the point $z = z_1$ if $\underset{z \to z_1}{Lt.} f(z) = f(z_1)$. Further, since $Z = u + iv$, each of the functions u and v will be continuous at $z = z_1$ if Z is continuous at that point, and conversely, if u and v are continuous at $z = z_1$, Z will also be continuous there. Again, Z is said to be continuous in a region D of the z-plane if it is continuous at every point in that region.

30. **Derivative of Function of Complex Variable.** Let Δz be the increment of z corresponding to increments Δx and Δy of x and y respectively.

Then, since $z = x + iy$, $z + \Delta z = (x + \Delta x) + i(y + \Delta y)$, and thus

$$\Delta z = \Delta x + i\Delta y \qquad . \quad . \quad . \quad \text{(IV.47)}$$

As the point z in the z-plane is not constrained to move on any particular curve, Δz can approach zero in an infinite number of ways. If, as Δz approaches zero *in any manner*, the ratio

$$\frac{f(z + \Delta z) - f(z)}{\Delta z}$$

approaches an unique value, then the function $Z = f(z)$, assumed single-valued and defined in a region D of the z-plane, is said to be differentiable at the point z in that region. The limit which the given ratio approaches is called the derivative of Z or $f(z)$ at the point z, and is denoted by $f'(z)$.

Thus, $$f'(z) = \underset{\Delta z \to 0}{Lt.} \frac{f(z + \Delta z) - f(z)}{\Delta z} \qquad . \quad . \quad \text{(IV.48)}$$

It is necessary to investigate the conditions under which the limit (IV.48) exists, i.e. under which $f(z)$ is differentiable at any point in the region D.

Since $f(z) = u + iv$, the existence of the limit (IV.48) implies the existence of the limit

$$\underset{\Delta z \to 0}{Lt.} \left(\frac{\Delta u}{\Delta z} + i \frac{\Delta v}{\Delta z} \right) \quad . \quad . \quad . \quad \text{(IV.49)}$$

where Δu and Δv are increments of u and v respectively corresponding to increments Δx, Δy of x, y.

As Δz can approach zero in any manner, it is justifiable to assume first that Δz is wholly real and then that Δz is wholly imaginary.

If Δz is wholly real, then from (IV.47) $\Delta y = 0$ and $\Delta z = \Delta x$, and (IV.49) becomes

$$\underset{\Delta x \to 0}{Lt.} \left(\frac{\Delta u}{\Delta x} + i \frac{\Delta v}{\Delta x} \right)$$

i.e.

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad . \quad . \quad . \quad \text{(IV.50)}$$

If Δz is wholly imaginary, then from (IV.47) $\Delta x = 0$ and $\Delta z = i\Delta y$, and (IV.49) becomes

$$\underset{\Delta y \to 0}{Lt.} \left(\frac{1}{i} \frac{\Delta u}{\Delta y} + i \cdot \frac{1}{i} \frac{\Delta v}{\Delta y} \right)$$

i.e.

$$\frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

i.e.

$$\frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \quad . \quad . \quad . \quad \text{(IV.51)}$$

The limits (IV.50) and (IV.51) must be identical, so that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad . \quad . \quad . \quad . \quad \text{(IV.52)}$$

and

$$\frac{\partial v}{\partial x} = - \frac{\partial u}{\partial y} \quad . \quad . \quad . \quad . \quad \text{(IV.53)}$$

These two relations are known as the *Cauchy-Riemann differential equations.*

It follows that a *necessary* condition for the existence of the derivative of the function $Z = f(z)$ is that the Cauchy-Riemann equations are satisfied.

Suppose now that the single-valued functions u and v together with their partial derivatives of the first order with respect to x and y are continuous at any point of the region D of the z-plane and that the conditions (IV.52) and (IV.53) are satisfied there.

Then, the limit in (IV.48) approaches the limit $\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$, or its equivalent $\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}$, as $\Delta z\to 0$ by any path whatever. There appears to be a single derivative $f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}$ associated with each point of the region D. To prove that this is so we assume that z is given the increment $\Delta z=\Delta x+i\Delta y$, where Δx and Δy are independent increments of x and y respectively. Using the extended form of Taylor's Theorem (Vol. I) and retaining only the first powers of Δx and Δy, we have

$$f(z+\Delta z)=u(x+\Delta x, y+\Delta y)+iv(x+\Delta x, y+\Delta y)$$

where $u(x, y)$ and $v(x, y)$ are the values of u and v corresponding to those of x and y respectively.

Then
$$\begin{aligned}f(z+\Delta z)&=u(x,y)+iv(x,y)+\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y\\&\quad+i\frac{\partial v}{\partial x}\Delta x+i\frac{\partial v}{\partial y}\Delta y\\&=f(z)+\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\Delta y\end{aligned}$$

Now, if (IV.50) and (IV.51) are each multiplied by i, and the results equated, we obtain

$$\begin{aligned}\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}&=-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\\&=i\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right),\end{aligned}$$

so that

$$\begin{aligned}f(z+\Delta z)-f(z)&=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\Delta x+i\Delta y)\\&=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta z\end{aligned}$$

Substituting this in (IV.48) and proceeding to the limit, we have

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y} \quad . \qquad \text{(IV.54)}$$

Thus, in order that $f(z)=u+iv$ may have a derivative at any point in a region D of the z-plane, it is *necessary* and *sufficient* that u and v and their partial derivatives of the first order with respect

to x and y are continuous at that point and that the Cauchy-Riemann differential equations are satisfied there. If a single-valued function $f(z)$ is differentiable at every point in a region D of the z-plane, the function is said to be *regular* (or *analytic* or *holomorphic*). If there are certain points in the region D at which $f(z)$ is not differentiable, such points are termed *singular points* or *singularities* of the function.

EXAMPLE 1

In each of the following cases determine where the Cauchy-Riemann equations are satisfied—

(1) $Z = 2 - 5z^2$ (2) $Z = \dfrac{1}{z}$ (3) $Z = |z|^2$

(1) $Z = 2 - 5(x + iy)^2 = 2 - 5x^2 + 5y^2 - i\,10xy$, so that $u = 2 - 5x^2 + 5y^2$ and $v = -10xy$, $\dfrac{\partial u}{\partial x} = -10x$, $\dfrac{\partial u}{\partial y} = 10y$, $\dfrac{\partial v}{\partial x} = -10y$, $\dfrac{\partial v}{\partial y} = -10x$.

Here $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$, and the Cauchy-Riemann equations are satisfied everywhere.

(2) $Z = \dfrac{1}{z} = \dfrac{1}{x + iy} = \dfrac{x - iy}{x^2 + y^2}$, so that $u = \dfrac{x}{x^2 + y^2}$ and $v = -\dfrac{y}{x^2 + y^2}$,

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2}, \frac{\partial v}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}, \frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

Here the Cauchy-Riemann equations are satisfied except at the point $z = 0$.

(3) $Z = |z|^2 =$ (absolute value of $z)^2 = x^2 + y^2$, so that $u = x^2 + y^2$ and $v = 0$, $\dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial u}{\partial y} = 2y$, $\dfrac{\partial v}{\partial x} = 0$, $\dfrac{\partial v}{\partial y} = 0$. Here the Cauchy-Riemann equations are satisfied only at the point $z = 0$.

EXAMPLE 2

Show that the functions e^z and $\cosh z$ are regular, and find their derivates.

If we write $e^z = e^{x+iy} = e^x(\cos y + i \sin y) = u + iv$, then $u = e^x \cos y$ and $v = e^x \sin y$. Here

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = e^x \sin y = -\frac{\partial u}{\partial y}$$

Since u, v, and the partial derivates are continuous and the Cauchy-Riemann equations are satisfied, the function e^z is regular for every value of z. Since $\cosh z = \cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$, then $u = \cosh x \cos y$, $v = \sinh x \sin y$, $\dfrac{\partial u}{\partial x} = \sinh x \cos y = \dfrac{\partial v}{\partial y}$, and $\dfrac{\partial v}{\partial x} = \cosh x \sin y = -\dfrac{\partial u}{\partial y}$. The

required conditions are satisfied, and cosh z is regular. The derivatives can be obtained from (IV.54). Thus, if $f(z) = \cosh z$, then

$$\begin{aligned} f'(z) &= \sinh x \cos y + i \cosh x \sin y \\ &= \sinh x \cosh iy + \cosh x \sinh iy \\ &= \sinh (x + iy) \end{aligned}$$

i.e. $$f'(z) = \sinh z$$

Again, if $f(z) = e^z$, then $f'(z) = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = e^{x+iy}$

i.e. $$f'(z) = e^z$$

The reader should verify that the ordinary standard forms for the derivatives of real functions hold good for the corresponding functions of a complex variable,

e.g. $\dfrac{d}{dz}(z^n) = nz^{n-1}$, $\dfrac{d}{dz}(\log_e z) = \dfrac{1}{z}$, $\dfrac{d}{dz}(\sin z) = \cos z$, and so on.

31. **Conjugate Functions.** If $f(z) = f(x + iy) = u + iv$ is a regular function, then u and v, which are real functions of the variables x and y, are termed *conjugate functions*. Let it be assumed that with $\phi(x, y)$ written for u and v in turn

$$\frac{\partial^2\phi(x, y)}{\partial x\, \partial y} = \frac{\partial^2\phi(x, y)}{\partial y\, \partial x} \quad . \quad . \quad . \quad \text{(IV.55)}$$

If (IV.52) be differentiated with respect to x and (IV.53) with respect to y, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\, \partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\, \partial x}$$

Adding, and using (IV.55),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad . \quad . \quad . \quad \text{(IV.56)}$$

Similarly, by differentiating (IV.52) with respect to y and (IV.53) with respect to x, and proceeding as above,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad . \quad . \quad . \quad \text{(IV.57)}$$

Thus, both the conjugate functions u and v satisfy Laplace's equation

$$\nabla^2\phi = \frac{\partial^2\phi(x, y)}{\partial x^2} + \frac{\partial^2\phi(x, y)}{\partial y^2} = 0 \quad . \quad . \quad \text{(IV.58)}$$

Solutions of the equation (IV.58) are called *harmonic* functions. This equation is of frequent occurrence in mathematical physics, e.g. in problems on heat flow, gravitational and electrical potential, and plane stream-line motion of a frictionless incompressible fluid.

EXAMPLE 1

Determine pairs of conjugate functions from (i) $w = iz^3$, and (ii) $w = \text{Log } z$, and verify that these conjugate functions satisfy Laplace's equation.

$$\text{(i) } w = iz^3 = i(x + iy)^3 = i(x^3 + i \,.\, 3x^2y - 3xy^2 - iy^3)$$
$$= (y^3 - 3x^2y) + i(x^3 - 3xy^2)$$

so that the conjugate functions are $u = y^3 - 3x^2y$ and $v = x^3 - 3xy^2$. It is easily verified in this case that both u and v satisfy Laplace's equation.

(ii) $w = \text{Log } z = \frac{1}{2}\log(x^2 + y^2) + i\left(2n\pi + \tan^{-1}\dfrac{y}{x}\right)$, so that the conjugate functions are $u = \frac{1}{2}\log(x^2 + y^2)$ and $v = 2n\pi + \tan^{-1}\dfrac{y}{x}$

Here $\dfrac{\partial u}{\partial x} = \dfrac{x}{x^2 + y^2}, \dfrac{\partial^2 u}{\partial x^2} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}, \dfrac{\partial u}{\partial y} = \dfrac{y}{x^2 + y^2}, \dfrac{\partial^2 u}{\partial y^2} = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$

and $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Again, $\dfrac{\partial v}{\partial x} = -\dfrac{y}{x^2 + y^2}, \dfrac{\partial^2 v}{\partial x^2} = \dfrac{2xy}{(x^2 + y^2)^2}, \dfrac{\partial v}{\partial y} = \dfrac{x}{x^2 + y^2}, \dfrac{\partial^2 v}{\partial y^2} = -\dfrac{2xy}{(x^2 + y^2)^2}$

and $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

The following example shows how the converse problem may be solved, i.e. to determine a regular function $w = u + iv$ when either u or v is given satisfying Laplace's equation.

EXAMPLE 2

Determine a regular function $w = f(z) = u + iv$ in each of the following cases—(i) when $v = 2xy$, and (ii) when $u = \sinh x \cos y$.

It is readily verified that the given functions v and u satisfy Laplace's equation.

In (i) $\qquad \dfrac{\partial v}{\partial y} = 2x$, and, since w is to be regular, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

Hence $\qquad \dfrac{\partial u}{\partial x} = 2x$, and, integrating with respect to x,

$$u = x^2 + \phi(y)$$

where $\phi(y)$ is an arbitrary function of y.

Now $\quad \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$, so that

$2y = -\phi'(y)$, whence $\phi(y) = -y^2 + c$, where c is an arbitrary constant

and $\quad u = x^2 - y^2 + c$

Thus, $\quad w = x^2 - y^2 + c + i \, . \, 2xy$

i.e. $\quad w = z^2 + c$

If we take $c = 0$, we obtain the simplest solution $w = z^2$.

In (ii) $\quad \dfrac{\partial u}{\partial x} = \cosh x \cos y = \dfrac{\partial v}{\partial y}$, since w is to be regular.

Integrating with respect to y,

$$v = \cosh x \sin y + \phi(x),$$

where $\phi(x)$ is an arbitrary function of x.

Now
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

so that $\quad \sinh x \sin y + \phi'(x) = \sinh x \sin y$, whence $\phi'(x) = 0$ and $\phi(x) = c$, where c is an arbitrary constant.

We obtain the simplest solution by taking $c = 0$.

Thus
$$\begin{aligned} w &= \sinh x \cos y + i \cosh x \sin y \\ &= \sinh x \cosh iy + \cosh x \sinh iy \\ &= \sinh (x + iy) \end{aligned}$$

i.e.
$$w = \sinh z$$

EXAMPLE 3

A regular function of z is such that two families of curves α and β in the z-plane correspond to two families of straight lines in the Z-plane parallel respectively to the u- and v-axes. Deduce from the Cauchy-Riemann equations that the families α and β intersect at right angles at all their points of intersection.

Here u and v have constant values so that $du = 0$ and $dv = 0$. From (V.5), Vol. I, page 152,

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

i.e.
$$\left(\frac{dy}{dx}\right)_\alpha = -\frac{\partial u}{\partial x} \bigg/ \frac{\partial u}{\partial y}$$

and similarly,
$$\left(\frac{dy}{dx}\right)_\beta = -\frac{\partial v}{\partial x} \bigg/ \frac{\partial v}{\partial y}$$

$\left(\dfrac{dy}{dx}\right)_\alpha$ and $\left(\dfrac{dy}{dx}\right)_\beta$ are the respective gradients of the α and β curves. From the Cauchy-Riemann equations,

$$-\frac{\partial u}{\partial x} \bigg/ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} \bigg/ \frac{\partial v}{\partial x}$$

whence
$$\left(\frac{dy}{dx}\right)_\alpha = -\frac{1}{\left(\frac{dy}{dx}\right)_\beta}, \text{ or} \left(\frac{dy}{dx}\right)_\alpha \left(\frac{dy}{dx}\right)_\beta = -1$$

and this is the condition that the families α and β intersect at right angles.

This result follows readily from the conformal property of a regular function established in Art. 29.

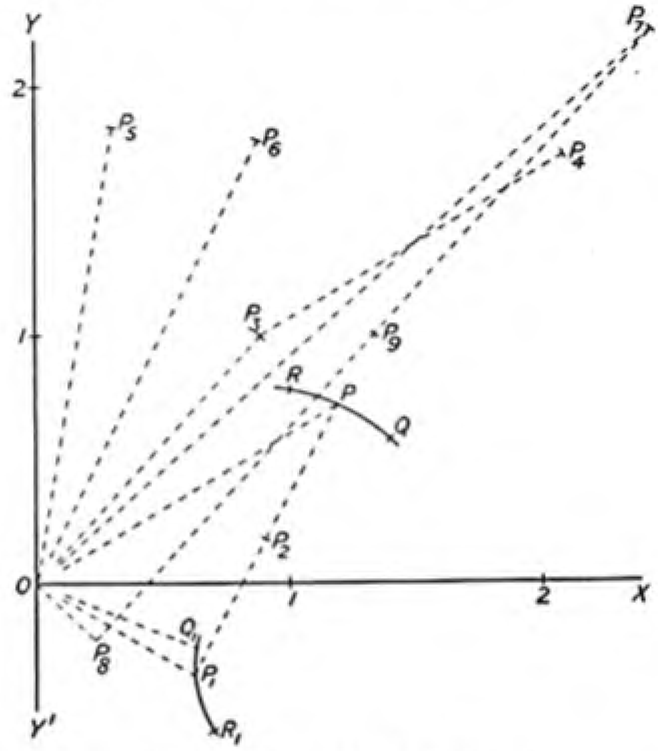


FIG. 23. COMPLEX FUNCTIONS

32. **Graphical Constructions for Simple Complex Functions.** Fig. 23 shows how to represent graphically a few of the simplest functions of $z = x + iy$, where x and y are given. x and y are plotted with the same scale unit. The point P represents $z_1 = 1{\cdot}2 + i\,0{\cdot}75$.

The angles XOP_1 and XOP are equal and $\overline{OP_1} = \frac{1}{\overline{OP}}$; then

$$P_1 \text{ represents } \frac{1}{z_1}$$

P_2 is the mid-point of PP_1, and it is easily seen that

$$P_2 \text{ represents } \tfrac{1}{2}\left(z_1 + \frac{1}{z_1}\right)$$

P_3 represents $z_2 = 0{\cdot}875 + i1$, and P_3P_4 is drawn equal and parallel to OP; then

$$P_4 \text{ represents } z_1 + z_2$$

The angle XOP_5 is equal to the sum of the angles XOP and XOP_3, and $\overline{OP_5} = \overline{OP} \times \overline{OP_3}$; then

$$P_5 \text{ represents } z_1 z_2$$

$\overline{OP_6} = \overline{OP}^2$, and the angle $XOP_6 = 2 \times$ angle XOP; then

$$P_6 \text{ represents } z_1^2$$

$\overline{OP_7} = e^{1\cdot 2}$, and angle $XOP_7 = 0{\cdot}75$ radian; then, since a point whose polar co-ordinates are (r, θ) is represented by $re^{i\theta}$,

$$P_7 \text{ represents } e^{1\cdot 2}e^{i0\cdot 75} = e^{z_1}$$

Angle $XOP_8 =$ angle XOP_7, and $\overline{OP_8} = \dfrac{1}{\overline{OP_7}}$; then

$$P_8 \text{ represents } e^{-z_1}$$

P_9 is the mid-point of P_7P_8, and, since P_9 represents half the sum of the complex numbers associated with P_7 and P_8, we see that

$$P_9 \text{ represents } \cosh z_1$$

As z varies, P moves along a curve in the z-plane, of which QPR is a portion; the points P_1, P_2, etc. will then describe curves in the Z-plane, which, for convenience, we assume to be superimposed on the z-plane. $Q_1P_1R_1$ is the curve traced out by P_1 as P traces out QPR.

33. **Conformal Transformation.** If $z = x + iy$ and $Z = f(z) = u + iv$, then z and Z can be represented on separate complex planes, the z-plane and the Z-plane, there being a correspondence between points of the one plane and points of the other, although this correspondence is not necessarily one-to-one. Let D be a region of the z-plane in which $Z = f(z)$ is regular. Then all the values of Z corresponding to points in D are contained in a region D' of the Z-plane, D being said to *map* into D'. If z moves on any curve in D, Z will move on a corresponding curve in D', the latter curve being the *map* of the former. In the case in which there is a one-to-one correspondence between the points in D and D' the mapping is said to be *one-to-one* or *bi-uniform.*

EXAMPLE

If $Z = z + \dfrac{a^2}{z}$, prove that, when z describes the circle $x^2 + y^2 = a^2$, Z describes a straight line, and find its length. Prove that, if z describes the circle $x^2 + y^2 = b^2$, where $b > a$, Z describes an ellipse whose foci are the extremities of the above line. (U.L.)

Here $\quad Z = x + iy + \dfrac{a^2}{x + iy} = x + iy + \dfrac{a^2(x - iy)}{x^2 + y^2}$, and since $x^2 + y^2 = a^2$,

$$Z = x + iy + x - iy = 2x + i \, . \, 0$$

Thus, $u = 2x$ and $v = 0$, so that Z moves on the u-axis between $u = -2a$ and $u = +2a$, the extreme values of x as z describes its circle being $-a$ and $+a$. The length of the line described by Z is, therefore, $4a$.

If $x^2 + y^2 = b^2$, then, from above,

$$Z = x + iy + \frac{a^2}{b^2}(x - iy) = \frac{b^2 + a^2}{b^2} x + i \frac{b^2 - a^2}{b^2} y$$

In this case $\quad u = \dfrac{b^2 + a^2}{b^2} x$ and $v = \dfrac{b^2 - a^2}{b^2} y$

whence $\quad x^2 = \dfrac{b^4 u^2}{(b^2 + a^2)^2}$ and $y^2 = \dfrac{b^4 v^2}{(b^2 - a^2)^2}$

Since $x^2 + y^2 = b^2$,

$$\frac{b^4 u^2}{(b^2 + a^2)^2} + \frac{b^4 v^2}{(b^2 - a^2)^2} = b^2$$

i.e. $$\frac{u^2}{\left(\dfrac{b^2 + a^2}{b}\right)^2} + \frac{v^2}{\left(\dfrac{b^2 - a^2}{b}\right)^2} = 1$$

Thus, Z describes an ellipse with semi-axes $\dfrac{b^2 + a^2}{b}$ and $\dfrac{b^2 - a^2}{b}$ along the u- and v-axes respectively.

The eccentricity e of this ellipse is given by

$$e^2 = 1 - \left(\frac{\dfrac{b^2 - a^2}{b}}{\dfrac{b^2 + a^2}{b}}\right)^2 = \frac{4a^2b^2}{(b^2 + a^2)^2}$$

whence $\quad e = \dfrac{2ab}{b^2 + a^2}$

The distances of the foci from the centre of the ellipse are $\pm \dfrac{b^2 + a^2}{b} \times e$, i.e. $\pm 2a$, so that these foci are the extremities of the path described by Z in the former case.

If any two curves drawn through a point P in D map into two curves through the corresponding point P' in D' so that the angle of intersection at P of the former pair of curves is equal to that at P' of the latter pair, then the transformation effected by the functional relationship $Z = f(z)$ is said to be *conformal*.

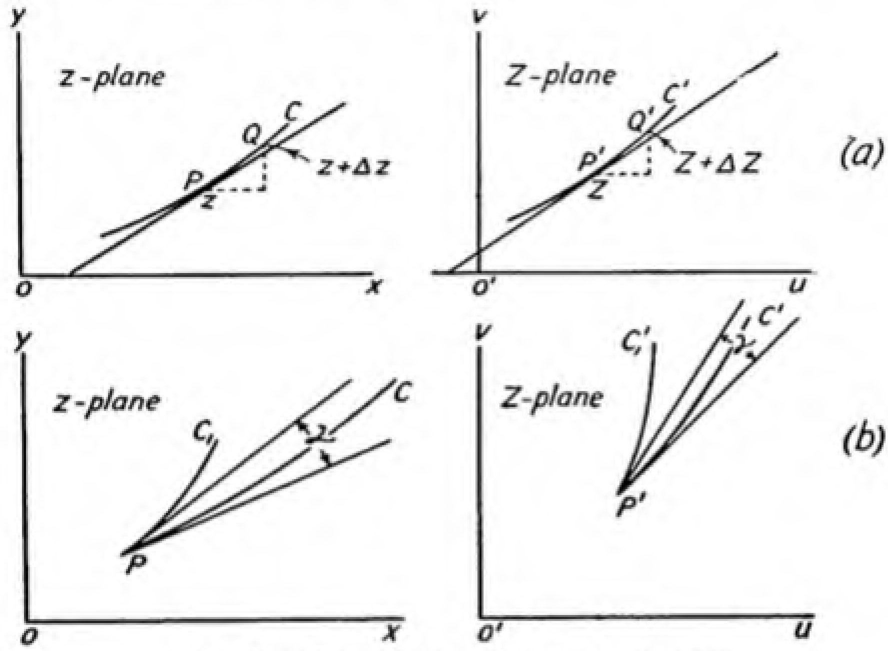


FIG. 24. CONFORMAL TRANSFORMATION

It will be shown that the transformation effected by a regular function $Z = f(z)$ is conformal at every point of the z-plane where $f'(z) \neq 0$.

Let P be a point z in the region D of the z-plane and P' the corresponding point Z in the region D' of the Z-plane, Fig. 24 (a). Suppose that z moves on a curve C, and Z moves on the corresponding curve C'. Let Q be a point $z + \Delta z$ near to P on the curve C, and Q' the corresponding point $Z + \Delta Z$ on the curve C'. Then $\overline{PQ} = \Delta z$ and $\overline{P'Q'} = \Delta Z$.

If r, r' and θ, θ' are the moduli and arguments respectively of Δz, ΔZ, α the inclination to the x-axis of the tangent at P to the curve C, and α' the inclination to the u-axis of the tangent at P' to the curve C', then

$$\Delta z = re^{i\theta} \text{ and } \Delta Z = r'e^{i\theta'}$$

Hence
$$\frac{\Delta Z}{\Delta z} = \frac{r'}{r} e^{i(\theta' - \theta)} \quad . \quad . \quad . \quad . \quad . \quad \text{(IV.59)}$$

As $\Delta z \to 0$, $\theta \to \alpha$, $\theta' \to \alpha'$, and $\dfrac{\Delta Z}{\Delta z} \to \dfrac{dZ}{dz}$ or $f'(z)$, so that

$$f'(z) = \left(Lt.\frac{r'}{r}\right) e^{i(\alpha' - \alpha)} \quad . \quad . \quad . \quad (\text{IV.60})$$

It is assumed here that $f'(z) \neq 0$. If ρ is the modulus and ϕ the argument of $f'(z)$, then

$$f'(z) = \rho e^{i\phi} \quad . \quad . \quad . \quad . \quad . \quad (\text{IV.61})$$

From these two relations

$$\rho = Lt.\frac{r'}{r} \quad . \quad . \quad . \quad . \quad . \quad (\text{IV.62})$$

and

$$\phi = \alpha' - \alpha \text{ or } \alpha' = \alpha + \phi \quad . \quad . \quad (\text{IV.63})$$

(IV.63) shows that the tangent at P to the curve C is rotated through angle ϕ under the given transformation. Since ϕ depends only on the function $f(z)$ and the point z, it is the same for all curves through P. Hence, if C_1 is a second curve passing through P in the z-plane and C_1' the corresponding curve passing through P' in the Z-plane, then ϕ is the angle between the direction of the tangent to C_1 at P and that to C_1' at P'. It follows that the angle of intersection of the curves C and C_1 at P is the same, both in size and sense, as the angle of intersection of the curves C' and C_1' at P'. This angle is γ in Fig. 24 (b).

Thus, *the transformation effected by the regular function* $Z = f(z)$ *is conformal at every point of the z-plane where* $f'(z) \neq 0$. The relation (IV.62) shows that in the transformation elements of arc passing through P in any direction are changed in length in the ratio $\rho : 1$, where $\rho = |f'(z)|$. This change in length will vary from point to point.

Consider, for example, the transformation $Z = z^2$. At the point $z = 1$ the linear dimensions are doubled and the angle of rotation $\phi = 0°$; at the point $z = 3 + j4$, the linear dimensions are increased tenfold and the angle of rotation $\phi = \tan^{-1}\frac{4}{3} = 53° \, 8'$.

34. **Geometrical Inversion.** If a straight line is drawn through the centre O of a circle of radius k and two points P and P_1 are taken on the line, both on the same side of O, such that $\overline{OP} \cdot \overline{OP_1} = k^2$, each of the points P and P_1 is called the inverse of the other. O is the *centre of inversion* and k the *radius of inversion*. As P describes

any curve L in the plane of the circle, P_1 describes a curve L_1. Each of these curves is the *inverse* of the other with respect to the circle (Fig. 25).

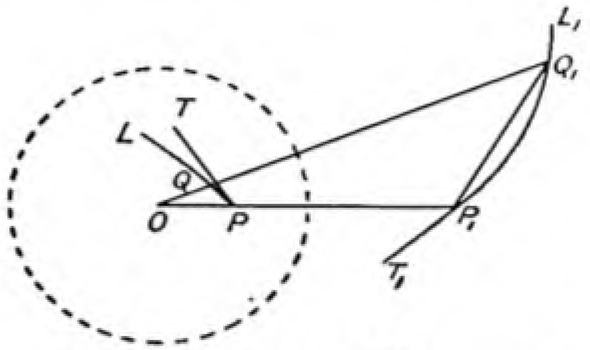


FIG. 25. GEOMETRICAL INVERSION

If P and Q are two points on L and P_1 and Q_1 their respective inverse points, then, since $OP \,.\, OP_1 = OQ \,.\, OQ_1$ or $\dfrac{OP}{OQ_1} = \dfrac{OQ}{OP_1}$, the triangles OPQ and OQ_1P_1 are similar and their angles at P and Q_1 are equal. If the angle POQ is small and is made to approach the limit zero, the straight line PQ will approach its limiting position PT, the tangent to L at P, and similarly the straight line P_1Q_1 will approach its limiting position P_1T_1, the tangent to L_1 at P_1. Thus, in the limit as angle POQ approaches zero, angle $OPQ \rightarrow$ angle OPT and angle $OQ_1P_1 \rightarrow$ angle OP_1T_1, so that angle $OPT =$ angle OP_1T_1, i.e. the tangents PT and P_1T_1 are equally inclined to the straight line OPP_1 on opposite sides of that line. It follows that, if two curves intersect, their angle of intersection is the same as that of their inverses with respect to any given circle.

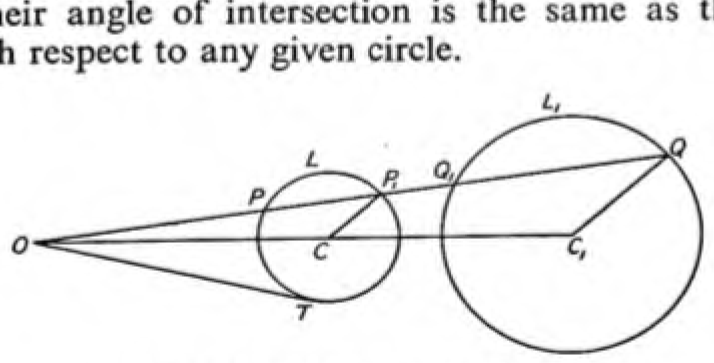


FIG. 26. INVERSE OF A CIRCLE

Let L be a circle, centre C (Fig. 26) and let L_1 be its inverse, O being the centre of inversion and k the radius of inversion. Then, provided that O is not on L, L_1 is also a circle. To prove this, let P be a point on L and Q its inverse on L_1, and let the straight line

OPQ cut L again in P_1. Through Q draw QC_1 parallel to P_1C to meet OC produced at C_1, and also draw OT the tangent to L from O. Then, $OP \,.\, OP_1 = OT^2$, a constant, and $OP \,.\, OQ = k^2$, so that $\dfrac{OQ}{OP_1} = \dfrac{k^2}{OT^2} =$ a constant ratio. Now, the triangles OC_1Q and OCP_1 are similar, and $\dfrac{OC_1}{OC} = \dfrac{C_1Q}{CP_1} = \dfrac{OQ}{OP_1} =$ constant (proved). Since OC is a fixed straight line, C_1 is a fixed point, and since CP_1 is of fixed length, C_1Q is of fixed length. Hence, the locus of Q, i.e. L_1, is a circle, centre C_1. The point Q_1 on L_1 is the inverse of P_1.

If O is on the circle L, the inverse of O is at infinity and L_1 becomes a straight line perpendicular to OC.

To sum up, if O is not on a given circle, the inverse of that circle is another circle; if O is on the given circle, the inverse is a straight line, and, conversely the inverse of a straight line is a circle passing through O.

35. **Examples on Transformations.** Certain elementary transformations are discussed below. The reader should draw figures where necessary in order to illustrate the argument.

(1) $Z = z^n$ (*n being a Positive Integer*). It is convenient here to express Z and z in polar form, thus

$$Z = Re^{i\phi} \text{ and } z = re^{i\theta}$$

Then, since $Z = z^n$, $Re^{i\phi} = r^n e^{i.n\theta}$, so that

$$R = r^n \quad . \quad . \quad . \quad . \quad \text{(IV.64)}$$

and

$$\phi = n\theta \quad . \quad . \quad . \quad . \quad \text{(IV.65)}$$

From the first of these relations it is seen that any circle in the z-plane having the origin as centre maps into a circle in the Z-plane with radius equal to the nth power of the radius of the original circle. Further, the latter relation shows that any straight line drawn through the z-plane origin maps into a straight line through the Z-plane origin whose inclination to the u-axis is n times the inclination of the original line to the x-axis. It follows that the transformation effected by $Z = z^n$ is not conformal at the origin, for if α is the inclination to each other of two radial lines in the z-plane, the inclination to each other of the two corresponding lines in the Z-plane is not α but $n\alpha$. In virtue of (IV.65) a sector of a circle drawn in the z-plane with central angle $\dfrac{2\pi}{n}$ at the origin maps into a complete circle in the Z-plane, and, if no restriction is placed

on the length of the radius, the sector is transformed into the whole Z-plane. Obviously, only one nth part of the z-plane maps into one complete Z-plane. The complete mapping of the z-plane on the Z-plane can be represented by the following method. Imagine n plane surfaces to be placed one on top of the other and a slit made in these surfaces along the u-axis from the origin to infinity. For convenience, let the surfaces be numbered 1 to n in order. The sector $\theta = 0$ to $\theta = \frac{2\pi}{n}$ on the z-plane maps into the whole surface 1, and ϕ then reaches the lower edge of the slit in that surface. Let ϕ now cross over to the upper edge of the slit in surface 2. The sector $\theta = \frac{2\pi}{n}$ to $\theta = \frac{4\pi}{n}$ on the z-plane then maps into the whole surface 2, and ϕ reaches the lower edge of the slit in this second surface. Again, let ϕ cross over to the upper edge of the slit in surface 3, and so on. Finally, the sector $\theta = \frac{2(n-1)\pi}{n}$ to $\theta = 2\pi$ maps into the whole surface n. These surfaces are known as *Riemann surfaces*.

(2) Consider now the particular case $Z = z^2$, which for complete mapping requires two Riemann surfaces.

Putting $n = 2$ in (IV.64) and (IV.65), we obtain

$$R = r^2 \text{ and } \phi = 2\theta \quad . \quad . \quad . \quad \text{(IV.66)}$$

It is easy to deduce from these two relations that under this transformation, if a circle of any radius c be drawn about the origin in the z-plane, the semicircular area above the x-axis maps into the entire circular area of radius c^2 about the origin in the Z-plane, and, further that the entire upper half of the z-plane maps into the entire Z-plane, the lower half of the z-plane mapping into the entire Z-plane on the second Riemann surface.

Using rectangular co-ordinates, we have $Z = (x + iy)^2 = x^2 - y^2 + i2xy$, so that

$$u = x^2 - y^2 \text{ and } v = 2xy \quad . \quad . \quad \text{(IV.67)}$$

Let x have a constant value c; then $y = \frac{v}{2c}$ and $y^2 = c^2 - u$ and the elimination of y gives $v^2 = 4c^2(c^2 - u)$.

Similarly, if y has a constant value k, it is easy to deduce that $v^2 = 4k^2(k^2 + u)$.

Thus, the straight lines $x = c$ and $y = k$ map into orthogonal parabolas, and, if the constants c and k are varied, a net of orthogonal

lines parallel to the axes of reference in the z-plane are transformed into a net of orthogonal parabolas in the Z-plane. Fig. 27 shows the parabolas in the Z-plane corresponding to the straight lines $x = \frac{1}{2}$, 1, $\frac{3}{2}$, $y = \frac{1}{2}$, 1, $\frac{3}{2}$ in the z-plane.

Again, if u has a constant value d, the resulting curve in the z-plane is the rectangular hyperbola $x^2 - y^2 = d$, which has the x-axis as its

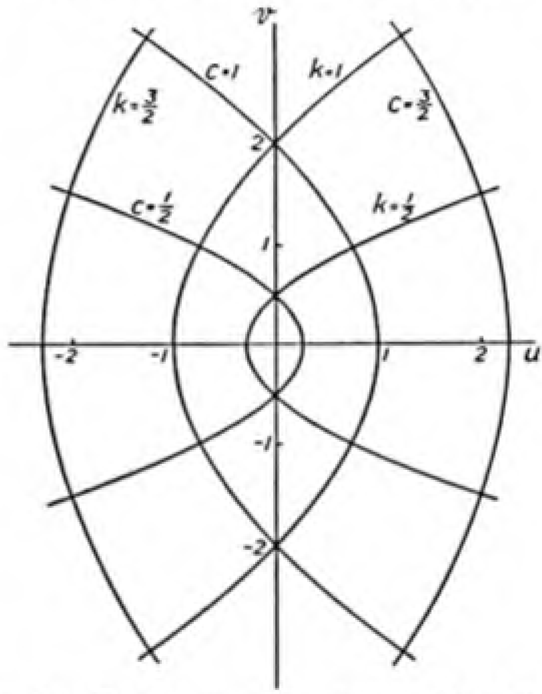


FIG. 27. ORTHOGONAL SYSTEMS OF PARABOLAS

transverse axis, and if v has a constant value l, the resulting curve in the z-plane is the rectangular hyperbola $xy = \frac{1}{2}l$, which has as its transverse axis the bisector of the angle between the positive directions of the x- and y-axes. The former rectangular hyperbola maps into a straight line in the Z-plane parallel to the v-axis, and the latter maps into a straight line parallel to the u-axis. If the constants d and l are varied, two families of rectangular hyperbolas in the z-plane, every member of the one family intersecting every member of the other family orthogonally, map into a net of orthogonal straight lines in the Z-plane parallel to the axes of reference. Fig. 28 shows the hyperbolas in the z-plane corresponding to the straight lines $u = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$, $v = -2, -1, 1, 2$ in the Z-plane.

It is easy to deduce that under this transformation the region between two rectangular hyperbolas $xy = \frac{1}{2}l_1$ and $xy = \frac{1}{2}l_2$ maps into the infinite strip between the lines $v = l_1$ and $v = l_2$.

Suppose now that z moves on the circumference of the circle

$(x-a)^2+y^2=b^2$, i.e. $z=a+b\cos\theta+ib\sin\theta$, where θ is the inclination to the positive direction of the x-axis of the radius of the circle in any position.

Then, if $Z=z^2$,

$$Z=(a+b\cos\theta)^2+2ib\sin\theta(a+b\cos\theta)-b^2\sin^2\theta$$
$$=a^2+2ab\cos\theta+b^2\cos^2\theta$$
$$+2ib\sin\theta(a+b\cos\theta)-b^2(1-\cos^2\theta)$$

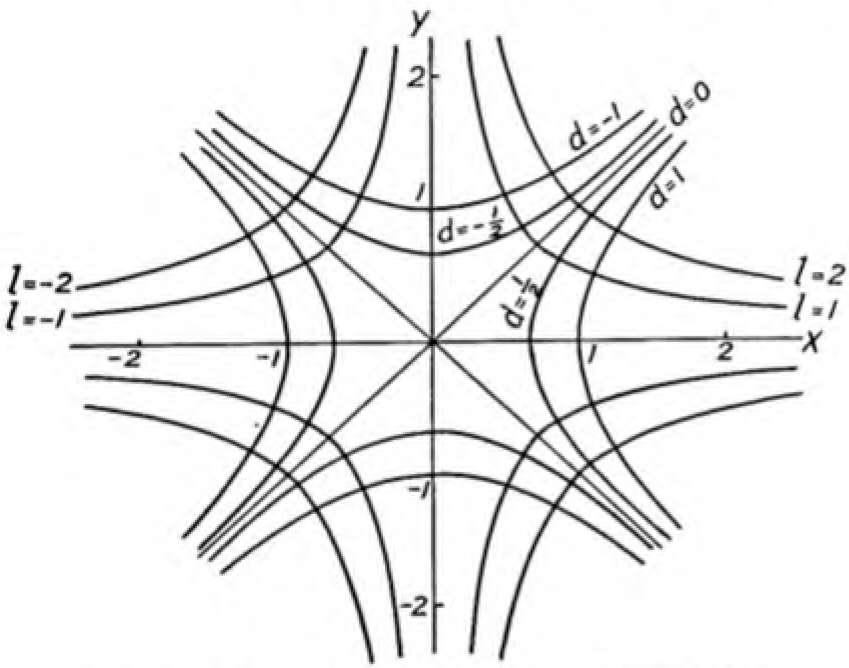


FIG. 28. ORTHOGONAL SYSTEMS OF HYPERBOLAS

$$\therefore Z-a^2+b^2=2ab\cos\theta+2b^2\cos^2\theta+2ib\sin\theta(a+b\cos\theta)$$
$$=2b\cos\theta(a+b\cos\theta)+2ib\sin\theta(a+b\cos\theta)$$
$$=2b(a+b\cos\theta)(\cos\theta+i\sin\theta)$$
$$=2b(a+b\cos\theta)e^{i\theta}$$

Let $Z-a^2+b^2=R'e^{i\phi'}$; then $\phi'=\theta$ and $R'=2b(a+b\cos\theta)$,

whence $$R'=2b(a+b\cos\phi')$$

which is the polar equation of a limaçon, the pole being at the point $Z=a^2-b^2$.

Thus, the circle $(x-a)^2+y^2=b^2$ in the z-plane maps into a limaçon in the Z-plane.

If $a = b$, the circle touches the y-axis at the origin, and maps into the cardioid $R' = 2a^2(1 + \cos \phi')$.

(3) $Z = \frac{1}{z}$. If Z and z be expressed in polar form, i.e. $Z = Re^{i\phi}$ and $z = re^{i\theta}$, then $Re^{i\phi} = \frac{1}{r} e^{-i\theta}$, so that

$$R = \frac{1}{r} \text{ and } \phi = -\theta \quad . \quad . \quad . \quad \text{(IV.68)}$$

Hence, a circle about the origin in the z-plane maps into a circle about the origin in the Z-plane described in the opposite sense and with radius equal to the reciprocal of the radius of the z-plane circle.

Let P and Q be two points on a straight line through the origin O in the z-plane such that $\overline{OP}.\overline{OQ} = 1$, i.e. Q is the inverse of P with respect to a circle of unit radius having centre at the origin, and let P_1 and Q_1 be the images of P and Q respectively in the real axis. Then, under the given transformation, P maps into Q_1 and Q maps into P_1. If P is outside the unit circle about O in the z-plane, Q_1 is inside the unit circle, and vice versa. It follows that points in the z-plane outside unit circle map into points in the Z-plane inside unit circle, and conversely.

When $z = 0$, Z is infinite, and in order that the origin O may be included in the one-to-one correspondence which exists between points inside and outside the unit circle, it is assumed that there is one point at infinity—called *the point at infinity*—which corresponds to the origin.

Since $z = x + iy$ and $Z = u + iv$, then $Z = \frac{1}{z}$ can be written as

$$u + iv = \frac{1}{x + iy}, \text{ or } x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

whence

$$x = \frac{u}{u^2 + v^2} \text{ and } y = -\frac{v}{u^2 + v^2} \quad . \quad . \quad \text{(IV.69)}$$

Now, the general equation of any circle in the z-plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad . \quad . \quad \text{(IV.70)}$$

which on substituting from (IV.69) and simplifying, becomes

$$c(u^2 + v^2) + 2gu - 2fv + 1 = 0 \quad . \quad \text{(IV.71)}$$

which is the equation of a circle.

If $c = 0$, the circle (IV.70) passes through the origin, and the equation (IV.71) becomes

$$2gu - 2fv + 1 = 0$$

which is that of a straight line.

Thus, any circle which passes through the origin in the z-plane maps into a straight line in the Z-plane, and any other circle in the z-plane maps into a circle in the Z-plane.

The relations (IV.69) give $\frac{y}{x} = -\frac{v}{u}$, so that any straight line through the origin in the z-plane maps into a straight line through the origin in the Z-plane.

(4) $Z = z + c$, *where c is a Complex Constant.* Let P be any point z and R be the point c in the z-plane. Then, if the parallelogram $OPQR$ is completed, O being the origin, the point Q obviously represents $z + c$. If now the x- and y-axes are assumed to be the u- and v-axes respectively of the Z-plane, Q is the point in that plane which under the given transformation corresponds to the point P in the z-plane, Accordingly, this transformation maps a figure in the z-plane into a figure in the Z-plane of the same shape and size, the latter figure being the former translated through the vector c.

(5) $Z = cz$, *where c is a Complex Constant.* Let $Z = Re^{i\phi}$, $z = re^{i\theta}$, and $c = r_1e^{i\theta_1}$, where r_1 and θ_1 are fixed. Then $Re^{i\phi} = rr_1e^{i(\theta + \theta_1)}$, so that $R = rr_1$ and $\phi = \theta + \theta_1$. Thus, by this transformation the radius vector of any point z in the z-plane is altered in length in the ratio $r_1 : 1$, and is rotated through angle θ_1. Accordingly, any figure in the z-plane is thereby transformed into a geometrically similar figure in the Z-plane.

For example, if $c = 1 + i1 = \sqrt{2}e^{i\frac{\pi}{4}}$, a rectangle $OABC$ whose sides $OA = a$ and $OC = b$ lie along the x- and y-axes respectively of the z-plane maps into a rectangle $O_1A_1B_1C_1$ in which $O_1A_1 = \sqrt{2}a$ and $O_1C_1 = \sqrt{2}b$ lie along the bisectors of the angles between the u- and v-axes of the Z-plane.

Also, under the transformation $Z = (0 + i1)z$, the infinite strip of the z-plane contained between the lines $x = 0$ and $x = a$ maps into the infinite strip of the Z-plane contained between the lines $v = 0$ and $v = a$, since $0 + i1 = e^{i\frac{\pi}{2}}$

(6) $Z = c_1z_1 + c_2$, *where c_1 and c_2 are Complex Constants.* This transformation is a combination of (4) and (5), and results in an expansion or a contraction of the radius vector of any point z in

the z-plane in the ratio $r_1 : 1$, where $r_1 = | c_1 |$, a rotation of the radius vector through angle $\theta_1 = \arg. c_1$, and, finally, a translation through the vector c_2.

For example, the function $Z = (0{\cdot}4 + i0{\cdot}3)z + 1 - i2$ transforms the square $OABC$ in the z-plane having the origin O and the point $2 + i2$ as diagonally opposite corners into the square $oabc$ in the Z-plane shown in Fig. 29.

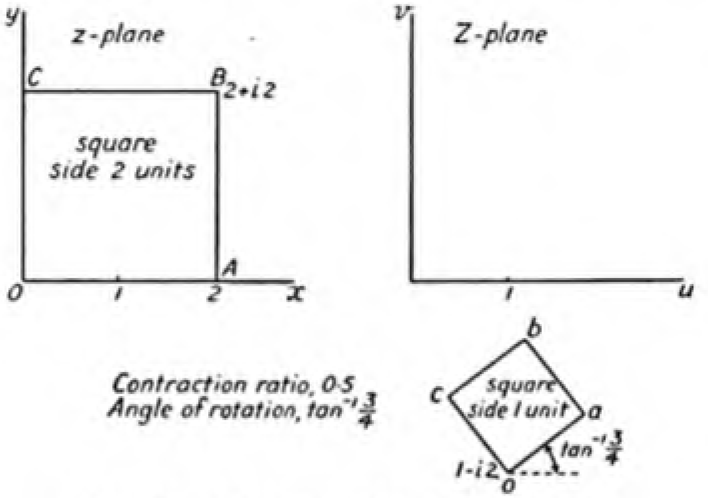


FIG. 29. TRANSFORMATION $Z = c_1z + c_2$

In cases (4) and (5) above graphical constructions for finding the point in the Z-plane corresponding to any point in the z-plane are simply the constructions for finding the sum and product of two complex numbers. Case (6) involves a combination of these two constructions.

(7) $Z = \dfrac{c_1z + c_2}{c_3z + c_4}$, *where* c_1, c_2, c_3, c_4 *are Complex Constants and* $c_1c_4 - c_2c_3 \neq 0$. This transformation is known as the *bilinear* or *Möbius transformation*. Note first the necessity for the restriction $c_1c_4 - c_2c_3 \neq 0$, for, if $c_1c_4 - c_2c_3 = 0$, $\dfrac{c_1z + c_2}{c_3z + c_4}$ either has a constant value or is meaningless. Under this transformation every point in the z-plane, except the point $z = -\dfrac{c_4}{c_3}$, has a unique corresponding point in the Z-plane.

By division the transformation can be expressed as

$$Z = \frac{c_1}{c_3} + \frac{c_2c_3 - c_1c_4}{c_3^{\,2}} \cdot \frac{1}{z + \dfrac{c_4}{c_3}}$$

i.e. $$Z - w_1 = \frac{w_2}{z - w_3} \qquad \text{(IV.72)}$$

where $$w_1 = \frac{c_1}{c_3},\; w_2 = \frac{c_2c_3 - c_1c_4}{c_3^{\,2}},\text{ and } w_3 = -\frac{c_4}{c_3}$$

Let $w_2 = ae^{i\phi}$ and $z - w_3 = re^{i\theta}$; then

$$Z - w_1 = \frac{a}{r}\, e^{i(\phi - \theta)} \qquad \text{(IV.73)}$$

Let P (Fig. 30) be any point z and P_0 the point w_3 in the z-plane; then $\overrightarrow{P_0P} = z - w_3$, i.e. the length $P_0P = r$ and the inclination of

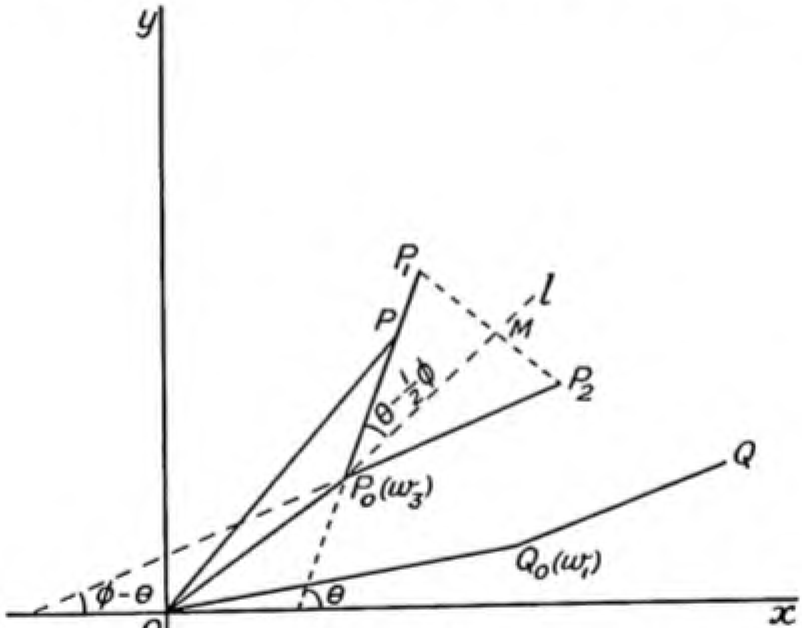


FIG. 30. BILINEAR TRANSFORMATION

P_0P to the real axis is θ. In P_0P (produced if necessary) find the point P_1 such that $\overline{P_0P}\,.\,\overline{P_0P_1} = a$; in other words, find the inverse of P with respect to the circle having centre P_0 and radius $= \sqrt{a}$. Then length $P_0P_1 = \frac{a}{r}$. Through P_0 draw a straight line l inclined at angle $\theta - \frac{1}{2}\phi$ to P_0P_1, and let P_1M perpendicular to the line l be produced its own length to P_2. P_2 is thus the reflection of P_1 in the line l. The inclination of the straight line P_0P_2 to the real axis is seen to be $\theta - (2\theta - \phi) = \phi - \theta$, and since length $P_0P_2 =$ length

$P_0P_1 = \frac{a}{r}$, $\overrightarrow{P_0P_2}$ must be equal and parallel to the vector $Z - w_1$ in the Z-plane. Hence, if Q_0 is the point w_1 in the Z-plane and $\overrightarrow{Q_0Q}$ is drawn equal and parallel to $\overrightarrow{P_0P_2}$, then $\overrightarrow{Q_0Q} = Z - w_1$, and Q is the point in the Z-plane which by this transformation corresponds to the point P in the z-plane. For convenience the x- and y-axes in Fig. 30 are now assumed to be the u- and v-axes respectively of the Z-plane. It follows that the bilinear transformation may be regarded as being effected by an inversion, a reflection, and a translation. Of these the last two do not alter the shape of a figure and the first [see transformation (3)] maps circles into circles, or, as limiting cases, straight lines. The given transformation, therefore, maps circles into circles with straight lines as limiting cases.

Suppose that Z_1, Z_2, Z_3, Z_4 are four points in the Z-plane corresponding respectively to four points z_1, z_2, z_3, z_4 in the z-plane. Using (IV.72), we obtain

$$Z_1 - Z_2 = w_2\left(\frac{1}{z_1 - w_3} - \frac{1}{z_2 - w_3}\right)$$

i.e.
$$Z_1 - Z_2 = -\frac{w_2}{(z_1 - w_3)(z_2 - w_3)}(z_1 - z_2)$$

with similar expressions for $Z_1 - Z_4$, $Z_3 - Z_2$, and $Z_3 - Z_4$

It is easy to deduce that

$$\frac{(Z_1 - Z_4)(Z_3 - Z_2)}{(Z_1 - Z_2)(Z_3 - Z_4)} = \frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)} \quad . \quad \text{(IV.74)}$$

The expression on the right-hand side is known as the *cross-ratio* of the four points z_1, z_2, z_3, z_4, and the equation shows that *a bilinear transformation leaves the cross-ratio unaltered.*

Dropping the suffix 4 in (IV.74), we have

$$\frac{(Z_1 - Z)(Z_3 - Z_2)}{(Z_1 - Z_2)(Z_3 - Z)} = \frac{(z_1 - z)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z)} \quad . \quad \text{(IV.75)}$$

This equation, when solved for Z, gives the bilinear transformation which maps the points z_1, z_2, z_3 into the points Z_1, Z_2, Z_3. Also since a circle is uniquely determined by three points on its circumference, the equation gives the transformation required to map a given circle in the z-plane into a given circle (or straight line) in the Z-plane.

EXAMPLE 1

Determine the bilinear transformation required in each of the following cases—

(i) To map the points $z = 1, -1, i$ into the points $Z = i, -i, -3$ respectively;
(ii) To map the points $z = 2, -1, -1 + i3$ into the points $Z = 0, \infty, -1 + i$ respectively.

(i) In terms of the ratios α, β, γ of c_1, c_2, c_3 respectively to c_4, the bilinear transformation may be written as $Z = \dfrac{\alpha z + \beta}{\gamma z + 1}$, and the substitution of the given values of z and Z in this relation gives

$$i = \frac{\alpha + \beta}{\gamma + 1}, \quad \text{i.e. } \alpha + \beta - i\gamma = i \quad . \quad . \quad . \quad (1)$$

$$-i = \frac{-\alpha + \beta}{-\gamma + 1}, \quad \text{i.e. } \alpha - \beta + i\gamma = i \quad . \quad . \quad . \quad (2)$$

$$-3 = \frac{i\alpha + \beta}{i\gamma + 1}, \quad \text{i.e. } i\alpha + \beta + i3\gamma = -3 \quad . \quad . \quad . \quad (3)$$

Adding (1) and (2), $\qquad 2\alpha = 2i$, so that $\alpha = i$

From (1) $\beta = i\gamma$, and substituting in (3), $-1 + i\gamma + i3\gamma = -3$, whence $\gamma = i\frac{1}{2}$ and $\beta = -\frac{1}{2}$.

Hence, the required transformation is $Z = \dfrac{iz - \frac{1}{2}}{i\frac{1}{2}z + 1}$ or $Z = \dfrac{2z + i}{z - i2}$

Otherwise, substituting the given values in (IV.75), we obtain

$$\frac{(i - Z)(-3 + i)}{i2(-3 - Z)} = \frac{(1 - z)(i + 1)}{2(i - z)}, \text{ which reduces to } Z = \frac{2z + i}{z - i2}$$

(ii) We can use the ratio method as in (i), but in this case there is an easier method. Since $Z = 0$ when $z = 2$ and $Z = \infty$ when $z = -1$, the required transformation must be of the form $Z = \lambda\dfrac{z - 2}{z + 1}$, where λ is a complex quantity whose value has yet to be found. Substituting $Z = -1 + i$ and $z = -1 + i3$ in this expression, we have

$$-1 + i = \lambda\frac{-3 + i3}{i3}, \text{ i.e. } -1 + i = \lambda(1 + i), \text{ so that } \lambda = \frac{-1 + i}{1 + i}, \text{ and this}$$

reduces to $\lambda = i$.

Hence, the required transformation is $Z = \dfrac{i(z - 2)}{z + 1}$

We can also use (IV.75) here, the left-hand side of this relation becoming in this case $\dfrac{Z_1 - Z}{Z_3 - Z}$, since Z_2 is infinite, and by substituting the given values we obtain $\dfrac{0 - Z}{-1 + i - Z} = \dfrac{(2 - z)(i3)}{3(-1 + i3 - z)}$, which reduces to $Z = \dfrac{i(z - 2)}{z + 1}$

EXAMPLE 2

If $z_1 = x_1 + iy_1$ is any given point in the upper half of the z-plane, i.e. if $y_1 > 0$, and $\bar{z}_1$ is the conjugate of z_1, i.e. $\bar{z}_1 = x_1 - iy_1$, show that the transformation $Z = \dfrac{z - z_1}{z - \bar{z}_1}$ maps the half-plane $y > 0$ into the unit circle $|Z| < 1$.

Since $z - z_1 = (x - x_1) + i(y - y_1)$, $z - \bar{z}_1 = (x - x_1) + i(y + y_1)$, and $|Z| = \dfrac{|z - z_1|}{|z - \bar{z}_1|}$ the condition $|Z| < 1$ is equivalent to $(x - x_1)^2 + (y - y_1)^2 < (x - x_1)^2 + (y + y_1)^2$, which reduces to $y_1 y > 0$. Now y_1 is given positive, so that it follows that $y > 0$.

(8) $Z = \frac{1}{2}\left(z + \dfrac{1}{z}\right)$. Compare worked example in Art. 28. Here Z becomes infinite when $z = 0$. Also, since $\dfrac{dZ}{dz} = \frac{1}{2}\left(1 - \dfrac{1}{z^2}\right)$, the derivative vanishes at the points $z = \pm 1$, which are, therefore critical points of the transformation.

If $z = r(\cos\theta + i\sin\theta)$, then $z + \dfrac{1}{z} = r(\cos\theta + i\sin\theta) + \dfrac{1}{r}(\cos\theta - i\sin\theta)$, and, since $Z = u + iv$,

$$u = \tfrac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta \text{ and } v = \tfrac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta$$

The elimination of θ from these relations leads to

$$\frac{u^2}{\frac{1}{4}\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\frac{1}{4}\left(r - \frac{1}{r}\right)^2} = 1 \quad . \quad . \quad \text{(IV.76)}$$

and the elimination of r leads to

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 1 \quad . \quad . \quad \text{(IV.77)}$$

From (IV.76) it is seen that each of the circles $r = c$ and $r = \dfrac{1}{c}$, where c is constant, maps into the same ellipse, and as c is varied, the resulting family of circles in the z-plane maps into a family of ellipses in the Z-plane. Since the distance between the foci of the ellipses is $\sqrt{\frac{1}{4}\left(r + \frac{1}{r}\right)^2 - \frac{1}{4}\left(r - \frac{1}{r}\right)^2} = 1 =$ constant, these ellipses are confocal.

As $r \to 0$, both the semi-major axis and the semi-minor axis of the ellipse $\to \infty$; as $r \to 1$, the semi-major axis $\to 1$ and the semi-minor axis $\to 0$; and as $r \to \infty$, both axes $\to \infty$. It follows that

both the inside and the outside of the unit circle in the z-plane map into the whole Z-plane, the unit circle itself mapping into the segment of the u-axis from $u = -1$ to $u = +1$.

From (IV.77) it is seen that straight lines through the origin in the z-plane map into a family of hyperbolas in the Z-plane. Since the distance between the foci of the hyperbola (IV.77) is $\sqrt{\cos^2\theta + \sin^2\theta} = 1 =$ constant, these hyperbolas are confocal. Further, since the radial lines and the circles in the z-plane cut orthogonally, each member of the family of hyperbolas cuts each member of the family of ellipses orthogonally.

(9) $Z = e^z$. If Z is expressed as $Re^{i\phi}$ and e^z as $e^{x+iy} = e^x . e^{iy}$, then $Re^{i\phi} = e^x . e^{iy}$, so that

$$R = e^x \quad . \quad . \quad . \quad . \quad \text{(IV.78)}$$

and

$$\phi = y \quad . \quad . \quad . \quad . \quad \text{(IV.79)}$$

The former relation shows that any straight line $x =$ constant, i.e. any straight line in the z-plane parallel to the y-axis, maps into a circle about the origin in the Z-plane, the radius R being > 1 or < 1 according as x is positive or negative, and being 1 when $x = 0$. The latter relation shows that any straight line $y =$ constant, i.e. any straight line in the z-plane parallel to the x-axis, maps into a straight line through the origin in the Z-plane.

It follows that the region of the z-plane between $y = 0$ and $y = 2\pi$ (or between $y = 0$ and $y = -2\pi$) maps into the whole Z-plane. Further, the region of the z-plane between $y = 0$ and $y = 2n\pi$ (or between $y = 0$ and $y = -2n\pi$), where n is a positive integer, maps into n Riemann surfaces in the Z-plane.

Here $\dfrac{dZ}{dz}$ is never zero, so that the representation is conformal throughout.

If x varies from $-\infty$ to 0 and y from 0 to 2π, then R varies from 0 to 1 and ϕ from 0 to 2π, and, therefore, the strip of the z-plane included between the lines $y = 0$ and $y = 2\pi$ and stretching from the y-axis to infinity in the negative direction maps into unit circle about the origin in the Z-plane, the semicircle on the upper side of the u-axis corresponding to the part of this strip between $y = 0$ and $y = \pi$ and that on the lower side to the part between $y = \pi$ and $y = 2\pi$.

It is easy to deduce that the exterior of the circle corresponds to the continuation of the strip from $x = 0$ to $x = \infty$.

(10) $Z = a \cosh z$, *where a is Real.* Here $Z = u + iv$ and $\cosh z = \cosh (x + iy) = \cosh x \cosh iy + \sinh x \sinh iy = \cosh x \cos y + i \sinh x \sin y$, so that

$$u = a \cosh x \cos y \qquad \text{(IV.80)}$$

and

$$v = a \sinh x \sin y \qquad \text{(IV.81)}$$

Elimination of y from these gives

$$\frac{u^2}{a^2 \cosh^2 x} + \frac{v^2}{a^2 \sinh^2 x} = 1 \qquad \text{(IV.82)}$$

and elimination of x gives

$$\frac{u^2}{a^2 \cos^2 y} - \frac{v^2}{a^2 \sin^2 y} = 1 \qquad \text{(IV.83)}$$

(IV.82) shows that any straight line x = constant in the z-plane, i.e. any straight line parallel to the y-axis, maps into an ellipse in the Z-plane, the semi-axes of this ellipse being $a \cosh x$ and $a \sinh x$, the foci at the points $(\pm a, 0)$ and the eccentric angle y. Further, the family of such straight lines obtained by varying x maps into a family of confocal ellipses.

When $x = 0$, the semi-major axis of the ellipse is a and the semi-minor axis 0. As x is increased numerically the ellipse becomes less "flat," and as $x \to \infty$, the ellipse approaches circular form.

(IV.83) shows that straight lines in the z-plane parallel to the x-axis map into confocal hyperbolas. Every member of the family of confocal hyperbolas intersects every member of the family of ellipses orthogonally.

If in (IV.82) x has either of the values $\pm \mu$, where μ is a constant, the same ellipse is obtained in the Z-plane, y being assumed to have any value from 0 to 2π. On the other hand, if in (IV.83) the value of y lies in the range 0 to $\frac{\pi}{2}$ while x may have any value from $-\infty$ to $+\infty$, then, since $\cosh x$ is always positive, (IV.80) shows that u is positive and, therefore, the right branch only of the hyperbola is obtained. It is seen similarly that, if the value of y lies in the range $\frac{\pi}{2}$ to π, the value of x being unrestricted as before, the left branch of the hyperbola is obtained.

(11) $Z = a \sin z$, *where a is Real.* Here

$$u + iv = a \sin (x + iy) = a(\sin x \cos iy + \cos x \sin iy)$$

i.e. $\quad u + iv = a(\sin x \cosh y + i \cos x \sinh y)$

Equating real and imaginary parts, we have

$$u = a \sin x \cosh y \qquad . \quad . \quad . \quad \text{(IV.84)}$$

and

$$v = a \cos x \sinh y \qquad . \quad . \quad . \quad \text{(IV.85)}$$

If y is constant, then, since $\sin^2 x + \cos^2 x = 1$,

$$\frac{u^2}{a^2 \cosh^2 y} + \frac{v^2}{a^2 \sinh^2 y} = 1 \qquad . \quad . \quad \text{(IV.86)}$$

Thus, the family of straight lines in the z-plane parallel to the x-axis corresponds to the family of confocal ellipses in the Z-plane obtained by varying y in (IV.86).

From (IV.84) and (IV.85) it is seen that, if y has a constant value $+ \mu$, then, as x varies from $-\frac{\pi}{2}$ through zero to $\frac{\pi}{2}$, u varies from $-a \cosh \mu$ through zero to $a \cosh \mu$, and v varies from zero to $a \sinh \mu$ and decreases again to zero. Thus, by this variation the half of the ellipse (IV.86) above the u-axis is covered. It is easy to deduce that, if y has a constant value $- \mu$, and x varies as above, the half of the ellipse below the u-axis is covered.

If, now, x is constant, then from (IV.84) and (IV.85), since $\cosh^2 y - \sinh^2 y = 1$,

$$\frac{u^2}{a^2 \sin^2 x} - \frac{v^2}{a^2 \cos^2 x} = 1 \qquad . \quad . \quad \text{(IV.87)}$$

Thus, the family of straight lines in the z-plane parallel to the y-axis corresponds to the family of confocal hyperbolas obtained by varying x in (IV.87).

36. **Practical Applications of Conformal Mapping.** The solution of a problem relating to the plane stream-line motion of a frictionless incompressible fluid yields two orthogonal families of plane curves — stream-lines and equipotential lines. If these curves are assumed to be in the z or (x, y) plane, where $z = x + iy$, and any regular function $w = u + iv$ is taken, where u and v are separate functions of x and y, the conformal map in the w or (u, v) plane also consists of two orthogonal families of curves. We have seen that u and v both satisfy (IV.58), and as this is the equation which must be satisfied by the stream-line function and the potential function, the curves in the w-plane give a second possible form for stream-line

flow of the fluid. As either family of curves may be taken as stream-lines, the curves give a third possible form. In stream-line flow we are usually given the value $z = c_0$ at a boundary where c_0 is constant, or zero; if the corresponding curve in the w-plane is a possible boundary, the stream-lines in the w-plane apply to an actual case. Similar considerations apply in problems on heat flow and gravitational and electrical potential. For a fuller treatment of this subject the reader should consult *The Theory and Use of the Complex Variable* by S. L. Green (Pitman).

EXAMPLES IV

(1) Prove that the amplitude (or argument) of the product of two complex numbers is the sum of their amplitudes (or arguments), and that the amplitude (or argument) of the quotient of two complex numbers is the difference of their amplitudes (or arguments).

If $\dfrac{Z-a}{Z+a}$ is pure imaginary, where Z is complex and a is a real constant, prove that in the Argand diagram Z lies on the circle on $a_1 - a$ as diameter. (U.L.)

(2) If $(a_1 + ib_1)(a_2 + ib_2) = x + iy$ and $\dfrac{a_1 + ib_1}{a_2 + ib_2} = u + iv$, show that $(a_1 - ib_1)(a_2 - ib_2) = x - iy$ and $\dfrac{a_1 - ib_1}{a_2 - ib_2} = u - iv$.

(3) If z_1 and z_2 are two complex quantities, show that

$$\text{(i) } |z_1 + z_2| < |z_1| + |z_2|, \text{ and (ii) } |z_1 - z_2| > |z_1| - |z_2|$$

State the conditions under which the equality sign can be used in each of these two cases.

(4) (i) If $Z = 1 + 2i$ and $z = 2 + i$, where $i^2 = -1$, exhibit on an Argand diagram Z, z, $p = Zz$, and $q = \dfrac{Z}{z}$

(ii) If $x + iy = u + \dfrac{a^2}{u}$ and $u = a(3 + e^{i\theta})/2$, a and θ being real, find the explicit expressions for x and y in terms of a and θ. (U.L.)

(5) (*a*) Find the cube roots of $1 + i$, and represent these roots on an Argand diagram.

(*b*) Solve completely the equation $x^3 + 8 = 0$.

(6) Find all the roots of each of the following equations—(i) $x^4 + 1 = 0$, (ii) $x^5 - 1 = 0$, (iii) $(1 + x)^3 + x^3 = 0$. Show that the points in the Argand diagram representing the roots in (iii) are collinear.

(7) Express $\sin ix$, $\cos ix$ in terms of $\sinh x$, $\cosh x$.

If $x + iy = \tan(u + iv)$, prove that

$$\coth 2v = \frac{x^2 + y^2 + 1}{2y}$$

$$\cot 2u = \frac{1 - x^2 - y^2}{2x} \qquad \text{(U.L.)}$$

(8) Using the exponential forms of $\sin\theta$ and $\cos\theta$, show that

(i) $32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

(ii) $32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$

(9) Show that, if n is a positive integer, the expression $x^{2n} + 2x^n\cos n\theta + 1$ can be expressed as the product of n quadratic factors of the form

$x^2 - 2x\cos\left[\theta + \frac{(2k+1)\pi}{n}\right] + 1$, k having the values $0, 1, 2, \ldots (n-1)$

(10) If $z = x + iy$, $Z = u + iv$, and $Z = w_1 z + w_2$, where $w_1 = r_1 e^{i\theta_1}$ and $w_2 = x_2 + iy_2$ are fixed complex numbers, show that

$$x = \frac{1}{r_1}[(u - x_2)\cos\theta_1 + (v - y_2)\sin\theta_1] \text{ and}$$

$$y = \frac{1}{r_1}[(v - y_2)\cos\theta_1 - (u - x_2)\sin\theta_1]$$

(11) Assuming $z = \cos\theta + i\sin\theta$ and using the formula which gives the sum of $n + 1$ terms of the geometric series $1 + z + z^2 + z^3 + \ldots$, deduce that

$$\text{(i)}\quad 1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = \frac{\cos\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}$$

$$\text{(ii)}\quad \sin\theta + \sin 2\theta + \ldots + \sin n\theta = \frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}$$

(12) Show that the nth roots of unity are represented by the vertices of a regular polygon of n sides inscribed in the circle $|z| = 1$, and show further that if these roots are $1, \omega, \omega^2, \ldots, \omega^{n-1}$, then $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$.

(13) If $z = x + iy$, express $\sin z$ and $\cos z$ in the form $u + iv$, and hence show that $\sin z$ vanishes if, and only if, $z = k\pi$, and $\cos z$ vanishes if, and only if, $z = (k + \frac{1}{2})\pi$, where k is zero or any positive or negative integer.

(14) If x and y are real, show that $\cosh(x + iy) + \cosh(x - iy)$ and $\sinh(x + iy) + \sinh(x - iy)$ are real, and find their values when $x = 0{\cdot}571$ and $y = 1{\cdot}117$.

(15) The functions $\cosh\sqrt{ZY}$ and $\sqrt{\frac{Y}{Z}}\sinh\sqrt{ZY}$ occur in transmission line calculations. Express each of these functions in the form $r[\theta]$ when $Z = 80 + j200$ and $Y = 0 + j0{\cdot}0025$.

[NOTE. In electrical engineering j is used in place of i.]

(16) Express in the form $u + iv$—

(i) $\sin(x - iy)$, (ii) $\cos(x - iy)$, (iii) $\tan(x + iy)$, (iv) $\tanh(x - iy)$.

Evaluate u and v in each case when $x = 1$ and $y = 1$.

(17) Prove that (i) $\tan\frac{1}{2}(x + iy) = \dfrac{\sin x + i\sinh y}{\cos x + \cosh y}$

$$\text{(ii)}\quad \tan x = \frac{1}{i}\left(\frac{e^{2ix} - 1}{e^{2ix} + 1}\right)$$

(18) Express in the form $A + iB$, where $i^2 = -1$,

(i) $\dfrac{(1+i)(2+i)}{3+i}$, (ii) $\sqrt{\dfrac{1+i}{1-i}}$, (iii) $\cos\left(\dfrac{\pi}{4} + \dfrac{i}{2}\right)$

If $x + iy = t + \dfrac{1}{t}$ and $t = re^{-i\theta}$, show that the locus in the (x, y) plane corresponding to r = constant is an ellipse, stating the lengths of its semi-axes, and determine the locus corresponding to $\theta = \dfrac{\pi}{4}$, when r varies. (U.L.)

(19) Express in the form $u + iv$ —

(i) Log 2, (ii) $\log(1 + i\sqrt{3})$, (iii) $\log(\pi - i\sqrt{2})$,
(iv) $\log(-1 + i2)$, (v) $\log(-3 - i4)$, (vi) $(1 + i)^{1-i}$, (vii) 3^i

(20) Establish the following relations—

(i) $\text{Log}(-3) = \log 3 + i(2n + 1)\pi$
(ii) $\text{Log}(i2) = \log 2 + i(2n + \frac{1}{2})\pi$
(iii) $x^i = e^{-2n\pi}[\cos(\log x) + i\sin(\log x)]$
(iv) $i^x = \cos\{(2n + \frac{1}{2})\pi x\} + i\sin\{(2n + \frac{1}{2})\pi x\}$

In (iii) and (iv) x is real.

(21) Show that

(i) $\log(-1) = i\pi$, (ii) $\log i = i\dfrac{\pi}{2}$

(iii) $\log(-i) = -i\dfrac{\pi}{2}$, (iv) $\log(\cos\phi + i\sin\phi) = i\phi$

(v) $\log\dfrac{x + iy}{x - iy} = 2i\tan^{-1}\dfrac{y}{x}$

(22) (*a*) Express $\tanh(x + iy)$ in the form $u + iv$.

(*b*) If $\tan\dfrac{x + iy}{2} = u + iv$, prove that

$$u = \frac{\sin x}{\cos x + \cosh y} \text{ and } v = \frac{\sinh y}{\cos x + \cosh y}$$

(23) (*a*) Prove that $[\sin(\theta + \phi) - e^{i\theta}\sin\phi]^n = e^{-in\phi}\sin^n\theta$.

(*b*) If $\tan(x + iy) = \sin(\alpha + i\beta)$, show that

$$\tan\alpha\sinh 2y = \tanh 2\beta\sin 2x$$

(24) (*a*) If $\sin(x + iy) = re^{i\theta}$, show that

$$r = \sqrt{\tfrac{1}{2}(\cosh 2y - \cos 2x)} \text{ and } \theta = \tan^{-1}(\cot x\tanh y)$$

and deduce that
$$y = \tfrac{1}{2}\log\frac{\cos(x + \theta)}{\cos(x - \theta)}$$

(*b*) If $\cos(x + iy) = re^{i\theta}$ show that

$$r = \sqrt{\tfrac{1}{2}(\cosh 2y + \cos 2x)} \text{ and } \theta = \tan^{-1}(-\tan x\tanh y)$$

and deduce that
$$y = \tfrac{1}{2}\log\frac{\sin(x - \theta)}{\sin(x + \theta)}$$

(25) If $\sin\theta = \operatorname{cosec} z$, where θ is real and $z = x + iy$, show that

$$z = \frac{2n+1}{2}\pi \pm i \log_e \cot \tfrac{1}{2}(n\pi + \theta)$$

(26) Show that, if r and y are real, $r^{iy} = \cos(y \log r) + i \sin(y \log r)$. Deduce that if $z = re^{i\theta}$, and x and y are real,

$$z^{x+iy} = r^x e^{-y\theta}[\cos(x\theta + y\log r) + i\sin(x\theta + y\log r)]$$

(27) Show that the principal value of Log sin $(x + iy)$ is

$$\tfrac{1}{2}\log\tfrac{1}{2}(\cosh 2y - \cos 2x) + i\tan^{-1}(\cot x \tanh y)$$

(28) State which regions in the z-plane are denoted by the following inequalities

(1) $|z| < a$, where a is a positive real number.

(2) $|z - z_1| < a$, where a is as in (1) and z_1 is a fixed complex number.

(3) $|z - 2| \leq 1$.

(4) $-\dfrac{\pi}{2} < \arg. z < \dfrac{\pi}{2}$

(5) $|4z - 1| > 5$.

(29) Given that $z_1 = 0{\cdot}6 + i\,0{\cdot}8$ and $z_2 = 1 + i\,0{\cdot}5$, draw a diagram showing the points $\dfrac{1}{z_1}$, $z_1 + \dfrac{1}{z_1}$, $z_1 + z_2$, z_1z_2, z_1^2. $\tfrac{1}{2}e^{z_1}$, $\tfrac{1}{2}e^{-z_1}$, $\tfrac{1}{2}\cosh z_1$

If a point z in the (x, y) plane traces out a circle having its centre at the origin and passing through the point z_1, draw the locus of each of the points representing $\dfrac{1}{z}$, z^2, and $\tfrac{1}{2}\left(z + \dfrac{1}{z}\right)$

(30) If $\alpha + i\beta$ is a root of the equation

$$a_0z^n + a_1z^{n-1} + a_2z^{n-2} + \ldots + a_n = 0$$

where a_0, a_1, a_2, etc., are real numbers, show that $\alpha - i\beta$ is also a root of this equation.

(31) A point P is represented by the complex number z. Determine the locus of P in each of the following cases—

(i) $\arg. z =$ constant,

(ii) $|z| =$ constant,

(iii) $|z - z_1| =$ constant, where z_1 represents a fixed point,

(iv) $|z - z_1| + |z - z_2| =$ constant, where z_1 and z_2 represent fixed points,

(v) $\left|\dfrac{z - z_1}{z - z_2}\right| =$ constant, z_1 and z_2 being as in (iv).

What does the locus in (v) become when the constant has the value unity?

(32) The vertices A, B, C, D of a square are represented by the complex numbers z_1, z_2, z_3, z_4 respectively.

(i) If $z_1 = -1 + i2$ and $z_2 = 2 - i2$, find the possible values of z_3 and z_4

(ii) If $z_1 = -2 + i3$ and $z_3 = 6 - i1$, find z_2 and z_4

(33) Find whether or not the following functions are continuous at the points indicated—

(i) $f(z) = (z - a)^2$, at the point $z = a$;

(ii) $f(z) = \dfrac{1}{z - a}$, at the point $z = a$;

(iii) $f(z) = \dfrac{1}{z}$, at the point $z = 0$;

(iv) $f(z) = z^3$, at the point $a + ib$, where neither a nor b is zero;

(v) The polynomial $a + bz + cz^2 + dz^3 + \ldots$, at any point in the finite part of the z-plane.

(34) Determine a regular function $Z = f(z) = u + iv$ in each of the following cases—

(i) $u = -y$, (ii) $u = \cos x \cosh y$, (iii) $v = y$, (iv) $v = -\dfrac{y}{x^2 + y^2}$

(35) If Z is a regular function of z and R and R' are the moduli of Z and $\dfrac{dZ}{dz}$ respectively, prove that

$$\frac{\partial^2(R^2)}{\partial x^2} + \frac{\partial^2(R^2)}{\partial y^2} = 4(R')^2$$

(36) Given that $z = x + iy$ and $Z = u + iv$, determine u and v in each of the following cases, and determine also the curves in the z-plane corresponding to u = constant and v = constant.

(i) $Z = az + b$, where a and b are real constants,

(ii) $Z = \dfrac{1}{z - 1}$, (iii) $Z = \dfrac{1}{z + i}$

(37) Given that $Z = f(z) = u + iv$ and that z moves on a curve $y = F(x)$ in the z-plane, determine the condition that this curve maps into the u-axis of the Z-plane.

(38) In an Argand diagram the point z moves along the real axis from $z = -1$ to $z = +1$. Find the corresponding motion of the point $\dfrac{1 - iz}{z - i}$ (U.L.)

(39) Express $w = \dfrac{z(z + i)}{z - i}$ in the form $a + ib$. Determine the regions of the plane within which the modulus of the function exp. (w) is greater than unity. (U.L.)

(40) Apply the transformation $Z = z^2$ to the region in the first quadrant of the z-plane bounded by the axes of reference and the circles $|z| = c_1$ and $|z| = c_2$, where $c_1 > c_2 > 0$. Discuss the case $c_2 = 0$, and show that in this case the transformation is not conformal at the origin.

(41) If $u + iv = (x + iy)^{\frac{1}{2}}$, where u and v are real, and the point (x, y) describes the circle $(x - 1)^2 + y^2 = 1$, find the polar equation of the locus of the point (u, v). (U.L.)

(42) Show that under the transformation $Z = i(z + c)$, where c is a real constant, the half of the z-plane to the right of the y-axis maps into the region of the Z-plane above the line $v = c$.

(43) Show that under the transformation $Z = 1 - iz$ the semi-infinite strip $x > 0$, $0 < y < 1$, maps into the region $v < 0$, $1 < u < 2$.

(44) If $u + iv = \frac{a}{z}$, where $z = x + iy$, $i^2 = -1$, and a is real, show that the curves in the (x, y) plane along which u and v are respectively constant are circles, and that they intersect orthogonally. (U.L.)

(45) Show that under the transformation $Z = \frac{c}{z}$, where c is real, (i) a circle about the origin in the z-plane transforms into a circle about the origin in the Z-plane, the latter circle being described in the opposite sense to the former, and (ii) a circle passing through the origin in the z-plane transforms into a straight line in the Z-plane.

(46) Show that under the transformation $Z = \frac{1}{z}$ the region in the z-plane defined by $x > \frac{1}{2}$, $y > 0$, maps into the region in the Z-plane defined by $|Z - 1| < 1$, $v < 0$.

(47) Show that under the transformation $Z = ze^{i\theta_1}$, where θ_1 is fixed, the figure in the Z-plane corresponding to a given figure in the z-plane is the latter figure rotated about the origin through angle θ_1.

(48) Show that under the transformation $Z = -\frac{1}{z + i}$ the real axis in the z-plane maps into a circle and the imaginary axis maps into a straight line.

(49) Show that under the transformation $Z = \log z$ any circle in the z-plane with centre at the origin and any straight line radiating from the origin map into straight lines parallel to the v- and u-axes respectively of the Z-plane.

(50) Given the transformation $Z = \log z$, show that there are singular points at the origin and at the point at infinity. Show also that, if z, starting from the point $-k + i0$, describes the circle $|z| = k$ once counter-clockwise, then under this transformation Z describes a certain straight line.

(51) If P is any point z in the z-plane, P_1 is the inverse of P with respect to a circle of radius a with centre at the origin, P_2 is the reflection of P_1 in the real axis, and Q is the mid-point of the straight line PP_2, show that Q is the point $\frac{1}{2}\left(z + \frac{a^2}{z}\right)$

Given that $Z = \frac{1}{2}\left(z + \frac{a^2}{z}\right)$, show that as z describes the circle $|z| = k$, where $k > a$, Z describes an ellipse. Find the eccentricity of this ellipse.

(52) Given that $Z = \frac{1}{2}\left[\frac{a + b}{a}z + \frac{a(a - b)}{z}\right]$, show that, if z moves on the circle $z = ae^{i\theta}$, then Z moves on an ellipse with semi-axes a and b and eccentric angle θ.

Discuss the above transformation when $b = 0$.

Show that in this latter case the transformation is equivalent to

$$\frac{Z + a}{Z - a} = \left(\frac{z + a}{z - a}\right)^2$$

(53) Determine the bilinear transformation required in each of the following cases—(i) to map the points $z = 1 + i$, $1 - i$, 1 into the points $Z = 0$, $2i$, $\frac{1}{2}(1 + i)$ respectively; (ii) to map the points $z = 1, -1, i$ into the points $Z = 0$, $\infty, -1 + i$ respectively.

(54) Given the transformation $Z = \dfrac{z-1}{z+1}$, show that the unit circles about the points $-2 + i0$ and $-1 + i1$ in the z-plane map into the straight lines $u = 2$ and $v = 1$ respectively in the Z-plane. Illustrate by a diagram that the two circles cut orthogonally.

(55) Prove that under the transformation $Z = \dfrac{1 - iz}{z - i}$ a certain semicircle in the z-plane maps into the part of the real axis of the Z-plane between $u = -1$ and $u = 1$.

(56) Show that by the transformation $Z = i\dfrac{1-z}{1+z}$ the unit circle about the origin in the z-plane maps into the u-axis of the Z-plane, the semi-circles above and below the x-axis mapping into the positive and negative halves respectively of the u-axis. Show also that a point in the upper half of the Z-plane corresponds to a point inside the unit circle in the z-plane.

(57) Show that under the transformation $Z = \dfrac{1+z}{1-z}$ the semicircle $|z| \leq 1$, $y \geq 0$, in the z-plane maps into the first quadrant of the Z-plane.

(58) If z_1 is any given point in the z-plane such that $|z_1| < 1$ and $\bar{z}_1$ is the conjugate of z_1, show that the transformation $Z = \dfrac{z - z_1}{\bar{z}_1 z - 1}$ maps the unit circle $z < 1$ into the unit circle $|Z| < 1$.

(59) Show that under the transformation $Z = e^z$ the family of straight lines $y = mx$ maps into the family of logarithmic spirals $R = e^{\frac{\phi}{m}}$

(60) Show that by the transformation $Z = e^z$ the rectangle in the z-plane bounded by the straight lines $x = a$, $x = b$, $y = c$, $y = d$, where $d - c < 2\pi$, maps into the region in the Z-plane bounded by circles about the origin of radii e^a and e^b and straight lines through the origin inclined at angles c and d to the u-axis. Show further that the area of the region in the Z-plane which corresponds to the rectangle in the z-plane bounded by the lines $x = 0$, $x = b$, $y = 0$, $y = \dfrac{\pi}{2}$ is $\dfrac{\pi}{4}(e^{2b} - 1)$

(61) A region in the z-plane is bounded by the lines $x = h$, $x = h + 2k$, $y = k$, $y = -k$, where $0 < k < \pi$. Determine the corresponding region in the Z-plane when the transformation is $Z = e^z$, and deduce that the ratio of the area of the latter region to that of the former tends to the limit e^{2h} as $k \to 0$.

(62) A rectangle in the z-plane is included by the straight lines $x = 0$, $x = 4$, $y = 0$, and $y = \dfrac{\pi}{4}$. Show that under the transformation $Z = \cosh z$ the area of the region in the Z-plane which corresponds to this rectangle is $\dfrac{\pi}{16}\sinh 8 - 1$.

(63) Show (i) that the transformation $Z = \cos z$ is the same as the transformation $Z = \sin z$ if each point in the z-plane be first translated to the right through a distance $\dfrac{\pi}{2}$, and (ii) that the transformation $Z = \sinh z$ is the same as the transformation $Z = \sin z$ if the axes in each plane be first rotated through the angle $\dfrac{\pi}{2}$

Deduce the similar relation between the transformations $Z = \cos z$ and $Z = \cosh z$.

CHAPTER V

VECTORS—SCALAR AND VECTOR PRODUCTS

37. It is assumed that the reader is acquainted with the distinction between scalar and vector quantities and with the parallelogram and the triangle laws for the combination of vectors. In this chapter a single letter printed in bold italic type denotes a vector, and the same letter printed in ordinary italics is used to denote the magnitude or module of the vector.

For example, $\boldsymbol{a}$ and a denote a vector and the magnitude of that vector respectively. The equation $\boldsymbol{a} = \boldsymbol{b}$ implies that the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal in magnitude and have the same direction and sense. By drawing a parallelogram $OABC$ and joining O to B, the reader can easily deduce that, if $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{AB} = \boldsymbol{b}$, then $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \boldsymbol{a} + \boldsymbol{b}$, and also $\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = \boldsymbol{b} + \boldsymbol{a}$ so that

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a} \quad . \quad . \quad . \qquad (V.1)$$

Thus, the *commutative law* of addition holds for vectors. Further, from a vector polygon $OABC$ in which $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{AB} = \boldsymbol{b}$, and $\overrightarrow{BC} = \boldsymbol{c}$, it is seen that

$\overrightarrow{OB} = \boldsymbol{a} + \boldsymbol{b}$, $\overrightarrow{AC} = \boldsymbol{b} + \boldsymbol{c}$, $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$ and also $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$

so that
$$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) \quad . \quad . \qquad (V.2)$$

Thus, the *associative law* of addition holds for vectors. If p is any scalar, positive or negative, integral or fractional, then $p\boldsymbol{a}$ denotes a vector of magnitude pa having the same direction as $\boldsymbol{a}$ if p is positive and the opposite direction to $\boldsymbol{a}$ if p is negative.

In Fig. 31 $\overrightarrow{OB} = \boldsymbol{a} + \boldsymbol{b}$, the vector sum of $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{AB} = \boldsymbol{b}$, A_1 is a point on OA such that $OA_1 = p \times OA$, and A_1B_1 is drawn parallel to AB such that $A_1B_1 = p \times AB$. It follows by elementary geometry that O, B, B_1 are collinear and that $OB_1 = p \times OB$.

From $\triangle OA_1B_1$, $$\overrightarrow{OB_1} = \overrightarrow{OA_1} + \overrightarrow{A_1B_1}$$

i.e. $$p \times \overrightarrow{OB} = p\mathbf{a} + p\mathbf{b}$$

i.e. $$p(\mathbf{a} + \mathbf{b}) = p\mathbf{a} + p\mathbf{b} \quad . \quad . \quad . \quad \text{(V.3)}$$

The generalized form of (V.3) can be readily deduced, namely,

$$p(\mathbf{a} + \mathbf{b} + \mathbf{c} + \ldots) = p\mathbf{a} + p\mathbf{b} + p\mathbf{c} + \ldots \quad . \quad \text{(V.4)}$$

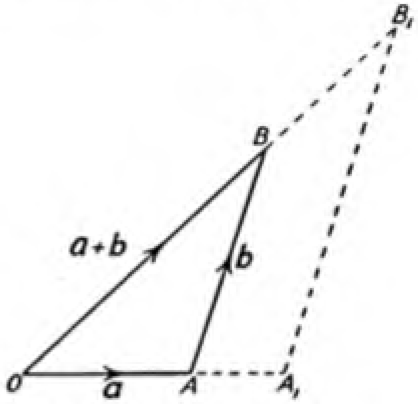


FIG. 31. VECTOR SUM

Thus, the *distributive law* holds for the product of a scalar and the vector sum of any number of vectors.

EXAMPLE

On the side BC of a triangle ABC a point O is taken such that $\frac{BO}{OC} = \frac{q}{p}$. Show that the resultant of the concurrent vectors $p\overrightarrow{AB}$ and $q\overrightarrow{AC}$ is $(p + q)\overrightarrow{AO}$.

The reader should draw the figure.

In vector notation,

$$\overrightarrow{AB} = \overrightarrow{AO} - \overrightarrow{BO}$$

$\therefore$ $$p\overrightarrow{AB} = p\overrightarrow{AO} - p\overrightarrow{BO} \quad . \quad . \quad . \quad . \quad (1)$$

Also, $$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$$

$\therefore$ $$q\overrightarrow{AC} = q\overrightarrow{AO} + q\overrightarrow{OC} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Adding (1) and (2),

$$p\overrightarrow{AB} + q\overrightarrow{AC} = (p + q)\overrightarrow{AO} + (q\overrightarrow{OC} - p\overrightarrow{BO})$$

Since $q\overrightarrow{OC} = p\overrightarrow{BO}$ (given), i.e. $q\overrightarrow{OC} - p\overrightarrow{BO} = 0$,

the required resultant is $(p + q)\overrightarrow{AO}$

38. **Scalar Product.** The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a} \cdot \mathbf{b}$ [alternatively by $(\mathbf{a}, \mathbf{b})$ or S $\mathbf{ab}$ or simply $\mathbf{ab}$], and is defined as the product of the numerical magnitude of any one of the vectors and the projection of the other vector upon it. Thus, this product is a scalar quantity, and not a vector quantity.

In Fig. 32, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ are two vectors of magnitudes

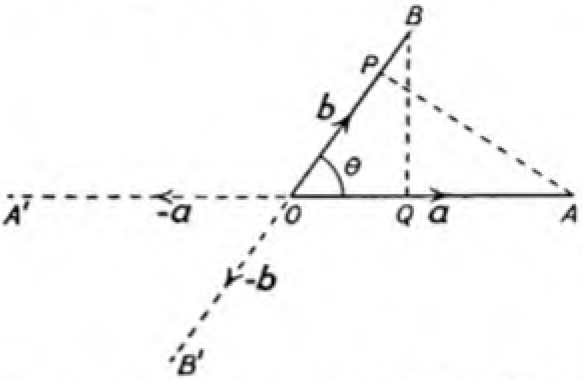


FIG. 32. SCALAR PRODUCT OF TWO VECTORS

a and b respectively, and AP and BQ are drawn perpendicular to OB and OA respectively. Then, by definition,

$$\mathbf{a} \cdot \mathbf{b} = OA \times OQ \text{ or } OB \times OP$$

If θ is the angle between the directions of the two vectors measured both outwards from O, or both inwards towards O, then $OQ = b \cos \theta$ and $OP = a \cos \theta$, so that

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta \qquad . \quad . \quad . \qquad (V.5)$$

Hence, the scalar product of two vectors may be defined as the product of their numerical magnitudes into the cosine of the angle between their directions.

It is obvious from the definition that

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \qquad . \quad . \quad . \qquad (V.6)$$

and the order of the factors is immaterial. This is known as the *commutative law* and holds good for scalar products.

If, in Fig. 32, AO and BO are produced their own lengths to A' and B' respectively, then $\overrightarrow{OA'} = -\mathbf{a}$ and $\overrightarrow{OB'} = -\mathbf{b}$.
By (V.5),

$$\mathbf{a} \cdot (-\mathbf{b}) = ab \cos \widehat{AOB'} = ab \cos (180° - \theta)$$
$$= -ab \cos \theta$$

so that $$\mathbf{a} \cdot (-\mathbf{b}) = -\mathbf{a} \cdot \mathbf{b} \qquad . \quad . \quad . \quad . \qquad (V.7)$$

Similarly, $(-\boldsymbol{a}) \cdot \boldsymbol{b} = -\boldsymbol{a} \cdot \boldsymbol{b}$ . . . . (V.8)

Also $(-\boldsymbol{a}) \cdot (-\boldsymbol{b}) = ab \cos \widehat{A'OB'} = ab \cos \theta$

whence $(-\boldsymbol{a}) \cdot (-\boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b}$ . . . . (V.9)

The relations (V.7) to (V.9) show that the rule of signs of ordinary algebra applies also to scalar products.

If neither $\boldsymbol{a}$ nor $\boldsymbol{b}$ is zero and $\theta = 90°$, whence $\cos \theta = 0$, then from (V.5)

$$\boldsymbol{a} \cdot \boldsymbol{b} = 0 \quad . \quad . \quad . \quad . \quad \text{(V.10)}$$

This is the condition that the two vectors are at right angles.

If the vectors have the same direction, i.e. if $\theta = 0°$, then (V.5) becomes

$$\boldsymbol{a} \cdot \boldsymbol{b} = ab \quad . \quad . \quad . \quad \text{(V.11)}$$

If $\boldsymbol{b} = \boldsymbol{a}$, i.e. if the vectors are equal in magnitude and have the same direction and sense, then (V.5) becomes

$$\boldsymbol{a} \cdot \boldsymbol{a} = a^2$$

or, as it is sometimes written,

$$\boldsymbol{a}^2 = a^2 \quad . \quad . \quad . \quad \text{(V.12)}$$

for θ is zero in this case and hence $\cos \theta$ is unity.

EXAMPLE

Calculate the scalar product in each of the following cases—

(i) $6_{24°}, 5_{87°}$, (ii) $2{\cdot}4_{59°}, 1{\cdot}5_{179°}$, (iii) $(8{\cdot}3_{39°})^2$

In (i) the angle between the vectors is $87° - 24°$, i.e. $63°$, so that the scalar product $= 6 \times 5 \times \cos 63° = 30 \times 0{\cdot}454 = 13{\cdot}62$. In (ii) the angle between the vectors is $179° - 59°$, i.e. $120°$, so that the scalar product $= 2{\cdot}4 \times 1{\cdot}5 \times \cos 120° = 3{\cdot}6 \times -0{\cdot}5 = -1{\cdot}8$. In (iii), by (V.12), the scalar product $= 8{\cdot}3^2 = 68{\cdot}89$.

39. **Distributive Law for Scalar Products.** Consider the scalar product $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$, where $\boldsymbol{b} + \boldsymbol{c}$ is the vector sum of $\boldsymbol{b}$ and $\boldsymbol{c}$.

By definition

$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = a \times (\text{projection of vector } \boldsymbol{b} + \boldsymbol{c} \text{ on vector } \boldsymbol{a})$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = a \times (\text{projection of vector } \boldsymbol{b} \text{ on vector } \boldsymbol{a})$$

and $$\boldsymbol{a} \cdot \boldsymbol{c} = a \times (\text{projection of vector } \boldsymbol{c} \text{ on vector } \boldsymbol{a})$$

Now, the projection of $\boldsymbol{b} + \boldsymbol{c}$ on any straight line is equal to the sum of the projections of $\boldsymbol{b}$ and $\boldsymbol{c}$ on this line, so that it follows that

$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c} \quad . \quad . \quad . \quad \text{(V.13)}$$

It is easy to deduce similarly that

$$\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}+\ldots)=\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}+\boldsymbol{a}\cdot\boldsymbol{d}+\ldots \qquad (V.14)$$

This is known as the *distributive law* and holds good for the scalar product of a vector and the sum of any number of vectors.

Consider now the scalar product $(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{c}+\boldsymbol{d})$. Since $\boldsymbol{a}+\boldsymbol{b}$ is itself a vector, then by (V. 13)

$$(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{c}+\boldsymbol{d})=(\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{c}+(\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{d}$$

Also, by (V.13),

$$(\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{c}=\boldsymbol{a}\cdot\boldsymbol{c}+\boldsymbol{b}\cdot\boldsymbol{c} \text{ and } (\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{d}=\boldsymbol{a}\cdot\boldsymbol{d}+\boldsymbol{b}\cdot\boldsymbol{d}$$

Hence, just as in ordinary algebra,

$$(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{c}+\boldsymbol{d})=\boldsymbol{a}\cdot\boldsymbol{c}+\boldsymbol{b}\cdot\boldsymbol{c}+\boldsymbol{a}\cdot\boldsymbol{d}+\boldsymbol{b}\cdot\boldsymbol{d} \qquad (V.15)$$

The result (V.15) can be readily extended to the scalar product of vectors each of which is the vector sum of a number of vectors.

As a particular case of (V.15),

$$(\boldsymbol{a}+\boldsymbol{b})^2=\boldsymbol{a}^2+2\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}^2 \quad . \quad . \qquad (V.16)$$

and by (V.12) and (V.5),

$$(\boldsymbol{a}+\boldsymbol{b})^2=a^2+2ab\cos\theta+b^2 \quad . \qquad (V.17)$$

where θ is the angle between the directions of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

As an application of (V.17), consider a triangle ABC with sides of lengths a, b, c respectively.

Let the vectors $\overrightarrow{BC}$, $\overrightarrow{BA}$, $\overrightarrow{AC}$ be denoted by $\boldsymbol{a}$, $\boldsymbol{c}$, $\boldsymbol{b}$ respectively.

Then $$\overrightarrow{BC}=\overrightarrow{BA}+\overrightarrow{AC}$$

i.e. $$\boldsymbol{a}=\boldsymbol{c}+\boldsymbol{b}$$

Squaring both sides,

$$\boldsymbol{a}^2=(\boldsymbol{c}+\boldsymbol{b})^2$$

i.e. $$a^2=c^2+2cb\cos(180^\circ-A)+b^2 \text{ [by (V.17)]}$$

the angle between the vectors $\boldsymbol{c}$ and $\boldsymbol{b}$ being $180^\circ - A$.

Hence, $$a^2=b^2+c^2-2bc\cos A$$

which is the cosine formula for the solution of triangles.

40. **Work and Power.** Consider the two vectors $\boldsymbol{P}$ and $\boldsymbol{s}$, where $\boldsymbol{P}$ is a given force of magnitude P and $\boldsymbol{s}$ is a given displacement of magnitude s.

If the vectors are inclined at any angle θ to each other, then the work done W by the force in the displacement $\mathbf{s}$ is defined by the relation

$$W = (\text{component of } \mathbf{P} \text{ in direction of } \mathbf{s}) \times s$$
$$= (P \cos \theta) \times s$$
$$= Ps \cos \theta$$

Since, from (V.5), $\qquad Ps \cos \theta = \mathbf{P} \cdot \mathbf{s}$

then $$W = \mathbf{P} \cdot \mathbf{s} \quad . \quad . \quad . \quad (V.18)$$

If the displacement is in the direction of the force, then by (V.11) $\mathbf{P} \cdot \mathbf{s} = Ps$, so that in this case

$$W = Ps$$

If $\mathbf{P}$ is perpendicular to $\mathbf{s}$, then by (V.10), $\mathbf{P} \cdot \mathbf{s} = 0$, so that $W = 0$, i.e. the force does no work. W may be regarded as the work done by the force $\mathbf{P}$ acting parallel to its original direction through the displacement $\mathbf{s}$.

If several forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, . . . act on a particle so as to move the particle through a displacement $\mathbf{s}$, the total work W done by the forces is given by

$$W = \mathbf{P}_1 \cdot \mathbf{s} + \mathbf{P}_2 \cdot \mathbf{s} + \mathbf{P}_3 \cdot \mathbf{s} + \ldots$$
$$= (\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \ldots) \cdot \mathbf{s}$$

i.e. $$W = \mathbf{R} \cdot \mathbf{s}$$

where $\mathbf{R} = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \ldots$ is the resultant of the given forces.

The total work done in this case is that which would be done by the single force $\mathbf{R}$.

Since velocity is displacement per unit time in a given direction, the above argument can be applied to the case of a force $\mathbf{P}$ which acts on a point moving with velocity $\mathbf{v}$, whether this velocity be in the direction of P or not.

In this case,

$$\mathbf{P} \cdot \mathbf{v} = \text{power, or rate at which the force is doing work.}$$

EXAMPLE

A body moves a distance 60 ft in a straight path under the action of a force 450 lb whose line of action is inclined at 35° to the direction of motion. Determine the work done by the force in this displacement of the body.

If the body moves at 2·5 ft/sec, determine the power.

Work done by force $= \mathbf{P} \cdot \mathbf{s} = Ps \cos \theta = 450 \times 60 \times \cos 35° = 2\,212$ ft-lb.

Power $= \mathbf{P} \cdot \mathbf{v} = Pv \cos \theta = 450 \times 2{\cdot}5 \times \cos 35° = 921{\cdot}5$ ft-lb/sec.

41. **Unit Vectors.** If $\boldsymbol{i}$ denotes a vector of magnitude 1 unit, then any vector, say $\boldsymbol{a}$, having the same direction as $\boldsymbol{i}$, can be expressed as

$$\boldsymbol{a} = a\boldsymbol{i} \text{ or } \boldsymbol{i} = \frac{\boldsymbol{a}}{a} \quad . \quad . \quad . \quad (V.19)$$

The unit vectors in the directions of rectangular axes OX and OY are denoted by $\boldsymbol{i}$ and $\boldsymbol{j}$ respectively.

It follows from (V.12) that

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = 1 \quad . \quad . \quad . \quad (V.20)$$

and from (V.10) that

$$\boldsymbol{i} \cdot \boldsymbol{j} = 0 \quad . \quad . \quad . \quad (V.21)$$

the vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ being at right angles.

Let (x, y) be the rectangular and (r, θ) the polar co-ordinates of a point P on a plane, and let PM be drawn perpendicular to OX.

Then, in vector notation,

$$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$$

i.e.

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} \quad . \quad . \quad . \quad . \quad (V.22)$$

Multiplying through in (V.22) by $\boldsymbol{i}$ and $\boldsymbol{j}$ in turn and using (V.20) and (V.21),

$$\boldsymbol{r} \cdot \boldsymbol{i} = x \quad . \quad . \quad . \quad . \quad (V.23)$$

and

$$\boldsymbol{r} \cdot \boldsymbol{j} = y \quad . \quad . \quad . \quad . \quad (V.24)$$

Thus, the scalar product of vector $\overrightarrow{OP}$ and unit vector in the direction OX gives the numerical value of the orthographic projection of $\overrightarrow{OP}$ on OX, and the scalar product of $\overrightarrow{OP}$ and unit vector in the direction OY gives the numerical value of the projection of $\overrightarrow{OP}$ on OY.

If P_1 is the point (x_1, y_1) or (r_1, θ_1), and P_2 is the point (x_2, y_2) or (r_2, θ_2), then from (V.22),

$$\boldsymbol{r}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j}$$

and $$\boldsymbol{r}_2 = x_2\boldsymbol{i} + y_2\boldsymbol{j}$$

Hence, $$\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = (x_1\boldsymbol{i} + y_1\boldsymbol{j}) \cdot (x_2\boldsymbol{i} + y_2\boldsymbol{j})$$

i.e. $$\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = x_1x_2 + y_1y_2 \quad . \quad . \quad . \quad . \qquad (V.25)$$

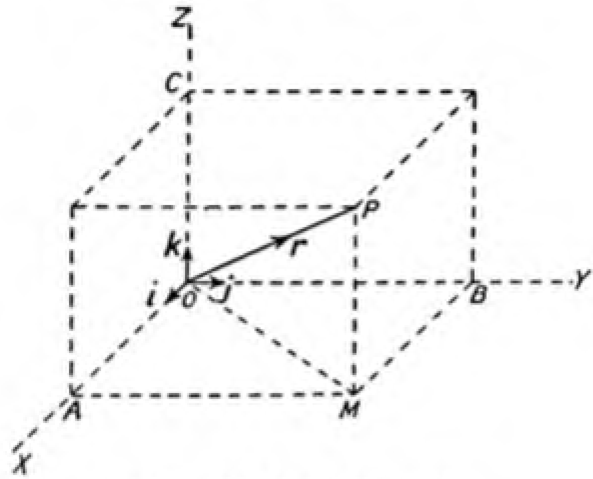


FIG. 33. UNIT VECTORS

since from (V.20) and (V.21)

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = 1 \text{ and } \boldsymbol{i} \cdot \boldsymbol{j} = 0$$

Now, $\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = r_1r_2 \cos \widehat{P_1OP_2} = r_1r_2 \cos (\theta_1 - \theta_2)$; also $x_1 = r_1 \cos \theta_1, y_1 = r_1 \sin \theta_1, x_2 = r_2 \cos \theta_2$, and $y_2 = r_2 \sin \theta_2$. Substitution of these values in (V.25) gives, after division by r_1r_2

$$\cos (\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \qquad (V.26)$$

Since θ_1 and θ_2 can have any values, the above proof of this well-known formula is quite general.

By changing θ_2 to $-\theta_2$, the reader can deduce from (V.26) the corresponding formula for $\cos (\theta_1 + \theta_2)$, and by using the relations $\sin \left(\alpha + \frac{\pi}{2}\right) = \cos \alpha$ and $\cos \left(\alpha + \frac{\pi}{2}\right) = -\sin \alpha$, he can deduce from the two formulae thus established the corresponding formulae for $\sin (\theta_1 - \theta_2)$ and $\sin (\theta_1 + \theta_2)$.

In Fig. 33 are represented three axes OX, OY, OZ at right angles to each other in space, OY and OZ being in the plane of the paper and OX perpendicular to the plane of the paper with its positive

direction from O towards the reader. With this method of representation a right-handed screw assumed to have its axis along OX, OY, OZ in turn would move in the directions OX, OY, OZ if rotated in the senses Y to Z, Z to X, X to Y respectively.

The unit vectors in the directions OX, OY, OZ are denoted by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ respectively, as indicated in the figure.

From (V.12),
$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = 1 \qquad \text{(V.27)}$$

and from (V.10),
$$\boldsymbol{i}\cdot\boldsymbol{j} = \boldsymbol{j}\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\boldsymbol{i} = 0 \qquad \text{(V.28)}$$

Let P be any point (x, y, z), and let $OP = r$, $\widehat{XOP} = \alpha$, $\widehat{YOP} = \beta$, and $\widehat{ZOP} = \gamma$. If OA, OB, OC are the projections of OP on OX, OY, OZ respectively, and M is the foot of the perpendicular from P on the XY-plane, then in vector notation,

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$

and
$$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$$

so that
$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AM} + \overrightarrow{MP}$$

i.e.
$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

i.e.
$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \qquad \text{(V.29)}$$

Further, from the definition of a scalar product,

$$\boldsymbol{r}\cdot\boldsymbol{i} = r \times 1 \times \cos\alpha$$

$$\left.\begin{array}{ll} \text{i.e.} & \boldsymbol{r}\cdot\boldsymbol{i} = x \\ \text{Similarly,} & \boldsymbol{r}\cdot\boldsymbol{j} = y \\ \text{and} & \boldsymbol{r}\cdot\boldsymbol{k} = z \end{array}\right\} \qquad \text{(V.30)}$$

Let P_1 be the point (x_1, y_1, z_1) and P_2 be the point (x_2, y_2, z_2), and let the inclinations of $OP_1 = r_1$ and $OP_2 = r_2$ to OX, OY, OZ be $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ respectively.

Then, from (V.29)

$$\boldsymbol{r}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$$

and
$$\boldsymbol{r}_2 = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$$

so that
$$\boldsymbol{r}_1\cdot\boldsymbol{r}_2 = (x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k})\cdot(x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k})$$

i.e.
$$\boldsymbol{r}_1\cdot\boldsymbol{r}_2 = x_1x_2 + y_1y_2 + z_1z_2 \qquad \text{(V.31)}$$

since by (V.12) and (V.10),

$$i^2 = j^2 = k^2 = 1 \text{ and } \boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{i} = 0$$

By substituting $\boldsymbol{r}_1 = \boldsymbol{r}_2 = \boldsymbol{r}$ in (V.31), or by squaring both sides of (V.29),

$$\boldsymbol{r}^2 = x^2 + y^2 + z^2$$

i.e.

$$r^2 = x^2 + y^2 + z^2 \qquad . \quad . \quad (V.32)$$

Now, if $\phi = P_1\widehat{O}P_2$, $\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = r_1 r_2 \cos\phi$; also $x_1 = r_1 \cos\alpha$, $y_1 = r_1 \cos\beta$, $z_1 = r\cos\gamma$, with similar relations for x_2, y_2, z_2. Substitution of these values in (V.31) gives, after division by $r_1 r_2$,

$$\cos\phi = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 \quad (V.33)$$

which is a well-known formula in three-dimensional geometry [see (X.23)]. If $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ are at right angles to each other, then by (V.10),

$$\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = 0$$

so that, from (V.31), the condition of perpendicularity is

$$x_1x_2 + y_1y_2 + z_1z_2 = 0 \qquad . \quad . \quad (V.34)$$

or, from (V.33),

$$\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 = 0 \quad (V.35)$$

EXAMPLE 1

The terminal points P, Q, R, S of two vectors $\overrightarrow{PQ} = \boldsymbol{a}$ and $\overrightarrow{RS} = \boldsymbol{b}$ are (2, 0, 5), (5, 12, 9), (– 3, – 1, 4), and (3, 2, 6) respectively. Express each of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in the form $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, and calculate the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$.

The components of $\boldsymbol{a}$ parallel to OX, OY, OZ are $5 - 2$, $12 - 0$, $9 - 5$, i.e. 3, 12, 4 respectively; and those of $\boldsymbol{b}$ are $3 - (-3)$, $2 - (-1)$, $6 - 4$, i.e. 6, 3, 2 respectively.

Hence, by (V.29),

$$\boldsymbol{a} = 3\boldsymbol{i} + 12\boldsymbol{j} + 4\boldsymbol{k}$$

and

$$\boldsymbol{b} = 6\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$$

By (V.31),

$$\boldsymbol{a} \cdot \boldsymbol{b} = 3 \times 6 + 12 \times 3 + 4 \times 2$$
$$= 18 + 36 + 8$$

i.e.

$$\boldsymbol{a} \cdot \boldsymbol{b} = 62$$

EXAMPLE 2

A particle is displaced from the point whose position vector is $5\boldsymbol{i} - 5\boldsymbol{j} - 7\boldsymbol{k}$ to the point whose position vector is $6\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$ under the action of constant forces $10\boldsymbol{i} - \boldsymbol{j} + 11\boldsymbol{k}$, $4\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k}$, and $-2\boldsymbol{i} + \boldsymbol{j} - 9\boldsymbol{k}$. Show that the total work done by the forces is 87 units.

The resultant force $\mathbf{R} = (10i - j + 11\mathbf{k}) + (4i + 5j + 6\mathbf{k}) + (-2i + j - 9\mathbf{k})$

$= 12i + 5j + 8\mathbf{k}$

The displacement $\mathbf{s} = (6i + 2j - 2\mathbf{k}) - (5i - 5j - 7\mathbf{k})$

$= i + 7j + 5\mathbf{k}$

$\therefore$ The total work done

$$= \mathbf{R} \cdot \mathbf{s} \quad \text{[by (V.18)]}$$
$$= (12i + 5j + 8\mathbf{k}) \cdot (i + 7j + 5\mathbf{k})$$
$$= 12 \times 1 + 5 \times 7 + 8 \times 5$$
$$= 12 + 35 + 40$$
$$= 87 \text{ units}$$

42. **Centre of Mass.** Let particles of masses m_1, m_2, m_3, etc., be situated at the points A, B, C, etc., respectively, and let O be any point chosen as origin. The positions of the points A, B, C, etc., may be defined by the vectors $\overrightarrow{OA} = \mathbf{r}_1$, $\overrightarrow{OB} = \mathbf{r}_2$, $\overrightarrow{OC} = \mathbf{r}_3$, etc. The centre of mass G_1 of the masses m_1 and m_2 is on the straight line $\overline{AB}$ such that $\overline{AG_1} = \dfrac{m^2}{m_1 + m_2}\overline{AB}$.

Now, in vector notation, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{r}_2 - \mathbf{r}_1$

Hence, $$\overrightarrow{AG_1} = \frac{m_2}{m_1 + m_2}(\mathbf{r}_2 - \mathbf{r}_1)$$

Also, $$\overrightarrow{AG_1} = \overrightarrow{OG_1} - \overrightarrow{OA} = \overrightarrow{OG_1} - \mathbf{r}_1$$

so that $$\overrightarrow{OG_1} - \mathbf{r}_1 = \frac{m_2}{m_1 + m_2}(\mathbf{r}_2 - \mathbf{r}_1)$$

$\therefore$ $$\overrightarrow{OG_1} = \mathbf{r}_1 + \frac{m_2}{m_1 + m_2}(\mathbf{r}_2 - \mathbf{r}_1)$$

i.e. $$\overrightarrow{OG_1} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \quad . \quad . \quad . \quad . \quad \text{(V.36)}$$

If G_2 is the centre of mass of masses $m_1 + m_2$ at G_1 and m_3 at C, then by (V.36),

$$\overrightarrow{OG_2} = \frac{(m_1 + m_2)\overrightarrow{OG_1} + m_3\mathbf{r}_3}{(m_1 + m_2) + m_3}$$

From (V.36),

$$(m_1 + m_2)\overrightarrow{OG_1} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2$$

Hence $$\overrightarrow{OG_2} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3}{m_1 + m_2 + m_3} \quad . \quad . \quad \text{(V.37)}$$

By repeated applications of this procedure the position of the centre of mass G of the whole system of masses is found to be given by the relation

$$\overrightarrow{OG} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

or $$\overrightarrow{OG} = \frac{\Sigma m\mathbf{r}}{\Sigma m} \quad . \quad . \quad . \quad . \quad . \quad \text{(V.38)}$$

The formula (V.38) is equivalent to the three formulae (V.40) below. Let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of G referred to rectangular axes OX, OY, OZ, and let (x, y, z) be the co-ordinates of any one of the masses m. Also let r be the distance of m from O.

Then, by (V.29), $$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and $$\overrightarrow{OG} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$$

Substitution in (V.38) gives

$$\bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k} = \frac{\Sigma m(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{\Sigma m}$$

i.e. $$\bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k} = \frac{(\Sigma mx)\mathbf{i}}{\Sigma m} + \frac{(\Sigma my)\mathbf{j}}{\Sigma m} + \frac{(\Sigma mz)\mathbf{k}}{\Sigma m} \quad . \quad \text{(V.39)}$$

Since the vector on the left-hand side of (V.39) is equal to that on the right-hand side, these vectors have equal components parallel to the axes of reference.

Thus, and
$$\left.\begin{aligned} \bar{x} &= \frac{\Sigma mx}{\Sigma m} \\ \bar{y} &= \frac{\Sigma my}{\Sigma m} \\ \bar{z} &= \frac{\Sigma mz}{\Sigma m} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad \text{(V.40)}$$

EXAMPLE 1

Masses of 5, 2, 3, 6 lb are situated at the points (1, 0, 1), (− 2, 3, 4), (5, − 2, − 1), (4, 2, 3) respectively. Find the co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the centre of mass of the given system of masses.

From (V.40), $\bar{x} = \dfrac{\Sigma mx}{\Sigma m} = \dfrac{5 \times 1 + 2 \times -2 + 3 \times 5 + 6 \times 4}{5 + 2 + 3 + 6}$

$$= \frac{5 - 4 + 15 + 24}{16}$$

$$= \frac{40}{16}$$

$$\therefore \quad \bar{x} = 2{\cdot}5$$

$$\bar{y} = \frac{\Sigma my}{\Sigma m} = \frac{5 \times 0 + 2 \times 3 + 3 \times -2 + 6 \times 2}{16}$$

$$= \frac{6 - 6 + 12}{16}$$

$$= \frac{12}{16}$$

$$\therefore \quad \bar{y} = 0{\cdot}75$$

$$\bar{z} = \frac{\Sigma mz}{\Sigma m} = \frac{5 \times 1 + 2 \times 4 + 3 \times -1 + 6 \times 3}{16}$$

$$= \frac{5 + 8 - 3 + 18}{16}$$

$$= \frac{28}{16}$$

$$\therefore \quad \bar{z} = 1{\cdot}75$$

$$\therefore \quad \bar{x} = 2{\cdot}5,\ \bar{y} = 0{\cdot}75, \text{ and } \bar{z} = 1{\cdot}75$$

EXAMPLE 2

If G is the centre of mass of masses $m_1, m_2, m_3, \ldots$ situated at the points $A, B, C, \ldots$ respectively, and O is any point in space, prove that

$$m_1OA^2 + m_2OB^2 + m_3OC^2 + \ldots$$
$$= m_1GA^2 + m_2GB^2 + m_3GC^2 + \ldots + (m_1 + m_2 + m_3 + \ldots)OG^2$$

In vector notation, $\overrightarrow{OA} = \overrightarrow{OG} + \overrightarrow{GA}$

Squaring and multiplying through by m_1,

$$m_1\overrightarrow{OA}^2 = m_1\overrightarrow{OG}^2 + 2m_1\overrightarrow{OG} \cdot \overrightarrow{GA} + m_1\overrightarrow{GA}^2$$

Similarly, $m_2\overrightarrow{OB}^2 = m_2\overrightarrow{OG}^2 + 2m_2\overrightarrow{OG} \cdot \overrightarrow{GB} + m_2\overrightarrow{GB}^2$

$$m_3\overrightarrow{OC}^2 = m_3\overrightarrow{OG}^2 + 2m_3\,\overrightarrow{OG}{\cdot}\overrightarrow{GC} + m_3\overrightarrow{GC}^2$$

and so on.

By addition,

$$m_1\overrightarrow{OA}^2 + m_2\overrightarrow{OB}^2 + m_3\overrightarrow{OC}^2 + \ldots$$
$$= (m_1 + m_2 + m_3 + \ldots)\overrightarrow{OG}^2 + 2\overrightarrow{OG}\cdot(m_1\overrightarrow{GA} + m_2\overrightarrow{GB} + m_3\overrightarrow{GC} + \ldots)$$
$$+ m_1\overrightarrow{GA}^2 + m_2\overrightarrow{GB}^2 + m_3\overrightarrow{GC}^2 + \ldots$$

If in (V.38), O and G are assumed to coincide, then $\Sigma m\mathbf{r} = 0$, so that

$$m_1\overrightarrow{GA} + m_2\overrightarrow{GB} + m_3\overrightarrow{GC} + \ldots = 0$$

Also by (V.12), $\overrightarrow{OA}^2 = OA^2$, $\overrightarrow{OB}^2 = OB^2$, and so on.

Hence, $m_1OA^2 + m_2OB^2 + m_3OC^2 + \ldots$
$= m_1GA^2 + m_2GB^2 + m_3GC^2 + \ldots + (m_1 + m_2 + m_3 + \ldots)OG^2$

43. **Vector Product.** The vector product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted by $\boldsymbol{a} \times \boldsymbol{b}$ (alternatively by $V\boldsymbol{ab}$ or $[\boldsymbol{ab}]$), and is defined as

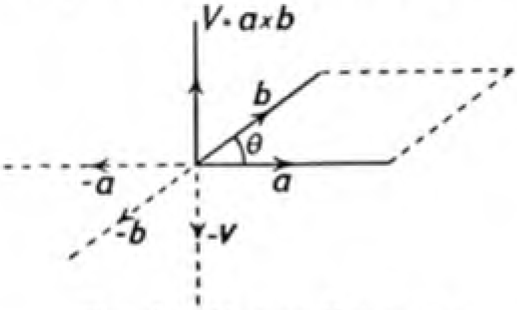


FIG. 34. VECTOR PRODUCT

the vector, say $\boldsymbol{V}$, which is normal to the plane of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and whose magnitude V is equal to $ab \sin \theta$, where θ is the angle between the directions of the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the sense of the direction of $\boldsymbol{V}$ being that in which a right-handed screw with its axis in that direction would move when rotated in a fixed nut in the sense in which the angle θ is swept out from $\boldsymbol{a}$ to $\boldsymbol{b}$ (Fig. 34). It is clear from the figure that the numerical valve of $\boldsymbol{V}$ is equal to the area of the parallelogram constructed with $\boldsymbol{a}$ and $\boldsymbol{b}$ as adjacent sides.

Thus, $$\boldsymbol{V} = \boldsymbol{a} \times \boldsymbol{b} \quad . \quad . \quad . \quad \text{(V.41)}$$

and $$V = ab \sin \theta \quad . \quad . \quad . \quad \text{(V.42)}$$

The numerical value of the vector product $\boldsymbol{b} \times \boldsymbol{a}$ is $ba \sin \theta$, the same as that of $\boldsymbol{a} \times \boldsymbol{b}$, but, since rotation in the sense $\boldsymbol{b}$ to $\boldsymbol{a}$ would move a right-handed screw in the sense opposite to that of $\boldsymbol{V}$ (see Fig. 34), then

$$\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{V}$$

i.e. $$\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b} \quad . \quad . \quad . \quad \text{(V.43)}$$

Thus, the *commutative law* does *not* hold good for vector products. $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{b} \times \boldsymbol{a}$ are vectors which are equal in numerical magnitude but opposite in sign.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same direction and sense, $\theta = 0°$, and if $\boldsymbol{a}$ and $\boldsymbol{b}$ have opposite senses, $\theta = 180°$. In either case $\sin\theta = 0$, so that

$$\boldsymbol{a} \times \boldsymbol{b} = 0 \quad . \quad . \quad . \quad (V.44)$$

This is thus the condition that the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (neither of which is assumed zero) have the same direction and the same sense or opposite senses.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are at right angles, $\theta = 90°$ and $\sin\theta = 1$, so that

$$V = ab \quad . \quad . \quad . \quad . \quad (V.45)$$

It is evident from the definition of vector product that the senses of the vectors $(-\boldsymbol{a}) \times \boldsymbol{b}$ and $\boldsymbol{a} \times (-\boldsymbol{b})$ are opposite to that of $\boldsymbol{a} \times \boldsymbol{b}$, while the sense of $(-\boldsymbol{a}) \times (-\boldsymbol{b})$ is the same as that of $\boldsymbol{a} \times \boldsymbol{b}$ (see Fig. 34).

Now, all these products have the same numerical magnitude $ab \sin\theta$.

Hence

$$(-\boldsymbol{a}) \times \boldsymbol{b} = -\boldsymbol{a} \times \boldsymbol{b}$$
$$\boldsymbol{a} \times (-\boldsymbol{b}) = -\boldsymbol{a} \times \boldsymbol{b}$$

and

$$(-\boldsymbol{a}) \times (-\boldsymbol{b}) = \boldsymbol{a} \times \boldsymbol{b}$$

so that the rule of signs of ordinary algebra applies to vector products.

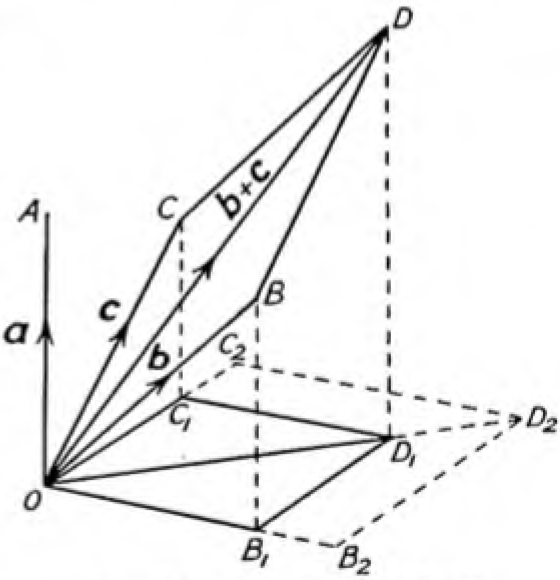


FIG. 35. DISTRIBUTIVE LAW

44. Distributive Law for Vector Products. Let $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$, and $\overrightarrow{OC} = \boldsymbol{c}$ be three vectors which do not lie in one plane. Complete the parallelogram $OBDC$ and join OD, Fig. 35.

Then, $$\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{OC} = \boldsymbol{b} + \boldsymbol{c}$$

Let $OB_1D_1C_1$ be the orthogonal projection of parallelogram $OBDC$ on the plane through O which is perpendicular to the vector $\boldsymbol{a}$.

Since equal and parallel straight lines project into equal and parallel straight lines, the figure $OB_1D_1C_1$, is a parallelogram. Increase the linear dimensions of the figure $OB_1D_1C_1$ in the ratio $a:1$, forming the parallelogram $OB_2D_2C_2$.

The numerical magnitudes of $\boldsymbol{a} \times \boldsymbol{b}$, $\boldsymbol{a} \times \boldsymbol{c}$, and $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c})$ are respectively $ab \sin \widehat{AOB}$, $ac \sin \widehat{AOC}$, and $a \times OD \times \sin \widehat{AOD}$

i.e. $\quad a \times OB_1$, $a \times OC_1$, and $a \times OD_1$

i.e. $\quad OB_2$, OC_2, and OD_2

Suppose now that the figure $OB_2D_2C_2$ is rotated through 90° in its own plane. Then $\overrightarrow{OB_2}$ in its new position will represent the vector product $\boldsymbol{a} \times \boldsymbol{b}$, for $\overrightarrow{OB_2}$ has the same direction and the same numerical magnitude as $\boldsymbol{a} \times \boldsymbol{b}$. Similarly, $\overrightarrow{OC_2}$ and $\overrightarrow{OD_2}$ in their new positions will represent the vector products $\boldsymbol{a} \times \boldsymbol{c}$ and $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c})$ respectively.

Hence, since $\quad \overrightarrow{OD_2} = \overrightarrow{OB_2} + \overrightarrow{OC_2}$

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c} \quad . \quad . \quad \text{(V.46)}$$

Thus, the *distributive law* holds good for vector products. The above argument is general and includes the particular case of three coplanar vectors, the figure $OB_1D_1C_1$ being in this case a collapsed parallelogram. It is left as an exercise for the reader to find an alternative proof of (V.46) when the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ lie in the same plane.

(V.46) can be readily extended to the vector product of two vectors each of which consists of the sum of a number of vectors, the method being similar to that employed in the case of scalar products.

Thus, $$(\boldsymbol{a} + \boldsymbol{b}) \times (\boldsymbol{c} + \boldsymbol{d}) = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{a} \times \boldsymbol{d} + \boldsymbol{b} \times \boldsymbol{d} \quad \text{(V.47)}$$

It is important to note that, as the commutative law does not apply to vector products, the order in which the terms of a product appear must remain unaltered throughout the operations unless the necessary change of sign has been made.

For example, it would be incorrect to write (V.46) as

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{c} \times \boldsymbol{a}, \text{ since } \boldsymbol{c} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{c}$$

45. **Free and Localized Vectors.** If a vector is completely specified by (1) its magnitude, (2) its direction, and (3) its sense, it is a *free vector*. The sum of a number of free vectors is found by taking them in any convenient order and joining the starting point of one to the end point of another until they are all joined to form a continuous track made up of straight lines with the arrows all pointing the same way along the track. The sum of the free vectors is the vector joining the first starting point to the last end point with the arrow pointing to the latter. If any of the vectors are to be subtracted, the senses of these are reversed and the vectors are then added. Changing the order of two consecutive vectors does not affect the sum vector, as it changes the two vectors for equivalent vectors forming the two sides of a parallelogram opposite to those formed by the two original vectors, and so gives the same end point. As the order of the vectors can be varied in all possible ways by changing the order of a sufficient number of consecutive pairs, it follows that the sum is independent of the order in which the vectors are taken. Changing the order involves moving vectors parallel to themselves. If, however, we are dealing with forces acting on a rigid body, then the moving of a force parallel to itself alters the turning effect on the body. In such a case the vector representing the force cannot be completely specified by (1), (2), and (3) above. In addition, its point of application, i.e. a point on its line of action, must be given. A vector whose line of action is fixed is a *localized vector*. In general, forces are localized vectors; it is only when the forces all meet in a point that they may be treated as free vectors.

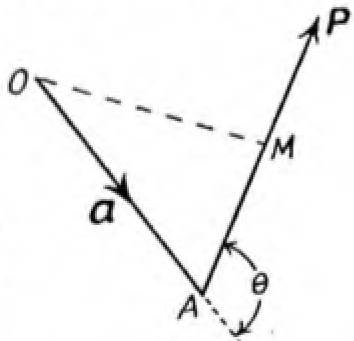


FIG. 36
LOCALIZED VECTOR

46. **Moment of a Localized Vector.** Let $\boldsymbol{P}$ be a localized vector, O a fixed point, and A any point on the line of action of $\boldsymbol{P}$. Let $\overrightarrow{OA} = \boldsymbol{a}$ (Fig. 36). Draw OM perpendicular to the line of action of $\boldsymbol{P}$, and let θ be the angle between the directions of $\boldsymbol{P}$ and $\boldsymbol{a}$. Consider the vector product $\boldsymbol{a} \times \boldsymbol{P}$. This product is a vector whose direction is perpendicular to the plane containing $\boldsymbol{P}$ and $\boldsymbol{a}$, or, in other words, where direction is that of the axis about which the vector $\boldsymbol{P}$ tends to produce rotation, and whose sense is given by the rule in Art. 43. Further, the numerical magnitude of $\boldsymbol{a} \times \boldsymbol{P}$ is $Pa \sin \theta$, i.e. $P \times \overline{OM}$, which is the numerical measure of the moment of P about O.

Thus, the vector product $\boldsymbol{a} \times \boldsymbol{P}$ represents very compactly the moment of the vector $\boldsymbol{P}$ about the point O.

If $\boldsymbol{P}$ represents a force and $\boldsymbol{a}$ a position vector, the vector $\boldsymbol{a} \times \boldsymbol{P}$ represents the moment of this force about O.

Let $\boldsymbol{R}$ be the resultant of several forces $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}_3$, etc., acting at the same point A. Then, if O is any point distant a from A and $\overrightarrow{OA}$ is denoted by $\boldsymbol{a}$, the moment of the resultant $\boldsymbol{R}$ about O

$$\begin{aligned}
&= \boldsymbol{a} \times \boldsymbol{R} \\
&= \boldsymbol{a} \times (\boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{P}_3 + \dots) \\
&= \boldsymbol{a} \times \boldsymbol{P}_1 + \boldsymbol{a} \times \boldsymbol{P}_2 + \boldsymbol{a} \times \boldsymbol{P}_3 + \dots \\
&= \text{sum of moments of the several forces about } O
\end{aligned}$$

47. **Unit Vectors.** It follows from (V.42) that

$$\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = 0 \qquad \text{(V.48)}$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the unit vectors defined in Art. 41 and represented in Fig. 33.

Also, from the definition of vector product,

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k};\ \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i};\ \text{and } \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j} \qquad \text{(V.49)}$$

the unit vectors being at right angles to one another.

Further, $\quad \boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k};\ \boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i};\ \text{and } \boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$ (V.50)

Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$ be two vectors in space, A and B being the points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, and O the origin of co-ordinates.

Then, by (V.29), $\quad \boldsymbol{a} = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$

and $\quad \boldsymbol{b} = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$

$$\begin{aligned}
\therefore \boldsymbol{a} \times \boldsymbol{b} &= (x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}) \times (x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}) \\
&= x_1x_2\boldsymbol{i} \times \boldsymbol{i} + y_1y_2\boldsymbol{j} \times \boldsymbol{j} + z_1z_2\boldsymbol{k} \times \boldsymbol{k} \\
&\quad + (x_1y_2\boldsymbol{i} \times \boldsymbol{j} + x_2y_1\boldsymbol{j} \times \boldsymbol{i}) + (z_2x_1\boldsymbol{i} \times \boldsymbol{k} + z_1x_2\boldsymbol{k} \times \boldsymbol{i}) \\
&\quad + (y_1z_2\boldsymbol{j} \times \boldsymbol{k} + y_2z_1\boldsymbol{k} \times \boldsymbol{j})
\end{aligned}$$

Using the relations (V.48), (V.49), and (V.50),

$$\boldsymbol{a} \times \boldsymbol{b} = (y_1z_2 - y_2z_1)\boldsymbol{i} + (z_1x_2 - z_2x_1)\boldsymbol{j} + (x_1y_2 - x_2y_1)\boldsymbol{k} \qquad \text{(V.51)}$$

or in determinant form,

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \quad . \quad . \quad . \quad \text{(V.52)}$$

Note that, if the parallelogram having OA, OB as adjacent sides is projected on to the co-ordinate planes, the areas of these projections are given by the components $y_1z_2 - y_2z_1$, $z_1x_2 - z_2x_1$, and $x_1y_2 - x_2y_1$ on the right-hand side of (V.51).

EXAMPLE

A force $\boldsymbol{P} = 4\boldsymbol{i} - 3\boldsymbol{k}$ passes through the point A whose position vector is $2\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$. Find the moment of $\boldsymbol{P}$ about the point B whose position vector is $\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$. (Force in lb-weight, distance in feet.)

The vector $\overrightarrow{BA} = (2\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}) - (\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}) = \boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$.

Then, the moment of $\boldsymbol{P}$ about $B = \overrightarrow{BA} \times \boldsymbol{P}$

$$= (\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}) \times (4\boldsymbol{i} - 3\boldsymbol{k})$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 4 \\ 4 & 0 & -3 \end{vmatrix} \quad \text{[by (V.52)]}$$

$$= -3\boldsymbol{i} + 19\boldsymbol{j} - 4\boldsymbol{k}$$

Magnitude of moment $= \sqrt{(9 + 361 + 16)} = 19{\cdot}65$ ft.

48. **Scalar and Vector Triple Products.** The vector product $\boldsymbol{b} \times \boldsymbol{c}$ is a vector which may be combined with another vector $\boldsymbol{a}$ to form a *scalar triple product* $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ or a *vector triple product* $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$. Consider first the scalar product $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$, which is usually denoted by $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$.

Let the components of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ parallel to rectangular axes of reference OX, OY, OZ be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) respectively.

Then, by (V.52), $\quad \boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

$$\therefore \qquad \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} \boldsymbol{a} \cdot \boldsymbol{i} & \boldsymbol{a} \cdot \boldsymbol{j} & \boldsymbol{a} \cdot \boldsymbol{k} \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Now, by (V.30), $\boldsymbol{a} \cdot \boldsymbol{i} = x_1$, $\boldsymbol{a} \cdot \boldsymbol{j} = y_1$, and $\boldsymbol{a} \cdot \boldsymbol{k} = z_1$, so that

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \Delta, \text{ say}$$

It is proved similarly that $\boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \Delta$ and $\boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \Delta$.

Thus, $$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) \quad . \quad \text{(V.53)}$$

or in the notation indicated above,

$$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}] = [\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}] \quad . \quad . \quad \text{(V.54)}$$

Note that in (V.54) the letters preserve cyclic order. If that order is altered, a change of sign in the product results. For example, $[\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{b}] = \boldsymbol{a} \cdot (\boldsymbol{c} \times \boldsymbol{b})$, and by (V.43), $\boldsymbol{c} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{c}$, so that $[\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{b}] = \boldsymbol{a} \cdot (-\boldsymbol{b} \times \boldsymbol{c}) = -[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$.

The determinant Δ above gives the volume of the parallelepiped having one corner at the origin of co-ordinates in terms of the co-ordinates of the other ends of the edges which meet at the origin. Thus, $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ = volume of parallelepiped having a, b, c as coterminous edges. This is established independently as follows. Let the parallelogram having the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ as adjacent sides be denoted by the parallelogram $\boldsymbol{bc}$ and the parallelepiped having $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ as coterminous edges by the parallelepiped $\boldsymbol{abc}$.

The numerical magnitude of the vector product $\boldsymbol{b} \times \boldsymbol{c}$ is equal to the area of the parallelogram bc, and the direction of this vector is perpendicular to the plane of $\boldsymbol{b}$ and $\boldsymbol{c}$.

From the definition of scalar product,

$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ = (numerical magnitude of $\boldsymbol{b} \times \boldsymbol{c}$) $\times$ (projection of $\boldsymbol{a}$ on $\boldsymbol{b} \times \boldsymbol{c}$)

= (area of parallelogram $\boldsymbol{bc}$) $\times$ (height of parallelepiped $\boldsymbol{abc}$)

= volume of parallelepiped $\boldsymbol{abc}$

Similarly, it is proved that each of the scalar triple products $\boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a})$ and $\boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b})$ is equal to the volume of this parallelepiped, and the relation (V.54) is again established.

Consider now the vector triple product $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$.

In terms of the components of the vectors parallel to the axes of reference, as above,

$$\boldsymbol{b} \times \boldsymbol{c} = (y_2z_3 - y_3z_2)\boldsymbol{i} + (z_2x_3 - z_3x_2)\boldsymbol{j} + (x_2y_3 - x_3y_2)\boldsymbol{k}$$

[from (V.51)]

from which it is seen that the components of $\boldsymbol{b} \times \boldsymbol{c}$ parallel to the axes of reference are $y_2z_3 - y_3z_2$, $z_2x_3 - z_3x_2$, and $x_2y_3 - x_3y_2$ respectively.

Hence, from (V.52),

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & z_1 \\ y_2z_3 - y_3z_2 & z_2x_3 - z_3x_2 & x_2y_3 - x_3y_2 \end{vmatrix}$$

$$= \Sigma\boldsymbol{i}\{y_1(x_2y_3 - x_3y_2) - z_1(z_2x_3 - z_3x_2)\}$$

$$= \Sigma\{\boldsymbol{i}x_2(x_1x_3 + y_1y_3 + z_1z_3) - \boldsymbol{i}x_3(x_1x_2 + y_1y_2 + z_1z_2)\}$$

Now, by (V.31), $x_1x_3 + y_1y_3 + z_1z_3 = \boldsymbol{a} \cdot \boldsymbol{c}$

and $x_1x_2 + y_1y_2 + z_1z_2 = \boldsymbol{a} \cdot \boldsymbol{b}$

Hence, $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \Sigma\{\boldsymbol{i}x_2 \cdot (\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{i}x_3 \cdot (\boldsymbol{a} \cdot \boldsymbol{b})\}$

$$= (x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}) \cdot (\boldsymbol{a} \cdot \boldsymbol{c}) - (x_3\boldsymbol{i} + y_3\boldsymbol{j} + z_3\boldsymbol{k}) \cdot (\boldsymbol{a} \cdot \boldsymbol{b})$$

i.e. $$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c} \cdot (\boldsymbol{a} \cdot \boldsymbol{b}) \quad . \quad \text{(V.55)}$$

Similarly, $$\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{b} \cdot \boldsymbol{a}) - \boldsymbol{a} \cdot (\boldsymbol{b} \cdot \boldsymbol{c}) \quad . \quad \text{(V.56)}$$

and $$\boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a} \cdot (\boldsymbol{c} \cdot \boldsymbol{b}) - \boldsymbol{b} \cdot (\boldsymbol{c} \cdot \boldsymbol{a}) \quad . \quad \text{(V.57)}$$

By addition of corresponding sides of (V.55), (V.56), and (V.57),

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = 0 \quad \text{(V.58)}$$

Note that, since the commutative law holds good for scalar products but not for vector products, $\boldsymbol{b} \cdot (\boldsymbol{a} \cdot \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \cdot \boldsymbol{a})$, for example, but $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq \boldsymbol{a} \times (\boldsymbol{c} \times \boldsymbol{b})$.

Further, note that $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ is not equal to $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$. For $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ is a vector whose direction is perpendicular to the directions of $\boldsymbol{a}$ and $\boldsymbol{b} \times \boldsymbol{c}$, and the direction of $\boldsymbol{b} \times \boldsymbol{c}$ is perpendicular to the plane of $\boldsymbol{b}$ and $\boldsymbol{c}$, so that $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ lies in the plane of $\boldsymbol{b} \times \boldsymbol{c}$. Similarily, $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ lies in the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$. In general, therefore, $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ and $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ are different vectors.

49. **Vector Representation of Rotational Motion.** Angular displacement, angular velocity, angular acceleration, couple, and angular momentum are vector quantities. First consider angular displacement. The direction associated with it is that of the axis of

rotation. As there are two senses of rotation, clockwise and anti-clockwise, there will be two senses of direction along the axis of rotation. Fig. 37 shows how these senses are related. In each case the curved arrow shows the sense of rotation in the angular displacement, and the arrow on the axis shows the sense of the vector. The length of the vector represents the magnitude of the displacement. If the closed curve be regarded as representing the head of a right-handed screw of which the threaded part (the axis) is screwed into a piece of wood, then the arrow on the vector shows the sense of motion of the screw into or out of the wood, according as the screw, viewed from above, is turned clockwise or anti-clockwise.

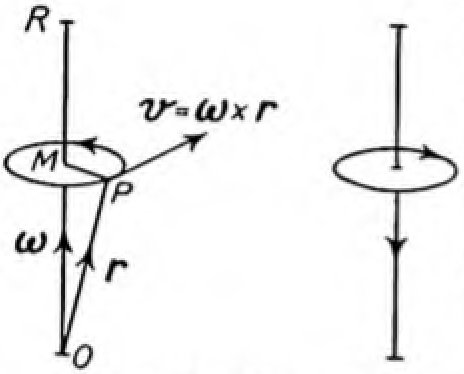


FIG. 37. VECTORS REPRESENTING ROTATIONS

If the angular displacement takes place at a uniform rate and in unit time, it becomes angular velocity. If the angular velocity is an increase of angular velocity in unit time, it becomes angular acceleration. Thus angular velocity and angular acceleration are vectors along the axis of rotation.

A couple $\mathbf{C}$ acting on a body of moment of inertia I about the axis of rotation produces an angular acceleration $\boldsymbol{\alpha}$, where $\boldsymbol{\alpha} = \frac{1}{I}\mathbf{C}$ and as $\frac{1}{I}$ is scalar, $\mathbf{C}$ is a vector with the same direction and sense as $\boldsymbol{\alpha}$.

Consider a rigid body rotating about an axis OR (Fig. 37) with angular velocity ω radn per sec. This angular velocity can be completely specified by a vector $\boldsymbol{\omega}$ of magnitude ω whose direction is that of the axis OR and whose sense is that given by the rule stated above. Let O be a point on the axis of rotation, and let any point P in the body be located by the vector $\overrightarrow{OP} = \mathbf{r}$. Also let $\mathbf{v}$ be the velocity of P.

Consider the vector product $\boldsymbol{\omega} \times \mathbf{r}$. Its numerical magnitude is $\omega r \sin \widehat{POR}$, i.e. $\omega \times \overline{PM}$, where $\overline{PM}$ is the perpendicular from P on OR, and its direction is perpendicular to the plane POR, as shown in Fig. 37.

Now $\mathbf{v}$ has magnitude $\omega \times \overline{PM}$, and its direction and sense are the same as those of $\boldsymbol{\omega} \times \mathbf{r}$.

Hence, $$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \quad . \quad . \quad . \quad (V.59)$$

The angular momentum, or moment of momentum, relative to the axis of rotation is $I\boldsymbol{\omega}$, where I is the moment of inertia of the body about that axis, and the vector $I\boldsymbol{\omega}$ has the same direction and sense as $\boldsymbol{\omega}$.

EXAMPLE

A rigid body is rotating at 5 radn/sec about an axis OR, where R is the point $3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}$ relative to O. Find the velocity of the particle of the body at the point $\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$. (All lengths are in feet.)

The distance $OR = \sqrt{3^2 + 6^2 + 2^2} = 7$ and the direction-cosines of OR are $\frac{3}{7}, \frac{6}{7}, \frac{-2}{7}$ respectively, so that the components of the angular velocity $\boldsymbol{\omega}$ of the body along the axes of reference are $5 \times \frac{3}{7}, 5 \times \frac{6}{7}, 5 \times -\frac{2}{7}$ respectively.

Hence
$$\boldsymbol{\omega} = \frac{5}{7}(3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k})$$

The position vector $\boldsymbol{r}$ of the particle is given by
$$\boldsymbol{r} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$$

The velocity $\boldsymbol{v}$ of the particle is then given by
$$\begin{aligned}\boldsymbol{v} &= \boldsymbol{\omega} \times \boldsymbol{r}\\ &= \frac{5}{7}(3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}) \times (\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})\\ &= \frac{5}{7}(4\boldsymbol{i} - 5\boldsymbol{j} - 9\boldsymbol{k})\end{aligned}$$

Hence, the velocity required has magnitude $\frac{5}{7}\sqrt{4^2 + 5^2 + 9^2}$, i.e. 7·89 ft/sec and its direction is that of the vector $4\boldsymbol{i} - 5\boldsymbol{j} - 9\boldsymbol{k}$.

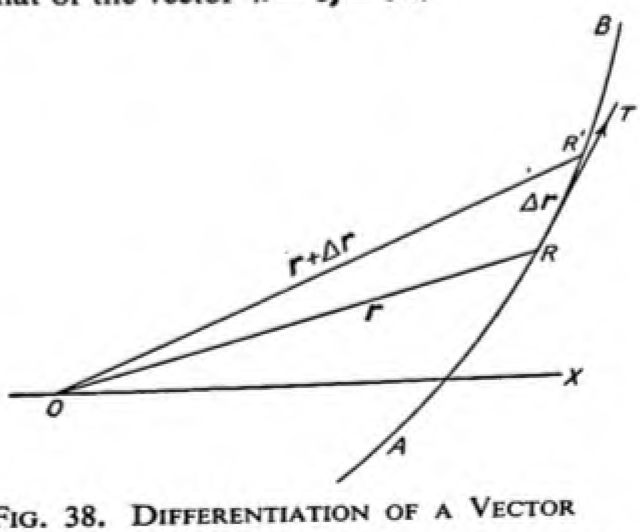


FIG. 38. DIFFERENTIATION OF A VECTOR

50. **Differentiation of a Vector.** Let the vector $\overrightarrow{OR} = \boldsymbol{r}$, Fig. 38, define the position of a point R relative to a fixed origin O, and let R

be assumed to move in a path ARB in one plane such that $\boldsymbol{r}$ is a continuous and single-valued function of a scalar variable s. If R moves to the position R' when s receives an increment Δs, and, therefore, $\boldsymbol{r}$ receives an increment $\Delta\boldsymbol{r}$, then

$$\overrightarrow{OR'} = \boldsymbol{r} + \Delta\boldsymbol{r}$$

$$\text{The change in } \boldsymbol{r} = \overrightarrow{OR'} - \overrightarrow{OR} = \overrightarrow{RR'}$$

i.e.

$$\Delta\boldsymbol{r} = \overrightarrow{RR'}$$

The average rate of change of $\boldsymbol{r}$ with respect to s in the displacement $\overrightarrow{RR'}$ is $\dfrac{\Delta\boldsymbol{r}}{\Delta s}$.

Note that, since $\Delta\boldsymbol{r}$ is a vector and Δs a scalar quantity, $\dfrac{\Delta\boldsymbol{r}}{\Delta s}$ is a vector and its direction and sense are those of $\overrightarrow{RR'}$. In the limit when Δs approaches the value zero, the direction of $\overrightarrow{RR'}$ approaches that of RT the tangent at R to the path ARB. The limiting value of $\dfrac{\Delta\boldsymbol{r}}{\Delta s}$ is denoted by $\dfrac{d\boldsymbol{r}}{ds}$, which is called the derivative (strictly, the first derivative) or differential coefficient of the vector $\boldsymbol{r}$ with respect to s.

Thus,

$$\frac{d\boldsymbol{r}}{ds} = \underset{\Delta s \to 0}{Lt.} \frac{\Delta\boldsymbol{r}}{\Delta s} \quad . \quad . \quad . \quad (V.60)$$

Note that $\dfrac{d\boldsymbol{r}}{ds}$ is a vector whose direction and sense are those of the vector $\overrightarrow{RT}$, where RT is the tangent at R to the path ARB.

Just as in algebraic calculus, $\boldsymbol{r}$ may possess a second derivative $\dfrac{d^2\boldsymbol{r}}{ds^2}$, a third derivative $\dfrac{d^3\boldsymbol{r}}{ds^3}$, and so on, all of which (if they exist) are vectors.

If in Fig. 38 s denotes the length of arc of the path measured from some fixed point A up to R, then

$$\Delta s = \text{length of arc } RR'$$

Now, from above, $\overrightarrow{RR'} = \Delta\boldsymbol{r}$, and, since in the limit when Δs approaches the value zero, the ratio of the length of the chord RR' to that of the arc RR' approaches the value unity, the limiting value of $\dfrac{\Delta\boldsymbol{r}}{\Delta s}$ in this case is a unit vector $\boldsymbol{\tau}$, say, in the direction of the tangent at R to the path ARB.

Thus, from (V. 60),

$$\frac{d\mathbf{r}}{ds} = \boldsymbol{\tau} \quad . \quad . \quad . \quad . \quad (V.61)$$

The following results are readily established—
If $\mathbf{r}$ is a constant vector,

$$\frac{d\mathbf{r}}{ds} = 0 \quad . \quad . \quad . \quad . \quad (V.62)$$

The derivative of the vector sum $\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \ldots$ is given by

$$\frac{d}{ds}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \ldots) = \frac{d\mathbf{r}_1}{ds} + \frac{d\mathbf{r}_2}{ds} + \frac{d\mathbf{r}_3}{ds} + \ldots \quad (V.63)$$

If the scalar variable s is itself a function of another scalar variable u, then

$$\frac{d\mathbf{r}}{du} = \frac{d\mathbf{r}}{ds} \cdot \frac{ds}{du} \quad . \quad . \quad . \quad (V.64)$$

The derivative of the scalar product $\mathbf{r}_1 \cdot \mathbf{r}_2$ is given by

$$\frac{d}{ds}(\mathbf{r}_1 \cdot \mathbf{r}_2) = \mathbf{r}_1 \cdot \frac{d\mathbf{r}_2}{ds} + \frac{d\mathbf{r}_1}{ds} \cdot \mathbf{r}_2 \quad . \quad . \quad (V.65)$$

The derivative of the vector product $\mathbf{r}_1 \times \mathbf{r}_2$ is given by

$$\frac{d}{ds}(\mathbf{r}_1 \times \mathbf{r}_2) = \mathbf{r}_1 \times \frac{d\mathbf{r}_2}{ds} + \frac{d\mathbf{r}_1}{ds} \times \mathbf{r}_2 \quad . \quad (V.66)$$

Note that in this last relation the order of the factors $\mathbf{r}_1$ and $\mathbf{r}_2$ in the terms must remain the same throughout, unless the necessary change of sign has been made.

If in (V.60) the scalar variable s denotes time t, i.e. $s = t$, then (V.60) becomes

$$\frac{d\mathbf{r}}{dt} = \underset{\Delta t \to 0}{Lt.} \frac{\Delta \mathbf{r}}{\Delta t} \quad . \quad . \quad . \quad (V.67)$$

$\frac{d\mathbf{r}}{dt}$ gives the instantaneous velocity of R, and its direction is along the tangent at R to the path of R.

$\frac{d^2\mathbf{r}}{dt^2}$ in this case gives the instantaneous acceleration of R.

s is a scalar function of any other scalar function which defines s.

Thus $\mathbf{r}$ is a function of r, $\theta = \widehat{ROX}$, time t, or any other independent variable that defines s. By (V.64) we have $\dfrac{d\mathbf{r}}{dr} = \dfrac{d\mathbf{r}}{ds} \cdot \dfrac{ds}{dr}$, $\dfrac{d\mathbf{r}}{d\theta} = \dfrac{d\mathbf{r}}{ds} \cdot \dfrac{ds}{d\theta}$, $\dfrac{d\mathbf{r}}{dt} = \dfrac{d\mathbf{r}}{ds} \cdot \dfrac{ds}{dt} = v\,\dfrac{d\mathbf{r}}{ds}$ where v is the speed of the particle at R.

If $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$, then the scalar product $\mathbf{r}_1 \cdot \mathbf{r}_2$ becomes $\mathbf{r}^2$, which by (V.12) is equal to r^2.

Substitution in (V.65) gives

$$\frac{d}{ds}(\mathbf{r}^2) = \mathbf{r} \cdot \frac{d\mathbf{r}}{ds} + \frac{d\mathbf{r}}{ds} \cdot \mathbf{r}$$

i.e.

$$\frac{d}{ds}(\mathbf{r}^2) = 2\mathbf{r} \cdot \frac{d\mathbf{r}}{ds}$$

Now $\dfrac{d}{ds}(\mathbf{r}^2) = \dfrac{d}{ds}(r^2) = 2r\,\dfrac{dr}{ds}$, so that it follows that

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{ds} = r\,\frac{dr}{ds} \quad . \quad . \quad . \quad . \quad \text{(V.68)}$$

EXAMPLE

Show that (i) $\dfrac{d}{dt}\left(\mathbf{r} \times \dfrac{d\mathbf{r}}{dt}\right) = \mathbf{r} \times \dfrac{d^2\mathbf{r}}{dt^2}$, and (ii) if $\mathbf{r} = \mathbf{a} \sin \omega t + \mathbf{b} \cos \omega t$, where $\mathbf{a}$, $\mathbf{b}$, ω are constants, then $\dfrac{d^2\mathbf{r}}{dt^2} = -\,\omega^2\mathbf{r}$ and $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt} = -\,\omega\mathbf{a} \times \mathbf{b}$.

(i) By (V.66), $\quad \dfrac{d}{dt}\left(\mathbf{r} \times \dfrac{d\mathbf{r}}{dt}\right) = \mathbf{r} \times \dfrac{d^2\mathbf{r}}{dt^2} + \dfrac{d\mathbf{r}}{dt} \times \dfrac{d\mathbf{r}}{dt}$

Since by (V.44) the term $\dfrac{d\mathbf{r}}{dt} \times \dfrac{d\mathbf{r}}{dt}$ is zero, then

$$\frac{d}{dt}\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) = \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}$$

(ii)

$$\frac{d\mathbf{r}}{dt} = \omega(\mathbf{a} \cos \omega t - \mathbf{b} \sin \omega t)$$

$$\frac{d^2\mathbf{r}}{dt^2} = -\,\omega^2(\mathbf{a} \sin \omega t + \mathbf{b} \cos \omega t)$$

i.e.

$$\frac{d^2\mathbf{r}}{dt^2} = -\,\omega^2\mathbf{r}$$

Also

$$\begin{aligned}\mathbf{r} \times \frac{d\mathbf{r}}{dt} &= (\mathbf{a} \sin \omega t + \mathbf{b} \cos \omega t) \times \omega(\mathbf{a} \cos \omega t - \mathbf{b} \sin \omega t)\\ &= \omega(-\mathbf{a} \times \mathbf{b} \sin^2 \omega t + \mathbf{b} \times \mathbf{a} \cos^2 \omega t)\\ &\qquad\qquad [\text{since } \mathbf{a} \times \mathbf{a} = \mathbf{b} \times \mathbf{b} = 0]\\ &= \omega(-\,\mathbf{a} \times \mathbf{b} \sin^2 \omega t - \mathbf{a} \times \mathbf{b} \cos^2 \omega t)\\ &= -\,\omega\mathbf{a} \times \mathbf{b}\,(\sin^2 \omega t + \cos^2 \omega t)\end{aligned}$$

Since $\sin^2 \omega t + \cos^2 \omega t = 1$, then $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt} = -\, \omega \mathbf{a} \times \mathbf{b}$.

51. **Motion along a Plane Curve.** In Fig. 39 $\overrightarrow{OR}$ is a vector rotating about O with angular velocity ω, whilst R traces the curve

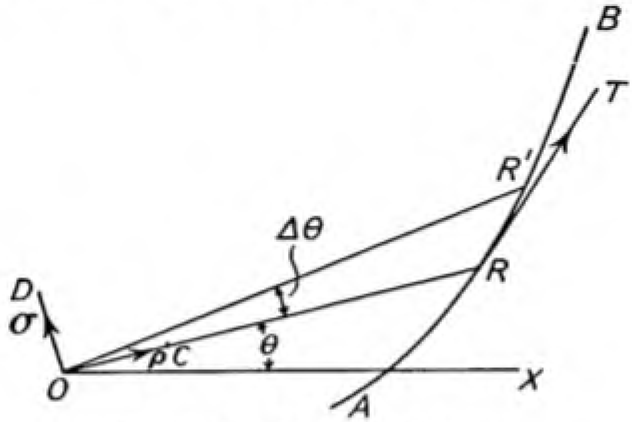


FIG. 39. ROTATING VECTOR

ARB. Let $\overrightarrow{OC} = \boldsymbol{\rho}$ and $\overrightarrow{OD} = \boldsymbol{\sigma}$ be unit vectors along OR and perpendicular to OR, as shown, these vectors rotating with OR and at the same speed as OR. Thus, the ends C and D move with equal speeds ω on the circumference of a circle of unit radius. By (V.67), the velocity of C is $\dot{\boldsymbol{\rho}}$, and that of D is $\dot{\boldsymbol{\sigma}}$. Now, the velocity of C is of magnitude ω in the direction and sense of $\boldsymbol{\sigma}$, so that

$$\dot{\boldsymbol{\rho}} = \omega \boldsymbol{\sigma} \qquad . \quad . \quad . \quad (V.69)$$

Also, the velocity of D is of magnitude ω in the direction of $\boldsymbol{\rho}$ but in the opposite sense, so that

$$\dot{\boldsymbol{\sigma}} = -\, \omega \boldsymbol{\rho} \qquad . \quad . \quad . \quad (V.70)$$

Let (r, θ) be the polar co-ordinates of R with respect to the origin O and the initial line OX. Let t and $t + \Delta t$ be the respective times in seconds at which the positions OR and OR' are occupied. Then, from (V.67), if $\mathbf{V}_R$ is the velocity of R,

$$\mathbf{V}_R = \frac{d\mathbf{r}}{dt}, \text{ or } \dot{\mathbf{r}} \qquad . \quad . \quad (V.71)$$

and the direction and sense of $\mathbf{V}_R$ are those of the vector $\overrightarrow{RT}$ shown in the figure, where RT is the tangent at R to the curve along which R moves.

The acceleration $\mathbf{A}_R$ of R is given by

$$\mathbf{A}_R = \frac{d^2\mathbf{r}}{dt^2}, \text{ or } \frac{d}{dt}(\mathbf{V}_R) \quad . \quad . \quad . \quad \text{(V.72)}$$

Expressions for $\mathbf{V}_R$ and $\mathbf{A}_R$ in terms of the components along OR and perpendicular to R will now be found.

Since $\boldsymbol{\rho}$ is unit vector along OR, then $\mathbf{r} = r\boldsymbol{\rho}$.

Hence,
$$\mathbf{V}_R = \frac{d}{dt}(r\boldsymbol{\rho})$$
$$= \dot{r}\boldsymbol{\rho} + r\dot{\boldsymbol{\rho}}$$

Using (V.69),
$$\mathbf{V}_R = \dot{r}\boldsymbol{\rho} + \omega r\boldsymbol{\sigma} \quad . \quad . \quad . \quad \text{(V.73)}$$

the unit vectors indicating the directions and senses of the components.

Also,
$$\mathbf{A}_R = \frac{d}{dt}(\mathbf{V}_R)$$
$$= \frac{d}{dt}(\dot{r}\boldsymbol{\rho} + \omega r\boldsymbol{\sigma})$$
$$= \ddot{r}\boldsymbol{\rho} + \dot{r}\dot{\boldsymbol{\rho}} + \dot{\omega}r\boldsymbol{\sigma} + \omega\dot{r}\boldsymbol{\sigma} + \omega r\dot{\boldsymbol{\sigma}}$$

Using (V.69) and (V.70),

$$\mathbf{A}_R = \ddot{r}\boldsymbol{\rho} + \dot{r}\omega\boldsymbol{\sigma} + \dot{\omega}r\boldsymbol{\sigma} + \omega\dot{r}\boldsymbol{\sigma} - \omega^2 r\boldsymbol{\rho}$$

i.e.
$$\mathbf{A}_R = (\ddot{r} - \omega^2 r)\boldsymbol{\rho} + (\dot{\omega}r + 2\omega\dot{r})\boldsymbol{\sigma} \quad . \quad . \quad \text{(V.74)}$$

This single relation shows that the radial acceleration is $\ddot{r} - \omega^2 r$ outwards, and the transverse acceleration is $\dot{\omega}r + 2\omega\dot{r}$ in the sense of θ increasing. The unit vectors indicate the directions and senses of the components.

If r is constant, the motion is circular, and

$$\mathbf{A}_R = -\,\omega^2 r\boldsymbol{\rho} + \dot{\omega}r\boldsymbol{\sigma} \quad . \quad . \quad \text{(V.75)}$$

If, in addition, ω is also constant, the motion is uniform circular motion in which

$$\mathbf{A}_R = -\,\omega^2 r\boldsymbol{\rho} \quad . \quad . \quad . \quad \text{(V.76)}$$

That is, the acceleration is entirely centripetal and of magnitude $\omega^2 r$.

If R represents a particle moving on the curve ARB, $\overrightarrow{OR}$ is its displacement vector, and $\mathbf{V}_R$ and $\mathbf{A}_R$ are its actual velocity and

acceleration respectively. If $\overrightarrow{OR}$ is not a displacement, the meanings of $\mathbf{V}_R$ and $\mathbf{A}_R$ will depend upon the nature of the vector $\overrightarrow{OR}$.

EXAMPLE 1

A rod is rotating about one end with angular velocity ω radn/sec and angular acceleration α radn/sec^2, both clockwise. At the instant when the rod is in its higher vertical position a slider on the rod is moving relative to it at v ft/sec and with a sliding acceleration of a ft/sec^2, both upwards. Find (1) the actual acceleration of the slider, and (2) the acceleration of the slider relative to the rod.

From (V.74), $\mathbf{A}_R =$ acceleration of slider

$$= (\ddot{r} - \omega^2 r)\boldsymbol{\rho} + (\dot{\omega} r + 2\omega\dot{r})\boldsymbol{\sigma}$$

where $\boldsymbol{\rho}$ points vertically upwards and $\boldsymbol{\sigma}$ to the right (the rotation being clockwise).

Here $\ddot{r} = a$, $\dot{r} = v$, and $\dot{\omega} = \alpha$, so that

$$\mathbf{A}_R = (a - \omega^2 r)\boldsymbol{\rho} + (\alpha r + 2\omega v)\boldsymbol{\sigma}$$

is the actual acceleration of the slider.

The point R' on the rod which instantaneously coincides with the slider is at a fixed distance r from O and its motion is circular. From (V.75),

$$\mathbf{A}_{R'} = -\omega^2 r\boldsymbol{\rho} + \dot{\omega} r\boldsymbol{\sigma}$$
$$= -\omega^2 r\boldsymbol{\rho} + \alpha r\boldsymbol{\sigma}$$

The acceleration of the slider relative to the rod is $\mathbf{A}_R - \mathbf{A}_{R'}$, where

$$\mathbf{A}_R - \mathbf{A}_{R'} = a\boldsymbol{\rho} + 2\omega v\boldsymbol{\sigma}$$

The beginner is apt to imagine that, because a is given as the acceleration along the rod, relative to it, this is the whole of the relative acceleration, whereas, as shown above, there is also a component relative acceleration $2\omega v$ in a transverse direction. This component is known as the "Coriolis" acceleration, and is very important in connection with mechanisms in which there is relative sliding combined with rotation.

EXAMPLES V

(1) P, Q, R are the mid-points of the sides BC, CA, AB respectively of a triangle ABC. Two systems of forces act at any point O, one represented by $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and the other by $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$. Show that the two systems are equivalent.

(2) M and N are the mid-points of the diagonals AC and BD respectively of a quadrilateral $ABCD$. Show that the resultant of the vectors $\overrightarrow{AB}$, $\overrightarrow{AD}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$ is $4\,\overrightarrow{MN}$.

(3) Forces of 2, 4, 3 units act along the sides AB, BC, CA respectively of an equilateral triangle ABC. Determine the resultant of these forces, and show that the line of action of the resultant cuts BC at a point O such that $BO = 3 \times BC$.

(4) At a certain instant two particles occupy positions A and B, 20 ft apart. The particle at A now moves towards that at B with uniform velocity 3 ft/sec, while the particle at B moves in a direction perpendicular to the line AB with uniform velocity 4 ft/sec. Determine (i) the velocity of the B particle relative to the A particle, (ii) the shortest distance apart of the particles, and (iii) the time taken to reach this shortest distance.

(5) Two points P and Q are located by vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ respectively, relative to a point O. R is a point on PQ produced such that $\overline{PR} = n\overline{PQ}$, and S is a point on QP produced such that $QS = k\overline{QP}$, n and k being constants. Express the location or position vectors of R and S in terms of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

(6) The angular points A, B, C of a quadrilateral $ABCD$ remain fixed while the point D varies in such a way that forces $\overrightarrow{DA}$ and $\overrightarrow{DC}$ acting at D have their resultant always along DB. Find the locus of the point D.

(7) Calculate the scalar product of each of the following pairs of vectors—

(i) $2_{20°}$, $3_{5°}$; (ii) $40_{110°}$, $25_{70°}$; (iii) $3{\cdot}5_{60°}$, $3{\cdot}5_{60°}$

(8) ABC is any triangle and H is the point in which the altitudes through A and B intersect. Let $\overrightarrow{HA} = \boldsymbol{r}_1$, $\overrightarrow{HB} = \boldsymbol{r}_2$, and $\overrightarrow{HC} = \boldsymbol{r}_3$. Express $\overrightarrow{BC}$ in terms of $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$ and $\overrightarrow{CA}$ in terms of $\boldsymbol{r}_3$ and $\boldsymbol{r}_1$. Noting that $\boldsymbol{r}_2$ and $\overrightarrow{CA}$ are perpendicular and $\boldsymbol{r}_1$ and $\overrightarrow{BC}$ are perpendicular, deduce that $\boldsymbol{r}_3$ and $\overrightarrow{AB}$ are perpendicular, thus proving that the altitudes of a triangle are concurrent.

(9) Establish the following results by vectorial methods—

(i) In a triangle ABC in which $A = 90°$, $a^2 = b^2 + c^2$.
(ii) In any triangle ABC, if M is the mid-point of BC, $AB^2 + AC^2 = 2\,AM^2 + 2\,BM^2$.

(10) A body is displaced a distance 18 ft in a straight path under the action of a force $\frac{1}{8}$ ton whose line of action is inclined at 46° to the direction of motion. Determine (in ft-lb) the work done by the force in this displacement of the body. If the body moves at 3 ft/sec along the straight path, determine the power.

(11) Constant forces $2\boldsymbol{i} - 5\boldsymbol{j} + 6\boldsymbol{k}$, $-\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, and $2\boldsymbol{i} + 7\boldsymbol{j}$ act on a particle. Determine the total work done by the forces in a displacement of the particle from the point $4\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$ to the point $6\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$.

(12) Show that (i) any vector can be uniquely resolved into two components in any two stated directions in the same plane with the vector, and (ii) any vector can be uniquely resolved into three components in any three stated non-coplanar directions.

(13) Show that, with the usual notation, a vector $\boldsymbol{r}$ can be expressed as

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

Deduce that the magnitude of the resultant of vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$, etc., acting at O is $\sqrt{\Sigma r^2 + 2\Sigma r_m r_n \cos\theta_{m,n}}$, where $\theta_{m,n}$ is the angle between the vectors r_m and r_n.

(14) Two vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are given by $\boldsymbol{r}_1 = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{r}_2 = 3\boldsymbol{i} - 5\boldsymbol{j} + 6\boldsymbol{k}$. Find (i) the angles between the direction of each vector and the axes of reference, and (ii) the angle between the directions of the vectors.

(15) Two vectors $\boldsymbol{r}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ and $\boldsymbol{r}_2 = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$ are inclined to each other at angle θ. Show that

$$\sin^2\theta = \frac{(y_1z_2 - y_2z_1)^2 + (z_1x_2 - z_2x_1)^2 + (x_1y_2 - x_2y_1)^2}{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)}$$

(16) The terminal points P, Q, R, S of two vectors $\overrightarrow{PQ} = \boldsymbol{a}$ and $\overrightarrow{RS} = \boldsymbol{b}$ are $(4, -1, 5)$, $(3, 2, -2)$, $(5, 0, 1)$, $(-1, 2, 3)$ respectively. Express each vector in the form $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, and deduce the scalar product $\boldsymbol{a}\cdot\boldsymbol{b}$ and the vector product $\boldsymbol{a}\times\boldsymbol{b}$.

(17) The velocity of a particle A relative to a particle B is $5\boldsymbol{i} + 3\boldsymbol{j}$, and the velocity of B relative to a third particle C is $6\boldsymbol{i} - 5\boldsymbol{j}$. Determine the magnitude and direction of the velocity of A relative to C, assuming that $\boldsymbol{i}$ and $\boldsymbol{j}$ represent velocities of 1 ft/sec horizontally and vertically respectively.

(18) Given two vectors $\overrightarrow{OA} = 3\boldsymbol{i} + 8\boldsymbol{j} - 5\boldsymbol{k}$ and $\overrightarrow{OB} = 10\boldsymbol{i} - 6\boldsymbol{j} + 9\boldsymbol{k}$, find the vector $\overrightarrow{AB}$, and deduce the inclinations of its direction to the axes of reference.

(19) Given three vectors $\overrightarrow{OP} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, $\overrightarrow{OQ} = -\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, and $\overrightarrow{OR} = 4\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$, find the vectors $\overrightarrow{PQ}$, $\overrightarrow{QR}$, and $\overrightarrow{RP}$, and deduce the lengths of the sides of the triangle PQR.

(20) Masses m, $2m$, $3m$, $4m$, $5m$, $6m$ are situated at the angular points O, A, B, C, D, E respectively of a regular hexagon $OABCDE$ of side a. Determine the distances of the centre of mass of the system of masses from OA and OD.

(21) Masses of 4, 8, 6, 9 lb are situated at the points $(0, 2, 3)$, $(3, -4, 6)$, $(1, 1, -1)$, $(-2, 3, 5)$ respectively. Find the co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the centre of mass of the given system of masses.

(22) Find the cosine of the angle between the directions of the vectors $\boldsymbol{a} = 4\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$, and find also the unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

(23) Find the numerical magnitude of the vector product of each of the following pairs of vectors, showing the direction of the product by the aid of a rough diagram—

(i) $4_{30°}$, $10_{114°}$; (ii) $1{\cdot}5_{233°}$, $0{\cdot}8_{96°}$.

(24) A force $\boldsymbol{P} = 3\boldsymbol{j} - 6\boldsymbol{k}$ passes through the point A whose position vector is $4\boldsymbol{i} - 2\boldsymbol{j} + 9\boldsymbol{k}$. Find the moment of $\boldsymbol{P}$ about the point B whose position vector is $6\boldsymbol{i} - 7\boldsymbol{k}$.

(25) Given that $\boldsymbol{a} = 2{\cdot}5_{145°}$ and $\boldsymbol{b} = 4{\cdot}8_{71°}$, find the scalar product $\boldsymbol{a}\cdot\boldsymbol{b}$, and the numerical magnitude of the vector product $\boldsymbol{a}\times\boldsymbol{b}$.

(26) Express the vector $\overrightarrow{AB}$ in the form $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ where A and B are the points $(5, -3, 0)$ and $(-2, 1, 5)$ respectively. If $\overrightarrow{CD}$ is a unit vector inclined at equal angles to the axes of reference, find the scalar and vector products of $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

(27) A and B are points whose position vectors relative to a point O are $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. A localized vector $\boldsymbol{P}$ acts at A. Show that the moment of $\boldsymbol{P}$ about B is $\boldsymbol{P}\times(\boldsymbol{a} - \boldsymbol{b})$.

(28) Show that the condition that the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are coplanar is $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = 0$.

(29) Show that

$$\text{(i) } \boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{b}) \equiv 0$$

$$\text{(ii) } (\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \times \boldsymbol{c}) \cdot (\boldsymbol{b} \times \boldsymbol{d}) - (\boldsymbol{b} \times \boldsymbol{c}) \cdot (\boldsymbol{a} \times \boldsymbol{d})$$

$$\text{(iii) } (\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) + (\boldsymbol{b} \times \boldsymbol{c}) \cdot (\boldsymbol{a} \times \boldsymbol{d}) + (\boldsymbol{c} \times \boldsymbol{a}) \cdot (\boldsymbol{b} \times \boldsymbol{d}) = 0$$

(30) A point moves on the circumference of a circle with variable angular velocity ω about the centre. Show that the point has component accelerations $\dot{\omega}r$ along the tangent and $\omega^2 r$ along the inward normal.

(31) A rigid body is rotating at 2·5 radn/sec about an axis OR, where R is the point $2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ relative to O. Find the velocity of the particle of the body at the point $4\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$. (All lengths are in feet.)

(32) A thin rod OA is at rest in a horizontal position and a slider of mass m on the rod is distant c from O. The rod is now rotated in a horizontal plane about the end O with uniform angular velocity ω. Show that at the instant when the slider is distant r from O the horizontal reaction of the rod on the slider is $2m\omega^2\sqrt{r^2 - c^2}$.

(33) If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are functions of a variable t, show that the derivatives of the scalar triple product $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ and the vector triple product $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ are given by

$$\frac{d}{dt}[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = \left[\frac{d\boldsymbol{a}}{dt}, \boldsymbol{b}, \boldsymbol{c}\right] + \left[\boldsymbol{a}, \frac{d\boldsymbol{b}}{dt}, \boldsymbol{c}\right] + \left[\boldsymbol{a}, \boldsymbol{b}, \frac{d\boldsymbol{c}}{dt}\right]$$

$$\text{and } \frac{d}{dt}\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \frac{d\boldsymbol{a}}{dt} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{a} \times \left(\frac{d\boldsymbol{b}}{dt} \times \boldsymbol{c}\right) + \boldsymbol{a} \times \left(\boldsymbol{b} \times \frac{d\boldsymbol{c}}{dt}\right)$$

(34) A force $\boldsymbol{P}$ is given in terms of its rectangular co-ordinates by the relation

$$\boldsymbol{P} = P_x\boldsymbol{i} + P_y\boldsymbol{j} + P_z\boldsymbol{k}$$

Show that the moment of $\boldsymbol{P}$ about the origin O is

$$(yP_z - zP_y)\boldsymbol{i} + (zP_x - xP_z)\boldsymbol{j} + (xP_y - yP_x)\boldsymbol{k}$$

where x, y, z are the co-ordinates of any point on the line of action of $\boldsymbol{P}$. Show also that the coefficients of $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ in this expression for the moment of $\boldsymbol{P}$ are the ordinary scalar moments of $\boldsymbol{P}$ about the co-ordinate axes.

(35) A rigid body is rotating about a fixed axis which passes through a fixed point O. The angular velocity of the body is a vector $\boldsymbol{\omega}$ and P is a point of the body such that OP is represented by the vector $\boldsymbol{r}$. Prove that the velocity of P is the vector product of $\boldsymbol{\omega}$ and $\boldsymbol{r}$.

A rigid body is rotating at the rate of 5 rev per sec, about an axis through the origin whose direction ratios are 2, − 1, 3.

Find the velocity (in ft/sec) of the point of the body whose co-ordinates are (2, 4, 3), the unit of length being 1 ft. [U.L.]

CHAPTER VI

ORDINARY DIFFERENTIAL EQUATIONS I

52. **Extended Use of Operators.** In Volume I we used operators in the solution of differential equations and showed that powers of the operator D in combination with constants may be manipulated, in a limited way, in accordance with some of the rules of algebra. Other rules which apply depend upon the nature of the function operated upon as well as upon that of the operating function. In applications of these differential equations we shall usually be concerned with first and second order equations. The general linear differential equation of the nth order with constant coefficients is

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = X \quad . \quad . \quad \text{(VI.1)}$$

where a_n, a_{n-1}, etc. are constants and X is a function of x. If we write D^r for $\frac{d^r}{dx^r}$, where $r = 1, 2, 3, \ldots (n-1), n$, (VI.1) becomes

$$(a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \ldots + a_1 D + a_0)y = X \quad \text{(VI.2)}$$

Without loss of generality we can divide through by a_n, and we shall assume, therefore, that $a_n = 1$. If, with this assumption, we write $F(D)$ for the expression in brackets in (VI.2), then

$$F(D) = D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \ldots + a_1 D + a_0 \quad \text{(VI.3)}$$

and (VI.2) may be written

$$F(D)y = X \quad . \quad . \quad . \quad \text{(VI.4)}$$

This tells us that, if the combined operation represented by $F(D)$ is performed on y, a function of x, the result is X. In the solution of a differential equation y is an unknown function and X a known function of x, so that the relation (VI.4) really asks the question— what function of x is y if the performance of the operation $F(D)$ on it

turns it into the function X? Before attempting to answer this question we must first consider the operation $F(D)y$.

We saw in Vol. I that, if $y = e^{\alpha x}$, $F(D)y = e^{\alpha x}F(\alpha)$, so that we have the following as a first rule.

RULE I
$$F(D)e^{\alpha x} = e^{\alpha x}F(\alpha) \qquad . \qquad . \qquad \text{(VI.5)}$$

This is really self-evident since $D^r(e^{\alpha x}) = e^{\alpha x}\alpha^r$, so that the operation $a_r D^r e^{\alpha x}$ gives $a_r \alpha^r e^{\alpha x}$, and giving r the values 1, 2, 3, etc., in turn, substituting in the left-hand side of (VI.5), and taking out the factor $e^{\alpha x}$, we obtain $e^{\alpha x}F(\alpha)$.

RULE II
$$F(D)e^{\alpha x}X = e^{\alpha x}F(D + \alpha)X \qquad . \qquad . \qquad \text{(VI.6)}$$

where X is a function of x. This is known as the *shift or shifting* transformation and is very useful, as we shall see shortly.

By differentiation,

$$De^{\alpha x}X = e^{\alpha x}(D + \alpha)X$$

$$D^2e^{\alpha x}X = e^{\alpha x}[D(D + \alpha)X + \alpha(D + \alpha)X]$$

$$= e^{\alpha x}(D + \alpha)^2X$$

$$D^3e^{\alpha x}X = e^{\alpha x}[D(D + \alpha)^2X + \alpha(D + \alpha)^2X]$$

$$= e^{\alpha x}(D + \alpha)^3X$$

and generally,

$$D^re^{\alpha x}X = e^{\alpha x}(D + \alpha)^rX$$

Now
$$\begin{aligned} F(D)e^{\alpha x}X &= D^n(e^{\alpha x}X) + a_{n-1}D^{n-1}(e^{\alpha x}X) \\ &\qquad + a_{n-2}D^{n-2}(e^{\alpha x}X) + \ldots + a_0e^{\alpha x}X \\ &= e^{\alpha x}[(D + \alpha)^nX + a_{n-1}(D + \alpha)^{n-1}X \\ &\qquad + a_{n-2}(D + \alpha)^{n-2}X + \ldots + a_0X] \end{aligned}$$

i.e.
$$F(D)e^{\alpha x}X = e^{\alpha x}F(D + \alpha)X$$

RULE III
$$\left.\begin{aligned} F(D^2)\sin(px + q) &= F(-p^2)\sin(px + q) \\ F(D^2)\cos(px + q) &= F(-p^2)\cos(px + q) \end{aligned}\right\} \qquad . \qquad \text{(VI.7)}$$

We established these relations in Vol. I. They are easily seen to be true since D^2 operating on $\sin(px+q)$ or $\cos(px+q)$ merely changes the sign of the operand and multiplies it by p^2. In the case of the operator $F(D^2)$ the operation merely replaces D^2 by $-p^2$ as in (VI.7).

Rule IV $a + bD$ operating on $\frac{\sin}{\cos}(px+q)$ has the effect of multiplying the function by $\sqrt{a^2 + p^2b^2}$ and increasing the angle $px + q$ by $\tan^{-1}\frac{pb}{a}$

In order to find the effect of $F(D)$ operating on $\sin(px+q)$ or $\cos(px+q)$ we divide the terms on the right of (VI.3) into odd and even powers of D; thus

$$F(D) = (a_0 + a_2D^2 + a_4D^4 + \ldots) + (a_1D + a_3D^3 + a_5D^5 + \ldots)$$

i.e. $$F(D) = (a_0 - a_2p^2 + a_4p^4 - a_6p^6 + \ldots) + (a_1 - a_3p^2 + a_5p^4 - \ldots)D$$

which reduces to

$$F(D) = a + bD \qquad \text{(VI.8)}$$

where a and b represent the quantities in brackets on the right-hand side.

Then,

$$\begin{aligned} F(D)\sin(px+q) &= (a+bD)\sin(px+q) \\ &= a\sin(px+q) + pb\cos(px+q) \end{aligned}$$

i.e. $$F(D)\sin(px+q) = \sqrt{a^2+p^2b^2}\,\sin\left(px + q + \tan^{-1}\frac{pb}{a}\right) \qquad \text{(VI.9)}$$

Similarly,

$$F(D)\cos(px+q) = \sqrt{a^2+p^2b^2}\,\cos\left(px + q + \tan^{-1}\frac{pb}{a}\right) \qquad \text{(VI.10)}$$

These four rules along with the rules for direct differentiation and integration are sufficient for our use in direct operations.

EXAMPLE 1

Show that $y = Ae^{-5x}$ and $y = Be^{-3x}$ both satisfy the equation

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 15y = 0$$

With $D = \dfrac{d}{dx}$, we have

$$(D^2 + 8D + 15)Ae^{-5x} = A[(-5)^2 + 8(-5) + 15] = 0$$

and $$(D^2 + 8D + 15)Be^{-3x} = B[(-3)^2 + 8(-3) + 15] = 0$$

Hence both $y = Ae^{-5x}$ and $y = Be^{-3x}$ satisfy the given equation.

EXAMPLE 2

Show that $\dfrac{d^n}{dx^n}(e^{ax}\cos bx) = Re^{ax}\cos(bx + n\alpha)$,

where $$R^2 = (a^2 + b^2)^n \text{ and } \alpha = \tan^{-1}\frac{b}{a}$$

With $D = \dfrac{d}{dx}$, we have

$$D^n(e^{ax}\cos bx) = e^{ax}(D + a)^n \cos bx \quad \text{[by Rule II]} \qquad . \quad . \quad . \; (1)$$

Now $$(D + a)\cos bx = a\cos bx - b\sin bx$$
$$= \sqrt{a^2 + b^2}\cos(bx + \alpha), \text{ where } \alpha = \tan^{-1}\frac{b}{a}$$

Similarly,

$$(D + a)^2 \cos bx = \sqrt{a^2 + b^2}\,(D + a)\cos(bx + \alpha)$$
$$= (\sqrt{a^2 + b^2})^2 \cos(bx + 2\alpha)$$

and $$(D + a)^3 \cos bx = (\sqrt{a^2 + b^2})^2\,(D + a)\cos(bx + 2\alpha)$$
$$= (\sqrt{a^2 + b^2})^3 \cos(bx + 3\alpha)$$

If this process is repeated n times in all, we have

$$(D + a)^n \cos bx = (\sqrt{a^2 + b^2})^n \cos(bx + n\alpha)$$

Substituting in (1), we obtain

$$D^n(e^{ax}\cos bx) = (\sqrt{a^2 + b^2})^n\, e^{ax}\cos(bx + n\alpha), \text{ as required.}$$

EXAMPLE 3

Show that $x = Re^{-5t}\cos(5t + e)$ satisfies the equation

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 50x = 0$$

With $D = \frac{d}{dt}$, we have

$$\begin{aligned}(D^2 + 10D + 50)x &= (D^2 + 10D + 50)Re^{-5t}\cos(5t + e)\\ &= Re^{-5t}[(D-5)^2 + 10(D-5) + 50]\cos(5t + e),\\ &\qquad\qquad\text{by Rule II}\\ &= Re^{-5t}(D^2 + 25)\cos(5t + e)\\ &= Re^{-5t}(-25 + 25)(\cos 5t + e), \text{ by Rule III}\\ &= 0\end{aligned}$$

Hence, the equation is satisfied.

EXAMPLE 4

Find the value of $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 11x$ if $x = A\cos(3t + 2)$

We have

$$\begin{aligned}(D^2 + 3D + 11)A\cos(3t + 2) &= (-9 + 3D + 11)A\cos(3t + 2), \text{ by Rule III}\\ &= (3D + 2)A\cos(3t + 2)\\ &= \sqrt{2^2 + 3^2 \times 3^2}\, A\cos\left(3t + 2 + \tan^{-1}\frac{9}{2}\right)\end{aligned}$$

Hence, $$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 11x = 9{\cdot}220A\cos(3t + 3{\cdot}353)$$

If we write pi instead of D on the right-hand side of (VI.3), where $i = \sqrt{-1}$, that side reduces to $a + bpi$ so that in place of (VI.8) we have

$$F(D) = a + bpi \qquad \text{(VI.11)}$$

and if D is restored in place of pi (VI.8) and (VI.11) are identical. Thus if we look upon pi as equivalent to D we may simplify $F(D)$, i.e. $F(pi)$ by the methods of complex algebra. In Example 4 above we could write

$$\begin{aligned}(D^2 + 3D + 11)A\cos(3t + 2) &= [(3i)^2 + 9i + 11]\,A\cos(3t + 2)\\ &= (9i + 2)A\cos(3t + 2)\\ &= (3D + 2)A\cos(3t + 2)\\ &= 9{\cdot}220A\cos(3t + 3{\cdot}353) \text{ as before.}\end{aligned}$$

Here there is no advantage in making the substitution but, if we assume that when applying Rule IV, $F(D)$ is not necessarily a polynomial in D, the substitution may greatly simplify the analysis.

EXAMPLE 5

The voltage V applied at one end of a long telephone line is $E_0 \sin pt$. The current C entering the line is given by $C = \sqrt{\dfrac{a + bD}{c + dD}} \cdot V$ where $D \equiv d/dt$. Find C as a function of t having given $a = 2$, $b = 0{\cdot}01$, $c = 5$, $d = 0{\cdot}005$ and $p = 8\,000$.

Writing pi for D,

$$\frac{a + bD}{c + dD} = \frac{a + bpi}{c + dpi} \text{ and putting in the given values}$$

$$\frac{a + bD}{c + dD} = \frac{2 + 80i}{5 + 40i} = \frac{\sqrt{6\,404}\,[\tan^{-1} 40]}{\sqrt{1\,625}\,[\tan^{-1} 8]} \text{ by complex algebra,}$$

$$= 1{\cdot}986[5{\cdot}69^\circ]$$

Hence $$\sqrt{\frac{a + bD}{c + dD}} = \sqrt{1{\cdot}986}\left[\frac{5{\cdot}69^\circ}{2}\right]$$

$$= 1{\cdot}409[\cos 2{\cdot}85^\circ + i \sin 2{\cdot}85^\circ]$$

$$= 1{\cdot}407 + 0{\cdot}0701i$$

and $$C = (1{\cdot}407 + 0{\cdot}0701i)E_0 \sin 8\,000t$$

$$= \sqrt{1{\cdot}407^2 + 0{\cdot}0701^2}\, E_0 \sin\left(8\,000t + \tan^{-1}\frac{0{\cdot}0701}{1{\cdot}407}\right)$$

$$= 1{\cdot}41 E_0 \sin (8\,000t + 0{\cdot}0497)$$

An alternative form of Rule IV is therefore—

$a + bpi$ operating on $\frac{\sin}{\cos}(px + q)$ has the effect of multiplying the function by $\sqrt{a^2 + b^2p^2}$ and increasing the angle by $\tan^{-1}\dfrac{pb}{a}$

Writing a single symbol c in place of bp this simplifies to—

$a + ci$ operating on $\frac{\sin}{\cos}(px + q)$ has the effect of multiplying the function by $\sqrt{a^2 + c^2}$ and increasing the angle by $\tan^{-1}\dfrac{c}{a}$

EXAMPLE 6

Simplify $(3 + 4i)6 \sin 5t$ where $i = \frac{1}{5}D$ and $D \equiv d/dt$. By the alternative form of Rule IV the expression simplifies to $6\sqrt{3^2 + 4^2} \sin (5t + \tan^{-1} \frac{4}{3})$ $= 30 \sin (5t + 0{\cdot}9274)$.

We now consider the operator $(D + \alpha)(D + \beta)$. The expression $D^2 + (\alpha + \beta)D + \alpha\beta$ regarded as an algebraic expression can be arranged in factors in two ways, thus—

$$D^2 + (\alpha + \beta)D + \alpha\beta \equiv (D + \alpha)(D + \beta) \equiv (D + \beta)(D + \alpha) \tag{VI.12}$$

We cannot assume, however, without further examination that these three operators are equivalent when operating on an operand X. Consider $(D + \alpha)(D + \beta)$ operating on X.

$$\begin{aligned}(D + \alpha)(D + \beta)X &\equiv (D + \alpha)(DX + \beta X)\\ &\equiv D^2X + D\beta X + \alpha DX + \alpha\beta X\\ &\equiv D^2X + (\alpha + \beta)DX + \alpha\beta X\\ &\equiv [D^2 + (\alpha + \beta)D + \alpha\beta]X\end{aligned}$$

Similarly, $\qquad (D + \beta)(D + \alpha)X \equiv [D^2 + (\alpha + \beta)D + \alpha\beta]X$

Thus, we have proved that, if $D \equiv \dfrac{d}{dx}$, the three operators in (VI.12) are equivalent when operating upon any function of x, and that any operator of the second order may be factorized and the factors taken in either order. The extension of this to three or more factors is clear. We see, then, that we may factorize an operating function and arrange the factors in any convenient order before applying the operating function to the operand.

53. **Inverse Operators.** Each of two operators is the inverse of the other if, when they operate in succession in either order on an operand, the latter is left unchanged. The operation of finding a logarithm and that of finding an antilogarithm are inverse operations because log. antilog. $x = x =$ antilog. log. x. The inverse operation of multiplication by 3 is division by 3. If we omit the added constant in indefinite integration, differentiation and integration with respect to the same independent variable are inverse operations. Thus, if I is the symbol for integration and D for differentiation, I may be written as $\dfrac{1}{D}$ and D as $\dfrac{1}{I}$, and $DIf(x) = D \times \dfrac{1}{D}f(x) = f(x)$. The inverse of the operation $F(D)$ is written $\dfrac{1}{F(D)}$, and the meaning of $\dfrac{1}{F(D)}$ operating on a function must be so interpreted that, if $F(D)$

and $\frac{1}{F(D)}$ operate in succession, in either order, the operand remains unchanged.

RULE V
$$\frac{1}{F(D)} e^{\alpha x} = \frac{1}{F(\alpha)} e^{\alpha x} \quad . \quad . \quad . \quad \text{(VI.13)}$$

This is clearly true since

$$F(D) \cdot \frac{1}{F(D)} e^{\alpha x} = F(D) \cdot \frac{e^{\alpha x}}{F(\alpha)} = \frac{F(\alpha)}{F(\alpha)} \cdot e^{\alpha x} = e^{\alpha x}$$

RULE VI
$$\frac{1}{F(D)} (e^{\alpha x} X) = e^{\alpha x} \frac{1}{F(D + \alpha)} X \quad . \quad . \quad \text{(VI.14)}$$

To prove this, consider Rule II which is

$$F(D)\, e^{\alpha x} X = e^{\alpha x} F(D + \alpha) X$$

Let X' denote $F(D + \alpha)X$; then X' is a function of x and

$$F(D) e^{\alpha x} X = e^{\alpha x} X'$$

so that
$$e^{\alpha x} X = \frac{1}{F(D)} e^{\alpha x} X' \quad . \quad . \quad . \quad \text{(VI.15)}$$

Again, $F(D + \alpha)X = X'$, and, therefore,

$$X = \frac{1}{F(D + \alpha)} X'$$

Substituting this value of X in (VI.15), we have

$$e^{\alpha x} \frac{1}{F(D + \alpha)} X' = \frac{1}{F(D)} e^{\alpha x} X' \quad . \quad . \quad \text{(VI.16)}$$

which is Rule VI with X' written instead of X.

RULE VII
$$\frac{1}{F(D^2)} \frac{\sin}{\cos} (px + q) = \frac{1}{F(-p^2)} \frac{\sin}{\cos} (px + q) \quad . \quad \text{(VI.17)}$$

This is clearly true as this operation cancels the effect of the direct operation.

Rule VIII $\dfrac{1}{a+bD}$ or $\dfrac{1}{a+bpi}$ operating on $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(px+q)$ has the effect of dividing the expression by $\sqrt{a^2+b^2p^2}$ and decreasing the angle by $\tan^{-1}\dfrac{bp}{a}$

Again, this is seen to be true as in either case the effect of the inverse operator cancels that of the direct operator.

EXAMPLE 1

Find $I_1 = \int e^{ax}\sin(px+q)dx$ and $I_2 = \int e^{ax}\cos(px+q)dx$

By Rule VI, $\dfrac{1}{D}\{e^{ax}\sin(px+q)\} = e^{ax}\dfrac{1}{D+a}\sin(px+q)$, and by

Rule VIII, $\dfrac{1}{D+a}\sin(px+q) = \dfrac{1}{\sqrt{a^2+p^2}}\sin\left(px+q-\tan^{-1}\dfrac{p}{a}\right)$

Hence, $\qquad = \dfrac{e^{ax}}{\sqrt{a^2+p^2}}\sin\left(px+q-\tan^{-1}\dfrac{p}{a}\right)$, and by the same method

$$I_2 = \frac{e^{ax}}{\sqrt{a^2+p^2}}\cos\left(px+q-\tan^{-1}\frac{p}{a}\right)$$

EXAMPLE 2

Find the result of integrating x^2e^{ax} n times, leaving out the constants of integration.

If $I \equiv \dfrac{1}{D}$, we are required to find $I^n(x^2e^{ax})$. We have

$$\begin{aligned} I^n(x^2e^{ax}) &= \frac{1}{D^n}(x^2e^{ax}) \\ &= e^{ax}\frac{1}{(D+a)^n}x^2 \\ &= \frac{e^{ax}}{a^n}\left(1+\frac{D}{a}\right)^{-n}x^2 \\ &= \frac{e^{ax}}{a^n}\Big[1-\frac{nD}{a}+\frac{n(n+1)}{\underline{|2}}\frac{D^2}{a^2} \\ &\qquad + \text{terms containing higher powers of } D\Big]x^2 \\ &= \frac{e^{ax}}{a^n}\left[x^2-\frac{2nx}{a}+\frac{n(n+1)}{a^2}+0+0+\ldots\right] \end{aligned}$$

Hence, $$I^n(x^2e^{ax}) = \frac{e^{ax}}{a^{n+2}}[a^2x^2 - 2nax + n(n+1)] \quad . \quad . \quad . \quad (1)$$

In expanding $\left(1 + \frac{D}{a}\right)^{-n}$ we have performed an operation the validity of which we shall discuss below. We can easily verify the result by using Leibnitz' theorem, thus—

$$D^n(e^{ax}x^2) = e^{ax}[a^nx^2 + 2na^{n-1}x + n(n-1)a^{n-2}]$$

$$D^n(e^{ax}x) = e^{ax}(a^nx + na^{n-1})$$

$$D^n(e^{ax}) = e^{ax} \,.\, a^n$$

Then, $$D^nI^n(e^{ax}x^2) = \frac{e^{ax}}{a^{n+2}}[a^2\{a^nx^2 + 2na^{n-1}x + n(n-1)a^{n-2}\} - 2na(a^nx + na^{n-1}) + n(n+1)a^n]$$

i.e. $D^nI^n(e^{ax}x^2) = e^{ax}x^2$, which shows that (1) is correct.

54. **Solution of Differential Equations by the Use of Operators.** The linear differential equation of the nth order is given in symbolic form by (VI.2) and (VI.4). The term X may be a constant or a function of x but does not involve y or any of its derivatives. If $X = 0$, we have

$$F(D)y = 0 \quad . \quad . \quad . \quad (\text{VI.18})$$

which is called the *reduced* equation of (VI.4). We saw in Vol. I that the complete solution of (VI.4) is given by

$$y = u + v \quad . \quad . \quad . \quad (\text{VI.19})$$

where u is the complete solution of the reduced equation and v is any function of x which satisfies (VI.4). u is called the *complementary function* and v the *particular integral*. We saw also in a few simple cases how operators are used to find particular integrals. We shall extend the use of operators to further cases of particular integrals, and shall also apply them to find complementary functions.

EXAMPLE 1

Solve $$\frac{dy}{dx} + ky = b \quad . \quad . \quad . \quad . \quad (\text{VI.20})$$

where k and b are constants. The solution of this equation can be obtained, as in Vol. I, by separation of the variables and also by the use of an integrating factor, and the reader should know both these methods.

By operators (VI.20) is written $(D + k)y = b$, so that

$$y = \frac{1}{D + k} b$$

i.e.

$$y = \frac{1}{D + k} e^{-kx} e^{kx} b$$

Note the device of introducing two factors of unity, e^{-kx} and e^{kx}, so as to allow us to apply Rule VI, thus $y = e^{-kx} \frac{1}{D} e^{kx} b$, and so transform the operator $\frac{1}{D + k}$ into $\frac{1}{D} = I = \int \ldots dx$.

This gives

$$y = be^{-kx} \int e^{kx} dx = be^{-kx}\left(\frac{1}{k} e^{kx}\right)$$

i.e.

$$y = \frac{b}{k} \qquad \text{(VI.21)}$$

This is the particular integral of (VI.20). To find the complementary function we write the reduced equation as $(D + k)\, y = 0$, from which

$$y = \frac{1}{D + k}\,(0)$$

i.e.

$$y = \frac{1}{D + k}\,(e^{-kx}\, 0)$$

Note that we introduce the factor e^{-kx} so as to be able to use Rule VI.

Then by this rule,

$$y = e^{-kx} \frac{1}{D}(0) = e^{-kx} \int 0\, dx$$

i.e. $y = Ae^{-kx}$, where A is an arbitrary constant.

The complete solution of (VI.20) is

$$y = Ae^{-kx} + \frac{b}{k} \qquad \text{(VI.22)}$$

The reader should memorize this result and should be able to write down the solution of (VI.20) by inspection. It is clear that the two operations in the above can be carried out simultaneously as in the next example.

EXAMPLE 2

Solve

$$a\frac{dy}{dt} + by = c \sin pt \qquad \text{(VI.23)}$$

Writing D for d/dt $$\left(D + \frac{b}{a}\right) y = \frac{c}{a} \sin pt$$

and $$y = \frac{c}{a} \cdot \frac{1}{D + \frac{b}{a}} \sin pt + \frac{1}{D + \frac{b}{a}} (0)$$

By Rules VI and VIII, $$y = \frac{c}{a\sqrt{p^2 + \frac{b^2}{a^2}}} \sin\left(pt - \tan^{-1}\frac{pa}{b}\right) + \frac{1}{D + \frac{b}{a}}\left(e^{-\frac{b}{a}t} 0\right)$$

Now $$\frac{1}{D + \frac{b}{a}}\left(e^{-\frac{b}{a}t} 0\right) = e^{-\frac{b}{a}t} \frac{1}{D}(0) = Ae^{-\frac{b}{a}t}$$

Hence, $$y = \frac{c}{\sqrt{a^2p^2 + b^2}} \sin\left(pt - \tan^{-1}\frac{pa}{b}\right) + Ae^{-\frac{b}{a}t} \quad \text{(VI.24)}$$

where A is an arbitrary constant.

EXAMPLE 3

Solve the differential equation $L\dfrac{di}{dt} + Ri = E$ . . . . . . (1)

where E is given by

$$E = \frac{E_0}{2}\left\{1 + \frac{4}{\pi}\left(\sin\frac{\pi t}{\tau} + \frac{1}{3}\sin\frac{3\pi t}{\tau} + \frac{1}{5}\sin\frac{5\pi t}{\tau} + \ldots\right)\right\}$$

Writing the equation as $\dfrac{di}{dt} + \dfrac{R}{L} i = \dfrac{E}{L}$ and using operators, we have

$$i = \frac{1}{D + \frac{R}{L}}(0) + \frac{1}{D + \frac{R}{L}}\left(\frac{E_0}{2L}\right) + \frac{2E_0}{\pi L}\sum \frac{1}{D + \frac{R}{L}}\left\{\frac{1}{2n+1}\sin\frac{(2n+1)\pi t}{\tau}\right\}$$

where $n = 0, 1, 2, 3$, etc.

Now $$\frac{1}{D + \frac{R}{L}}(0) = \frac{1}{D + \frac{R}{L}}\left(e^{-\frac{R}{L}t} \cdot 0\right) = e^{-\frac{R}{L}t} \cdot \frac{1}{D}(0) = Ae^{-\frac{R}{L}t}$$

$$\frac{1}{D + \frac{R}{L}}\left(\frac{E_0}{2L}\right) = \frac{1}{D + \frac{R}{L}}\left(e^{-\frac{R}{L}t} \cdot e^{\frac{R}{L}t} \cdot \frac{E_0}{2L}\right) = e^{-\frac{R}{L}t} \cdot \frac{1}{D}\left(\frac{E_0}{2L} e^{\frac{R}{L}t}\right) = \frac{E_0}{2R}$$

and $$\frac{1}{D+\frac{R}{L}}\left\{\sin\frac{(2n+1)\pi t}{\tau}\right\} = \frac{\frac{R}{L}-D}{\frac{R^2}{L^2}-D^2}\left\{\sin\frac{(2n+1)\pi t}{\tau}\right\}$$

$$= \frac{\frac{R}{L}\sin\frac{(2n+1)\pi t}{\tau} - \frac{(2n+1)\pi}{\tau}\cos\frac{(2n+1)\pi t}{\tau}}{\frac{R^2}{L^2}+\frac{(2n+1)^2\pi^2}{\tau^2}}$$

$$= \frac{\sin\left\{\frac{(2n+1)\pi t}{\tau}-\alpha_{2n+1}\right\}}{\sqrt{\frac{R^2}{L^2}+\frac{(2n+1)^2\pi^2}{\tau^2}}}$$

where $$\alpha_{2n+1} = \tan^{-1}\frac{(2n+1)\pi L}{R\tau}$$

The complete solution of (1) is

$$i = \frac{E_0}{2R} + Ae^{-\frac{R}{L}t} + \frac{2E_0}{\pi}\left[\frac{\sin\left(\frac{\pi t}{\tau}-\alpha_1\right)}{\sqrt{R^2+\frac{\pi^2L^2}{\tau^2}}} + \frac{\sin\left(\frac{3\pi t}{\tau}-\alpha_3\right)}{3\sqrt{R^2+\frac{9\pi^2L^2}{\tau^2}}} + \frac{\sin\left(\frac{5\pi t}{\tau}-\alpha_5\right)}{5\sqrt{R^2+\frac{25\pi^2L^2}{\tau^2}}} + \dots\right]$$

where $$\alpha_1 = \tan^{-1}\frac{\pi L}{R\tau},\ \alpha_3 = \tan^{-1}\frac{3\pi L}{R\tau},\ \alpha_5 = \tan^{-1}\frac{5\pi L}{R\tau},\ \text{etc.}$$

55. **Determination of Complementary Functions.** (1) Consider the equation

$$\frac{d^2y}{dx^2} + (a+b)\frac{dy}{dx} + aby = 0 \qquad . \qquad . \quad \text{(VI.25)}$$

where a and b are constants.

By operators, $[D^2 + (a+b)D + ab]y = 0$

i.e. $$(D+a)(D+b)y = 0$$

$$\therefore \qquad y = \frac{1}{(D+a)(D+b)}(0)$$

$$= \frac{1}{(D+a)(D+b)}(e^{-bx}0)$$

$$= \frac{1}{D+a} e^{-bx} \frac{1}{D}(0) \text{ [by Rule VI]}$$

$\therefore$
$$y = \frac{1}{D+a} Ae^{-bx},$$

where A is an arbitrary constant.

Again
$$y = A \cdot \frac{1}{D+a} \cdot e^{-ax} e^{(a-b)x}$$

$$= Ae^{-ax} \frac{1}{D} e^{(a-b)x}$$

and by Rule V
$$y = Ae^{-ax} \cdot \frac{1}{a-b} \left\{ e^{(a-b)x} + B \right\}$$

where B is an arbitrary constant.

Hence
$$y = \frac{A}{a-b} e^{-bx} + \frac{AB}{a-b} e^{-ax}$$

i.e.
$$y = A_1 e^{-ax} + B_1 e^{-bx} \quad . \quad . \quad . \quad \text{(VI.26)}$$

A_1 and B_1 being arbitrary constants.

The reader should compare this result with that found by means of the auxiliary equation, as in Vol. I. A shorter method than the above is to use partial fractions.

Thus
$$y = \frac{1}{(D+a)(D+b)}(0)$$

$$= \frac{1}{b-a}\left[\frac{1}{D+a} - \frac{1}{D+b}\right](0)$$

$$= \frac{1}{b-a}\left[\frac{1}{D+a}(e^{-ax} \cdot 0) - \frac{1}{D+b}(e^{-bx} \cdot 0)\right]$$

$$= \frac{1}{b-a}\left[e^{-ax}\frac{1}{D}(0) - e^{-bx}\frac{1}{D}(0)\right]$$

i.e.
$$y = A_1 e^{-ax} + B_1 e^{-bx}, \text{ as before.}$$

(2) Consider now the following equation in which the auxiliary equation has a repeated root (compare Case 2, p. 396, Vol. I).

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + a^2y = 0 \qquad . \qquad . \qquad \text{(VI.27)}$$

By operators,

$$\begin{aligned} y &= \frac{1}{(D+a)^2}(0) \\ &= \frac{1}{(D+a)^2}(e^{-ax}0) \\ &= e^{-ax}\frac{1}{D^2}(0) \\ &= e^{-ax}\iint 0\,dxdx \end{aligned}$$

i.e.

$$y = (Ax + B)e^{-ax} \qquad . \qquad . \qquad . \qquad \text{(VI.28)}$$

(3) Next consider the equation

$$(D+a)^r y = 0 \qquad . \qquad . \qquad . \qquad \text{(VI.29)}$$

where r is a positive integer, the auxiliary equation having in this case r equal roots.

We have

$$\begin{aligned} y &= \frac{1}{(D+a)^r}(0) \\ &= \frac{1}{(D+a)^r}(e^{-ax}0) \\ &= e^{-ax}\frac{1}{D^r}(0) \end{aligned}$$

i.e.

$$y = e^{-ax}(A_0 + A_1x + A_2x^2 + \ldots + A_{r-1}x^{r-1}) \qquad \text{(VI.30)}$$

(4) To solve the equation

$$(D+a)(D+b)(D+c)^2y = 0 \qquad . \qquad . \qquad \text{(VI.31)}$$

We have

$$y = \frac{1}{(D+a)(D+b)(D+c)^2}(0)$$

and by partial fractions,

$$y = \left[\frac{A}{D+a} + \frac{B}{D+b} + \frac{C}{D+c} + \frac{E}{(D+c)^2}\right](0)$$

where A, B, C, E are constants whose values are found by the rules given in Vol. I. We shall see that their values do not need to be

found in determining complementary functions as they are absorbed into the arbitrary constants which occur on integration.

We have then

$$y = \frac{A}{D+a}(0) + \frac{B}{D+b}(0) + \frac{C}{D+c}(0) + \frac{E}{(D+c)^2}(0)$$

$$= \frac{A}{D+a}(e^{-ax}0) + \frac{B}{D+b}(e^{-bx}0) + \frac{C}{D+c}(e^{-cx}0) + \frac{E}{(D+c)^2}(e^{-cx}0)$$

$$= Ae^{-ax}\frac{1}{D}(0) + Be^{-bx}\frac{1}{D}(0) + Ce^{-cx}\frac{1}{D}(0) + Ee^{-cx}\frac{1}{D^2}(0)$$

$$= A_1e^{-ax} + B_1e^{-bx} + C_1e^{-cx} + (E_1 + E_2x)e^{-cx}$$

where A_1, B_1, C_1, E_1, E_2 are arbitrary constants.

We combine the third and fourth terms, and obtain

$$y = A_1e^{-ax} + B_1e^{-bx} + F_1e^{-cx} + E_2xe^{-cx} \quad . \quad \text{(VI.32)}$$

where F_1 is also an arbitrary constant.

The reader should examine this result and should be able to write down by inspection the complete solution of an equation such as (VI.31).

(5) Complex factors: To solve

$$(D^2 + a^2)(D + b)y = 0 \quad . \quad . \quad \text{(VI.33)}$$

Using complex factors

$$y = \frac{1}{(D+ia)(D-ia)(D+b)}(0)$$

where $i = \sqrt{-1}$, and by partial fractions,

$$y = \left(\frac{A}{D+ia} + \frac{B}{D-ia} + \frac{C}{D+b}\right)(0)$$

Then $$y = A\,.\,\frac{1}{D+ia}(e^{-iax}0) + B\,.\,\frac{1}{D-ia}(e^{iax}0) + C\frac{1}{D+b}(e^{-bx}0)$$

$$= Ae^{-iax}\frac{1}{D}(0) + Be^{iax}\frac{1}{D}(0) + Ce^{-bx}\frac{1}{D}(0)$$

or, since each integration produces an arbitrary constant,

$$y = A_1e^{-iax} + B_1e^{iax} + C_1e^{-bx}$$

and by complex algebra,

$$y = A_1(\cos ax - i \sin ax) + B_1(\cos ax + i \sin ax) + C_1e^{-bx}$$

The solution is

$$y = L \cos ax + M \sin ax + C_1e^{-bx} \quad . \quad (\text{VI.34})$$

where L, M, C_1 are arbitrary constants.

(6) Repeated complex factors: To solve

$$\frac{d^4y}{dx^4} + 2a^2\frac{d^2y}{dx^2} + a^4y = 0 \quad . \quad . \quad (\text{VI.35})$$

With $D \equiv \dfrac{d}{dx}$, the equation is

$$(D^4 + 2a^2D^2 + a^4)y = 0$$

i.e.

$$(D^2 + a^2)^2y = 0$$

and

$$y = \frac{1}{(D + ia)^2\,(D - ia)^2}(0)$$

Using partial fractions,

$$y = \left[\frac{A}{D + ia} + \frac{B}{(D + ia)^2} + \frac{C}{D - ia} + \frac{E}{(D - ia)^2}\right](0)$$

$$= \frac{A}{D + ia}(e^{-iax}0) + \frac{B}{(D + ia)^2}(e^{-iax}0) + \frac{C}{D - ia}(e^{iax}0) + \frac{E}{(D - ia)^2}(e^{iax}0)$$

$$= Ae^{-iax}\frac{1}{D}(0) + Be^{-iax}\frac{1}{D^2}(0) + Ce^{iax}\frac{1}{D}(0) + Ee^{iax}\frac{1}{D^2}(0)$$

$$= Ae^{-iax}F + Be^{-iax}(G + Hx) + Ce^{iax}I + Ee^{iax}(J + Kx)$$

$$= Le^{-iax} + Mxe^{-iax} + Ne^{iax} + Pxe^{iax}$$

where $L = AF + BG$, $M = BH$, $N = CI + EJ$, and $P = EK$ are arbitrary constants.

By complex algebra,

$$y = (L + Mx)(\cos ax - i \sin ax) + (N + Px)(\cos ax + i \sin ax)$$

i.e. $y = \{L + N + (M + P)x\} \cos ax + i\{N - L + (P - M)x\} \sin ax$

i.e. $y = (Q + Rx) \cos ax + (S + Tx) \sin ax$ . . . (VI.36)

where Q, R, S, T are arbitrary constants.

(7) To solve the equation

$$\frac{d^2y}{dx^2} + 2a \frac{dy}{dx} + (a^2 + b^2)y = 0 \quad . \quad . \quad \text{(VI.37)}$$

This is the case in which the auxiliary equation has complex roots.

We have $$(D^2 + 2aD + a^2 + b^2)y = 0$$

i.e. $$[(D + a)^2 + b^2]y = 0$$

Hence

$$y = \frac{1}{(D + a + ib)(D + a - ib)}(0)$$

$$= \frac{A}{D + a + ib}(0) + \frac{B}{D + a - ib}(0)$$

$$= \frac{A}{D + a + ib}\{e^{-(a+ib)x}0\} + \frac{B}{D + a - ib}\{e^{-(a-ib)x}0\}$$

$$= Ae^{-(a+ib)x}\frac{1}{D}(0) + Be^{-(a-ib)x}\frac{1}{D}(0)$$

$$= e^{-ax}[ACe^{-ibx} + BEe^{ibx}]$$

where C and E are constants of integration.

Substituting F for AC and G for BE,

$$y = e^{-ax}[Fe^{-ibx} + Ge^{ibx}]$$

and by complex algebra,

$$y = e^{-ax}[(F + G) \cos bx + i(G - F) \sin bx]$$

i.e. $$y = e^{-ax}(L \cos bx + M \sin bx) \quad . \quad . \quad . \quad \text{(VI.38)}$$

where $L = F + G$ and $M = i(G - F)$ are arbitrary constants. (VI.38) can be written in the form

$$y = Re^{-ax} \begin{matrix}\sin \\ \cos\end{matrix} (bx + \alpha) \quad . \quad . \quad . \quad \text{(VI.39)}$$

where R and α are arbitrary constants and sin or cos may be taken at choice. (VI.37) is the equation of motion for damped vibrations, and we shall deal with its applications later in this chapter.

56. **Justification of the Use of Partial Fractions and of the Use of the Binomial Expansion with Operators.** Consider the relation

$$F(D) \equiv (D-a_1)(D-a_2)(D-a_3)^2$$

where a_1, a_2, a_3 are constants, real, imaginary, or complex. If $\dfrac{1}{F(D)}$ operates on X, we have

$$\frac{1}{F(D)}X \equiv \frac{1}{(D-a_1)(D-a_2)(D-a_3)^2}X$$

and, assuming we may split up $\dfrac{1}{F(D)}$ into partial fractions,

$$\frac{1}{F(D)}X = \left[\frac{A}{D-a_1} + \frac{B}{D-a_2} + \frac{C}{D-a_3} + \frac{E}{(D-a_3)^2}\right]X \quad \text{(VI.40)}$$

If the use of partial fractions is justified, then operating on (VI.40) with $F(D)$ will lead to the identity $X \equiv X$. This operation gives

$$X = \left[\frac{A\,.\,F(D)}{D-a_1} + \frac{B\,.\,F(D)}{D-a_2} + \frac{C\,.\,F(D)}{D-a_3} + \frac{E\,.\,F(D)}{(D-a_3)^2}\right]X$$

i.e. $$X = [A(D-a_2)(D-a_3)^2 + B(D-a_1)(D-a_3)^2 + C(D-a_1)(D-a_2)(D-a_3) + E(D-a_1)(D-a_2)]X \quad \text{(VI.41)}$$

Now, the expression in the square brackets in (VI.41) is the actual expression we make identically equal to unity when evaluating A, B, C, E, and (VI.41) reduces to the identity $X \equiv X$. Thus we see that the use of partial fractions is justified in this case, and it is clear that the proof may be extended to cover the general case in which $F(D)$ is any polynomial in D.

In Art. 53, Example 2, we expanded the operator $\left(1+\dfrac{D}{a}\right)^{-n}$ by the binomial theorem, the operand being of the form x^r, where r is a positive integer. Consider the case $n = 1$, and assume that we may expand $\left(1+\dfrac{D}{a}\right)^{-1}$ by the binomial theorem or by long division.

We have $\dfrac{1}{1+\dfrac{D}{a}}x^r = \left(1+\dfrac{D}{a}\right)^{-1}x^r = \left(1-\dfrac{D}{a}+\dfrac{D^2}{a^2}-\dfrac{D^3}{a^3}\right.$

$$\left. + \ldots \text{ to } \overline{r+1} \text{ terms}\right)x^r$$

so that

$$\frac{1}{1+\dfrac{D}{a}}x^r = x^r - \frac{r}{a}x^{r-1} + \frac{r(r-1)}{a^2}x^{r-2} - \frac{r(r-1)(r-2)}{a^3}x^{r-3}$$
$$+ \ldots \pm \frac{\lfloor r}{a^r} \quad . \quad . \quad (\text{VI.42})$$

Now, if we are justified in expanding $\left(1+\dfrac{D}{a}\right)^{-1}$ in a series, the operator $1+\dfrac{D}{a}$ applied to both sides of (VI.42) should reduce it to the identity $x^r \equiv x^r$.

We have then

$$\frac{1+\dfrac{D}{a}}{1+\dfrac{D}{a}}x^r = \left(1+\frac{D}{a}\right)\left[x^r - \frac{r}{a}x^{r-1} + \frac{r(r-1)}{a^2}x^{r-2} - \ldots \pm \frac{\lfloor r}{a^r}\right]$$

$$= x^r - \frac{r}{a}x^{r-1} + \frac{r(r-1)}{a^2}x^{r-2} - \ldots \pm \frac{\lfloor r}{a^r}$$

$$+ \frac{r}{a}x^{r-1} - \frac{r(r-1)}{a^2}x^{r-2} + \ldots \mp \frac{\lfloor r}{a^r}$$

i.e. $\quad x^r \equiv x^r$, and the method is justified.

Similarly, it can be shown that the expanded form of $\left(1+\dfrac{D}{a}\right)^{-2}$ is the inverse of $\left(1+\dfrac{D}{a}\right)^{2}$, and the proof may be extended to show that for all positive integral values of n, the operator $\left(1+\dfrac{D}{a}\right)^{-n}$ may be expanded in series form before operating on an integral positive power of x. The expansion produces an infinite series in ascending powers of D, but as $D^n x^r = 0$ when $n > r$, the series terminates after $r + 1$ terms.

57. Determination of Particular Integrals and Complete Solutions. We shall first show the general method, which, however, is not usually the most convenient. To find a particular integral of (VI.4), we have

$$y = \frac{1}{F(D)} X \quad . \quad . \quad . \quad \text{(VI.43)}$$

where $F(D)$ is a polynomial in D with constant coefficients and X is any function of x. Suppose that the polynomial is of degree n, and let its factors be $(D - a_1)(D - a_2)(D - a_3) \ldots (D - a_{n-1})(D - a_n)$, where the constants a_1, a_2, a_3, etc. are real, imaginary, or complex.

Then (VI.43) becomes

$$y = \frac{1}{(D-a_1)(D-a_2)\ldots(D-a_{n-1})(D-a_n)} X \, . \quad \text{(VI.44)}$$

or by partial fractions,

$$y = \left(\frac{A_1}{D-a_1} + \frac{A_2}{D-a_2} + \ldots + \frac{A_{n-1}}{D-a_{n-1}} + \frac{A_n}{D-a_n}\right) X \quad \text{(VI.45)}$$

If r is any of the numbers 1 to n, we have for the rth term

$$y_r = \frac{A_r}{D-a_r} X \quad . \quad . \quad . \quad \text{(VI.46)}$$

The value of A_r is determined by leaving out the factor $(D - a_r)$ from the coefficient of X in (VI.44) and substituting a_r for D. Inserting the factors $e^{a_r x}$ and $e^{-a_r x}$, whose product is unity, on the right-hand side of (VI.46), we obtain

$$y_r = \frac{A_r}{D-a_r} e^{a_r x} e^{-a_r x} X$$

and by Rule VI, $y_r = A_r e^{a_r x} \dfrac{1}{D}(e^{-a_r x} X) = A_r e^{a_r x} \int e^{-a_r x} X dx$ (VI.47)

Substituting $r = 1, 2, 3, \ldots$ up to n in succession in (VI.47) and adding the results, we have for the particular integral

$$y = A_1 e^{a_1 x} \int e^{-a_1 x} X dx + A_2 e^{a_2 x} \int e^{-a_2 x} X dx + A_3 e^{a_3 x} \int e^{-a_3 x} X dx$$
$$+ \ldots + A_n e^{a_n x} \int e^{-a_n x} X dx \quad . \quad \text{(VI.48)}$$

The complementary function is found by putting $X = 0$ in (VI.48), and since $\int 0dx$ is a constant, the complementary function is

$$y = C_1e^{a_1x} + C_2e^{a_2x} + C_3e^{a_3x} + \ldots + C_ne^{a_nx} \quad . \quad \text{(VI.49)}$$

where $C_1, C_2, C_3, \ldots, C_n$ are arbitrary constants.

The complete solution of (VI.4) is the sum of the values of y given by (VI.49) and (VI.48). This method of solving a differential equation is known as the *method of quadratures*. The integrals may be evaluated by direct integration, when possible, or by approximate graphical or numerical methods.

EXAMPLE 1

Find a particular integral of $D^3y + 2D^2y - Dy - 2 = x^2$ . . . . (1)

We have
$$y = \frac{1}{D^3 + 2D^2 - D - 2}x^2$$
$$= \frac{1}{(D-1)(D+1)(D+2)}x^2$$
$$= \left[\frac{A}{D-1} + \frac{B}{D+1} + \frac{C}{D+2}\right]x^2$$

Evaluated by the usual method,

$$A = \frac{1}{2 \times 3} = \frac{1}{6},\ B = \frac{1}{-2 \times 1} = -\frac{1}{2},\ \text{and } C = \frac{1}{-3 \times -1} = \frac{1}{3}$$

so that
$$y = \left[\frac{1}{6}\cdot\frac{1}{D-1} - \frac{1}{2}\cdot\frac{1}{D+1} + \frac{1}{3}\cdot\frac{1}{D+2}\right]x^2 \quad . \quad . \quad . \quad (2)$$

Using (VI.48), $y = \frac{1}{6}e^x \int x^2e^{-x}dx - \frac{1}{2}e^{-x}\int x^2e^xdx + \frac{1}{3}e^{-2x}\int x^2e^{2x}dx$ . . (3)

To simplify this we integrate by parts $\int x^2e^{cx}dx$;

thus
$$\int x^2e^{cx}dx = \frac{1}{c}x^2e^{cx} - \frac{2}{c}\int xe^{cx}dx$$
$$= \frac{1}{c}x^2e^{cx} - \frac{2}{c^2}xe^{cx} + \frac{2}{c^2}\int e^{cx}dx$$
$$= \frac{e^{cx}}{c^3}(c^2x^2 - 2cx + 2)$$

Substituting $c = -1, 1, 2$ in succession, we obtain for the integrals in (3), $-e^{-x}(x^2 + 2x + 2)$, $e^x(x^2 - 2x + 2)$, $\frac{1}{2}e^{2x}(x^2 - x + \frac{1}{2})$, and from (3),

$$y = -\tfrac{1}{6}(x^2 + 2x + 2) - \tfrac{1}{2}(x^2 - 2x + 2) + \tfrac{1}{6}(x^2 - x + \tfrac{1}{2})$$

whence $y = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{5}{4}$ is the required particular integral.

This method is based on the general expression (VI.48). Otherwise, we can expand the operational expression $\dfrac{1}{D^3 + 2D^2 - D - 2}$ as follows. By the binomial theorem, or by division,

$$y = -\tfrac{1}{2}\left(1 + \frac{D}{2} - D^2 - \frac{D^3}{2}\right)^{-1} x^2$$

$$= -\tfrac{1}{2}\left[1 + \left(\frac{D}{2} - D^2\right)\right]^{-1} x^2$$

(neglecting powers of D above the second, since $D^3x^2 = 0$)

$$= -\tfrac{1}{2}\left[1 - \frac{D}{2} + D^2 + \frac{D^2}{4}\right] x^2$$

and $$y = -\tfrac{1}{2}x^2 + \tfrac{1}{2}x - \tfrac{5}{4}, \text{ as before.}$$

When the power of x is a larger one, the expansion must be carried further and may become involved. In such a case it is better to expand each of the functions of D in equation (2).

The above methods of solution can be used when the terms of X are integral powers of x, or products of the type x^re^{ax}, where r is a positive integer.

As $\sin(px + q)$ and $\cos(px + q)$ can be expressed in the forms $\dfrac{1}{2i}[e^{i(px+q)} - e^{-i(px+q)}]$ and $\tfrac{1}{2}[e^{i(px+q)} + e^{-i(px+q)}]$ respectively, products of either of these and a power of x may be dealt with by the methods of this section.

EXAMPLE 2

Find a particular integral of $\dfrac{d^2y}{dx^2} + 4y = x\sin x$ . . . . . . (1)

We have $$y = \frac{1}{D^2 + 4}x\sin x \quad . . . . . . (2)$$

We shall find a particular integral of

$$y = \frac{1}{D^2 + 4}e^{ix}x \quad . . . . . . (3)$$

and, as $e^{ix} = \cos x + i\sin x$, the imaginary part of this particular integral will be the required solution of (2),

By Rule VI, $$y = e^{ix}\frac{1}{(D + i)^2 + 4}x$$

$$= e^{ix}\frac{1}{3 + 2iD + D^2}x$$

$$= \tfrac{1}{3}e^{ix}[1 + (\tfrac{2}{3}iD + \tfrac{1}{3}D^2)]^{-1}x$$

$$= \tfrac{1}{3}e^{ix}[1 - (\tfrac{2}{3}iD + \tfrac{1}{3}D^2) + \ldots]x$$

$$= \tfrac{1}{3}e^{ix}(1 - \tfrac{2}{3}iD)x, \text{ since } D^2x = 0,\ D^3x = 0, \text{ etc.}$$

$$= \tfrac{1}{3}(\cos x + i\sin x)(x - \tfrac{2}{3}i)$$

i.e. $$y = \tfrac{1}{3}(x\cos x + \tfrac{2}{3}\sin x) + \tfrac{1}{3}i(x\sin x - \tfrac{2}{3}\cos x) \quad . . . (4)$$

The coefficient of i in this expression is a particular solution of (1). This is

$$y = \tfrac{1}{3}(x \sin x - \tfrac{2}{3} \cos x) \qquad . \quad . \quad . \quad . \quad (5)$$

The complementary function of (1) is obtained by substituting $a = 0$, $b = 2$ in (VI.37) and (VI.38),

i.e.
$$y = L \cos 2x + M \sin 2x \qquad . \quad . \quad . \quad . \quad (6)$$

Then, the full solution of (1) is

$$y = L \cos 2x + M \sin 2x + \tfrac{1}{3}(x \sin x - \tfrac{2}{3} \cos x)$$

If in (1) $\cos x$ is substituted for $\sin x$, a particular integral is the real part of y in (4), i.e. $\frac{1}{3}(x \cos x + \frac{2}{3} \sin x)$.

EXAMPLE 3

Find a particular integral of $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = x^3e^{3x}$ . . . . . (1)

We have
$$y = \frac{1}{2 + 3D + D^2}(x^3e^{3x})$$

$$= e^{3x}\frac{1}{2 + 3(D + 3) + (D + 3)^2}x^3$$

$$= e^{3x}\frac{1}{20 + 9D + D^2}x^3$$

$$= \frac{1}{20}e^{3x}\left[1 + \left(\frac{9}{20}D + \frac{1}{20}D^2\right)\right]^{-1}x^3$$

$$= \frac{1}{20}e^{3x}\left[1 - \left(\frac{9}{20}D + \frac{1}{20}D^2\right) + \left(\frac{9}{20}D + \frac{1}{20}D^2\right)^2 - \left(\frac{9}{20}D\right)^3\right]$$

the expansion being carried as far as D^3 since $D^4x^3 = 0$.

$$\therefore \qquad y = \frac{1}{20}e^{3x}\left[1 - \frac{9}{20}D + \frac{61}{400}D^2 - \frac{369}{8\,000}D^3\right]x^3$$

i.e.
$$y = \frac{1}{20}e^{3x}\left[x^3 - \frac{27}{20}x^2 + \frac{183}{200}x - \frac{1\,107}{4\,000}\right]$$

This is the particular integral of (1). The complementary function is easily seen to be $y = Ae^{-x} + Be^{-2x}$, where A and B are arbitrary constants. The complete solution of (1) is

$$y = Ae^{-x} + Be^{-2x} + \frac{1}{20}e^{3x}\left(x^3 - \frac{27}{20}x^2 + \frac{183}{200}x - \frac{1\,107}{4\,000}\right)$$

EXAMPLE 4

Solve completely
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 6y = x^2e^{2x} \qquad . \quad . \quad . \quad . \quad (1)$$

We have $$y = \frac{1}{D^2 - 2D + 6} e^{2x}x^2$$

and by Rule VI, $$y = e^{2x} \frac{1}{(D + 2)^2 - 2(D + 2) + 6} x^2$$

$$= e^{2x} \frac{1}{6 + 2D + D^2} x^2$$

$$= \tfrac{1}{6}e^{2x} \left[1 + \left(\frac{D}{3} + \frac{D^2}{6} \right) \right]^{-1} x^2$$

$$= \tfrac{1}{6}e^{2x} \left[1 - \left(\frac{D}{3} + \frac{D^2}{6} \right) + \left(\frac{D}{3} + \frac{D^2}{6} \right)^2 \right] x^2$$

(no further expansion being necessary)

$$= \tfrac{1}{6}e^{2x} \left[1 - \frac{D}{3} - \frac{D^2}{18} \right] x^2$$

i.e. $$y = \tfrac{1}{6}e^{2x} (x^2 - \tfrac{2}{3}x - \tfrac{1}{9}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(2) is the particular integral. The complementary function is readily found to be $y = e^x(A \sin \sqrt{5}x + B \cos \sqrt{5}x)$, and the complete solution of (1) is

$$y = e^x(A \sin \sqrt{5}x + B \cos \sqrt{5}x) + \tfrac{1}{6}e^{2x}(x^2 - \tfrac{2}{3}x - \tfrac{1}{9})$$

where A and B are arbitrary constants.

EXAMPLE 5

Solve completely $$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 3e^{2x} \sin (4x + 3) \quad . \quad . \quad . \quad (1)$$

For a particular integral, $y = 3 \cdot \dfrac{1}{(D + 2)^2} e^{2x} \sin (4x + 3)$

and by Rule VI, $$y = 3e^{2x} \frac{1}{(D + 4)^2} \sin (4x + 3)$$

$$= 3e^{2x} \frac{1}{D^2 + 8D + 16} \sin (4x + 3)$$

$$= 3e^{2x} \cdot \frac{1}{8D} \sin (4x + 3)$$

and integrating, $$y = -\tfrac{3}{32} e^{2x} \cos (4x + 3)$$

The complementary function is

$$y = e^{-2x}(Ax + B)$$

and the complete solution of (1) is

$$y = e^{-2x}(Ax + B) - \tfrac{3}{32} e^{2x} \cos (4x + 3)$$

EXAMPLE 6

Solve $$\frac{d^2y}{dt^2} + 2a\frac{dy}{dt} + n^2y = c\sin(pt+q) \quad . \quad . \quad . \quad (1)$$

First consider the reduced equation $\frac{d^2y}{dt^2} + 2a\frac{dy}{dt} + n^2y = 0 \quad . \quad . \quad . \quad (2)$

Putting $D = \frac{d}{dt}$, we have $y = \frac{1}{D^2 + 2aD + n^2}(0)$

Now $$D^2 + 2aD + n^2 = (D+a)^2 - (a^2 - n^2)$$
$$= (D + a - \sqrt{a^2 - n^2})(D + a + \sqrt{a^2 - n^2})$$

$$\therefore \quad y = \frac{1}{(D + a - \sqrt{a^2 - n^2})(D + a + \sqrt{a^2 - n^2})}(0)$$

$$= \frac{1}{2\sqrt{a^2 - n^2}}\left[\frac{1}{D + a - \sqrt{a^2 - n^2}} - \frac{1}{D + a + \sqrt{a^2 - n^2}}\right](0)$$

$$= \frac{1}{2\sqrt{a^2 - n^2}}\left[\frac{1}{D + a - \sqrt{a^2 - n^2}}e^{-(a - \sqrt{a^2 - n^2})t}(0)\right.$$
$$\left. - \frac{1}{D + a + \sqrt{a^2 - n^2}}e^{-(a + \sqrt{a^2 - n^2})t}(0)\right]$$

$$= \frac{1}{2\sqrt{a^2 - n^2}}\left[e^{-(a - \sqrt{a^2 - n^2})t}\frac{1}{D}(0) - e^{-(a + \sqrt{a^2 - n^2})t}\frac{1}{D}(0)\right]$$

Hence $$y = \frac{e^{-at}}{2\sqrt{a^2 - n^2}}(Ae^{\sqrt{a^2 - n^2}\,.\,t} - Be^{-\sqrt{a^2 - n^2}\,.\,t})$$

or, if C is written for $\frac{A}{2\sqrt{a^2 - n^2}}$ and E for $\frac{-B}{2\sqrt{a^2 - n^2}}$

$$y = e^{-at}(Ce^{\sqrt{a^2 - n^2}\,.\,t} + Ee^{-\sqrt{a^2 - n^2}\,.\,t}) \quad . \quad . \quad . \quad (3)$$

(3) is the complementary function of (1), and, if $a^2 > n^2$, it is in its simplest form.

Case I, $a^2 > n^2$. $y = Ce^{(-a + \sqrt{a^2 - n^2})t} + Ee^{(-a - \sqrt{a^2 - n^2})t} \quad . \quad . \quad (4)$

and y is the sum of two decay functions.

Case II, $a^2 = n^2$. In this case (3) becomes $y = e^{-at}(C + E)$, which is not the complete solution, since $C + E$ is a single arbitrary constant.

We have $$y = \frac{1}{(D+a)^2}(0) = \frac{1}{(D+a)^2}(e^{-at} \times 0)$$

and by Rule VI, $$y = e^{-at}\frac{1}{D^2}(0)$$

i.e. $$y = e^{-at}(At + B) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Case III, $a^2 < n^2$. Here we write $i\sqrt{n^2 - a^2}$ for $\sqrt{a^2 - n^2}$, and (3) becomes

$$y = e^{-at}(Ce^{i\sqrt{n^2-a^2}t} + Ee^{-i\sqrt{n^2-a^2}t})$$

$$= e^{-at}[(C + E)\cos\sqrt{n^2 - a^2}t + i(C - E)\sin\sqrt{n^2 - a^2}t]$$

$$\left.\begin{aligned} \text{i.e.} \quad & y = e^{-at}[L\cos\sqrt{n^2 - a^2}t + M\sin\sqrt{n^2 - a^2}t] \\ \text{or} \quad & y = Re^{-at}\sin(\sqrt{n^2 - a^2}t + \beta) \\ \text{or} \quad & y = Re^{-at}\cos(\sqrt{n^2 - a^2}t + \gamma) \end{aligned}\right\} \quad . \quad . \quad (6)$$

where L, M, R, β, γ are arbitrary constants.

Thus we see that the complementary function of (1) may take any one of the forms (4), (5), and (6). We shall discuss these later in connection with damped vibrations.

To find a particular integral we have from (1),

$$y = c \cdot \frac{1}{D^2 + 2aD + n^2}\sin(pt + q)$$

and by Rule VII, $\quad y = c \cdot \dfrac{1}{2aD + n^2 - p^2}\sin(pt + q)$

By Rule VIII, $\quad y = \dfrac{c\sin(pt + q - \alpha)}{\sqrt{(n^2 - p^2)^2 + 4a^2p^2}}$, where $\alpha = \tan^{-1}\dfrac{2ap}{n^2 - p^2} \quad . \quad (7)$

The complete solution of (1) is the sum of (7) and the complementary function.

If $a = 0$, (1) becomes $\dfrac{d^2y}{dt^2} + n^2y = c\sin(pt + q) \quad . \quad . \quad . \quad . \quad (8)$

and its complete solution is

$$y = \frac{c}{n^2 - p^2}\sin(pt + q) + R\sin(nt + \beta) \quad . \quad . \quad . \quad (9)$$

where R and β are arbitrary constants, an alternative to the last term in (9) being $R\cos(nt + \gamma)$.

58. Cases of Failure when finding Particular Integrals. Consider the equation

$$\frac{d^2y}{dt^2} + n^2y = a\sin nt \quad . \quad . \quad . \quad (\text{VI.50})$$

With $D \equiv \dfrac{d}{dt}$, the particular integral is given by

$$y = a\frac{1}{D^2 + n^2}\sin nt \quad . \quad . \quad . \quad (\text{VI.51})$$

By Rule VII, $\qquad y = a\dfrac{\sin nt}{-n^2+n^2}$

which is infinite, and the method fails to give a particular integral. Since $\sin nt$ is the imaginary part of $e^{int} \equiv \cos nt + i\sin nt$ we can write e^{int} instead of $\sin nt$ on the right-hand side of (VI.51) if we take the imaginary part of the resulting particular integral as the particular integral required. Thus we have

$$y = a\frac{1}{D^2+n^2}e^{int} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{VI.52})$$

i.e. $\quad y = a\dfrac{1}{(D-in)(D+in)}e^{int}$

and by partial fractions,

$$y = \frac{a}{2in}\left(\frac{1}{D-in} - \frac{1}{D+in}\right)e^{in}$$

By Rules VI and V,

$$\begin{aligned}
y &= \frac{a}{2in}\left(e^{int}\frac{1}{D}(1) - \frac{1}{2in}e^{int}\right)\\
&= \frac{ae^{int}}{2in}\left(t - \frac{1}{2in}\right)\\
&= \frac{a}{2in}(\cos nt + i\sin nt)\left(t - \frac{1}{2in}\right)\\
&= \frac{a}{2in}\left[\left(t\cos nt - \frac{1}{2n}\sin nt\right) + i\left(t\sin nt + \frac{1}{2n}\cos nt\right)\right]\\
&= \frac{a}{2n}\left[\left(t\sin nt + \frac{1}{2n}\cos nt\right) + i\left(\frac{1}{2n}\sin nt - t\cos nt\right)\right]
\end{aligned}$$

This is a particular integral of (VI.52), and the particular integral of (VI.51) is its imaginary part, i.e.

$$y = \frac{a}{2n}\left(\frac{1}{2n}\sin nt - t\cos nt\right) \quad . \quad . \quad (\text{VI.53})$$

The complementary function of (VI.50) is $L\cos nt + M\sin nt$. The term $\dfrac{a}{4n^2}\sin nt$ is contained in this, so that the complete solution of (VI.50) is

$$y = L\cos nt + M\sin nt - \frac{at}{2n}\cos nt \quad . \quad (\text{VI.54})$$

where L and M are arbitrary constants.

Again, when finding a particular integral of $F(D)y = e^{\alpha x}$, we have by Rule V,

$$y = \frac{e^{\alpha x}}{F(\alpha)}$$

and the method fails if $F(\alpha) = 0$, i.e. if α is a root of $F(D) = 0$. We deal with this case by using Rule VI thus,

$$y = \frac{1}{F(D)} e^{\alpha x}$$

$$= e^{\alpha x} \frac{1}{F(D+\alpha)}(1) \quad . \quad . \quad . \quad \text{(VI.55)}$$

which leads to a solution, since, if $F(\alpha) = 0$, $F(D + \alpha) \neq 0$.

Consider the equation $\dfrac{d^2y}{dx^2} + (a + b)\dfrac{dy}{dx} + aby = e^{-bx}$. (VI.56)

We have
$$y = \frac{1}{(D+a)(D+b)} e^{-bx} \quad . \quad . \quad \text{(VI.57)}$$

By Rule V,
$$y = \frac{e^{-bx}}{(a-b)(0)}$$

and the method fails. By partial fractions (VI.57) becomes

$$y = \frac{1}{a-b}\left[\frac{1}{D+b} e^{-bx} - \frac{1}{D+a} e^{-bx}\right]$$

$$= \frac{1}{a-b}\left[e^{-bx}\frac{1}{D}(1) - \frac{1}{-b+a} e^{-bx}\right]$$

$$= \frac{1}{a-b}\left[xe^{-bx} - \frac{1}{a-b} e^{-bx}\right] \quad . \quad . \quad \text{(VI.58)}$$

The complementary function is $y = Ae^{-ax} + Be^{-bx}$, and the part of (VI.58) not included in this is $\dfrac{1}{a-b} xe^{-bx}$

The complete solution is, therefore,

$$y = Ae^{-ax} + Be^{-bx} + \frac{1}{a-b} xe^{-bx} . \quad . \quad \text{(VI.59)}$$

59. **Simple Harmonic Motion.** The definition of simple harmonic motion given in Vol. I limits it to rectilinear motion. Taking x as the displacement of the particle from its mean position at time t we proved the relations

$$x = r\cos(\omega t + \alpha) \text{ and } \frac{d^2x}{dt^2} = -\omega^2 x \quad . \quad \text{(VI.60)}$$

where r, ω, and α are arbitrary constants. The former relation may also be written $x = r\sin(\omega t + \alpha)$. If the vibrating particle moves in a circular or curved path we still call the motion simple harmonic if the above relations are satisfied, x now being measured along the path and d^2x/dt^2 being the component acceleration along the path. In general, if a rigid body or a rigidly connected system of particles in plane motion is such that its position at any time t can be specified by a single variable x, and x and t satisfy either of the relations (VI.60), the motion is simple harmonic motion. Though the engineer often has to produce vibrations in machinery, as for instance, in the simple engine mechanism or in the sewing machine, he is also concerned with the reduction of vibrations and their effects caused by disturbances due to periodic forces or couples. Thus he designs the spring system of a motor-car so that vibrations of the chassis due to road inequalities, etc., shall have as little effect as possible on the car body, and he attempts to balance the moving parts of engines and machines which run at high speeds. We saw in the last chapter how periodic disturbances can be expressed in simple harmonic form and it appears, therefore, that a thorough understanding of simple harmonic motion is a necessary part of the engineer's equipment. Except in gravity systems like the pendulum and other similar cases in which there are fields of force, the important properties of material vibrating systems are (*a*) the inertial mass of the vibrating body and (*b*) the spring force exerted on the body by its constraint, which tends to force the body back to its position of equilibrium. We refer to single mass systems only. The corresponding properties in an electric vibrating system are (*a*) the inductance of the circuit and (*b*) the reciprocal of the capacity in

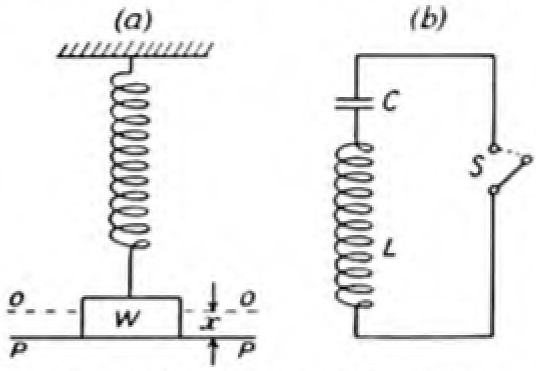


FIG. 40. VIBRATING SYSTEMS

the circuit. The electrical engineer is concerned with the production of vibrations as in generating and distributing alternating current, but he must also pay close attention to the elimination or reduction of unwanted vibrations. In Fig. 40 (*a*) we show a mass of W lb weight suspended from the lower end of a vertical spiral spring of which the upper end is fixed. In the stationary position the pull of gravity W lb is balanced by the tension in the spring. The mass is pulled downwards and then released so that it begins to vibrate. Consider the mass when it is at a distance x ft below the position of equilibrium and, assuming Hooke's law to apply, let sx lb be the additional tension in the spring. In the figure OO is the position of equilibrium of the lower end of the weight and PP that when the time is t sec and the displacement x ft. The unbalanced force is sx lb acting upwards and the equation of motion is

$$\text{Force} = \text{mass} \times \text{acceleration}$$

i.e.

$$-\,sx = \frac{W}{g}\frac{d^2x}{dt^2}$$

or

$$\frac{d^2x}{dt^2} + n^2 x = 0 \qquad . \quad . \quad . \quad . \quad . \quad (\text{VI}.61)$$

where $n^2 = \dfrac{sg}{W}$. The solution of this is by comparison with (VI.60)

$$\left.\begin{aligned} x &= R \,{\sin \atop \cos}\,(nt + \alpha) \\ \text{or} \qquad x &= A \sin nt + B \cos nt \end{aligned}\right\} \quad . \quad . \quad (\text{VI}.62)$$

where R and α or A and B are arbitrary constants and sin or cos may be taken at choice in the former expression. x varies continually with t, oscillating between the values $x = -\,R$ and $x = +\,R$ or $x = -\sqrt{A^2 + B^2}$ and $+\sqrt{A^2 + B^2}$. The motion is simple harmonic.

Now consider the system represented in Fig. 40 (*b*) in which a condenser of capacity C is connected in series with a coil of inductance L and a switch S. With the switch open the condenser is charged and the switch is then closed. Let q be the charge at time t sec after the switch is closed and i the current in the circuit at that instant. Then $i = -\dfrac{dq}{dt}$. The voltage drop across the condenser is

$\frac{q}{C}$. The back electromotive force due to the inductance is $L\frac{di}{dt}$, so that the voltage drop across the inductance in the direction in which the current is flowing is $-L\frac{di}{dt} = L\frac{d^2q}{dt^2}$. The total voltage drop is zero because the outer ends of the coil and the condenser are directly connected by a perfect conductor, and we have

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0 \quad . \quad . \quad . \quad \text{(VI.63)}$$

or

$$-\frac{q}{C} = L\frac{d^2q}{dt^2} \quad . \quad . \quad \text{(VI.64)}$$

Comparing (VI.64) with (VI.61), we see that q takes the place of x as the dependent variable, $\frac{1}{C}$ takes the place of the spring constant s, and L takes the place of the mass $\frac{W}{g}$. Thus $\frac{1}{C}$ can be regarded as the electric spring constant and L as the inertial mass of the circuit.

From (VI.62) we see that the solution of (VI.64) is

$$\left.\begin{aligned} q &= q_0 \begin{matrix}\sin\\ \cos\end{matrix}(nt + \alpha) \\ \text{or} \quad q &= A\sin nt + B\cos nt \end{aligned}\right\} \quad . \quad . \quad \text{(VI.65)}$$

where $n = \frac{1}{\sqrt{LC}}$, and q_0 and α or A and B are arbitrary constants. The electrical oscillations die away because of slight resistances in the circuit just as the mechanical oscillations die away because of air resistance and internal friction or hysteresis in the material vibrating system. The above vibrations are known as *free or natural vibrations* as distinguished from vibrations which are maintained by the action of external forces and which are called *forced vibrations*.

60. **The Energy Method in Elastic Vibrating Systems.** In systems such as the above in which there is no gain or loss of energy by the system due to external forces or internal resistances the equation of motion may be readily obtained by first writing down the expression for the total energy E in the system. Since the energy is

constant, $\frac{dE}{dt} = 0$, and this is the equation of motion. In the electrical circuit above we have

$$E = \tfrac{1}{2}Li^2 + \tfrac{1}{2}\frac{q^2}{C} \quad . \quad . \quad . \quad (\text{VI}.66)$$

and differentiating with respect to t,

$$\frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = 0$$

But $i = -\frac{dq}{dt}$ and therefore

$$i\frac{di}{dt} = \frac{dq}{dt}\frac{d^2q}{dt^2}$$

Substituting these and simplifying we obtain (VI.63).

When we know that the motion is simple harmonic motion, we do not need to form the equation of motion, and we use the energy equation in its simplest form. We shall show this in the case of the material vibrating system represented in Fig. 40 (a). The total energy in the system is constant. At the end of a swing the energy is all elastic whilst in the middle of a swing it is all kinetic. Thus, the energy equation takes the form

$$\text{Maximum kinetic energy} = \text{maximum elastic energy} \quad (\text{VI}.67)$$

Now from (VI.62) we see that the amplitude and the maximum velocity are R ft and nR ft/sec respectively. The maximum kinetic energy is $\frac{W}{2g}n^2R^2$ ft-lb and the maximum elastic energy is $\frac{1}{2}Rs \times R$ ft-lb. Substituting these values in (VI.67), we have

$$\frac{W}{2g}n^2R^2 = \tfrac{1}{2}R^2s$$

so that

$$n = \sqrt{\frac{g}{\frac{W}{s}}} \quad . \quad . \quad . \quad . \quad (\text{VI}.68)$$

Now $\dfrac{W}{s}$ ft, or $\dfrac{12W}{s}$ in., is the static deflection, i.e. the deflection which a force of W lb weight would produce when at rest. Hence

$$n = \frac{\sqrt{12g}}{\sqrt{\text{static deflection in inches}}} \quad . \quad . \quad \text{(VI.69)}$$

The time of a complete vibration is $\dfrac{2\pi}{n}$ sec, and the frequency of vibration is $\dfrac{n}{2\pi}$ per sec. n is called the *angular frequency* or the *circular frequency*. It is usual to calculate the frequency in vibrations per minute, so that, if N is this frequency, then

$$N = \frac{60n}{2\pi} = \frac{30}{\pi} \cdot \frac{\sqrt{12g}}{\sqrt{\text{static deflection in inches}}}$$

i.e. $$N = \frac{188}{\sqrt{\text{static deflection in inches}}} \text{ vibrations per min} \qquad \text{(VI.70)}$$

EXAMPLE 1

A light uniform horizontal beam freely supported at its ends carries a load of W lb weight at the middle of its span of L in. If E is Young's modulus of elasticity in lb/in.^{2} and I is the moment of inertia in in.^{4} units of the cross-section about its neutral axis, find the frequency of small vertical vibrations.

From *Strength of Materials* the deflection under the load is $\dfrac{WL^3}{48EI}$ in., and from (VI.70),

$$N = 188\sqrt{\frac{48EI}{WL^3}} \text{ vibrations per min}$$

Assuming the span to be 24 ft, $E = 30 \times 10^6$ lb/in.^{2}, $I = 210$ in.^{4} units, and the load 1 ton, we have

$$N = 188\sqrt{\frac{48 \times 30 \times 10^6 \times 210}{2\,240 \times 288^3}}$$

i.e. $$N = 447 \text{ vibrations per min}$$

EXAMPLE 2

A steel shaft 24 in. long and 2 in. diameter is fixed at one end and carries a mass of 600 lb weight at the other end. Find the frequency of the free axial vibrations.

Let Δ in. be the static extension due to an axial tension of 600 lb. The area of the cross-section of the shaft is π in.^{2}, so that the stress is $\frac{600}{\pi}$ lb/in.^{2} The strain is $\frac{\Delta}{24}$. Now $\frac{\text{stress}}{\text{strain}}$ = Young's modulus of elasticity, which is here 30×10^6 lb/in.^{2}

Hence, $$\frac{600}{\pi} \times \frac{24}{\Delta} = 30 \times 10^6$$

which gives $$\Delta = \frac{600 \times 24}{30\pi \times 10^6} = 0{\cdot}0001528$$

From (VI.70), $$N = \frac{188}{\sqrt{\Delta}} = \frac{188}{\sqrt{0{\cdot}0001528}} \text{ vibrations per min}$$

Hence, required frequency = 15 200 vibrations per min

The reader should work these two examples by writing down and solving the equations of motion.

Systems such as those in this and the preceding section are known as *undamped* systems because the effects of friction and of other energy dissipating agencies are assumed to be negligible, and the vibrations, once started, are assumed to continue indefinitely. Such vibrations are called *undamped* vibrations.

61. **Further Examples of Simple Harmonic Motion.** Consider the case of a heavy pulley of moment of inertia I engineers' units about its axis mounted on the end S_1 of a light stiff shaft whose other end S is rigidly fixed (Fig. 41). Suppose that the pulley is vibrating freely about its axis and consider the motion at time t sec when a radius OA of the pulley is displaced from its position of rest through an angle θ radians. If C is the couple in lb-ft in the shaft when the

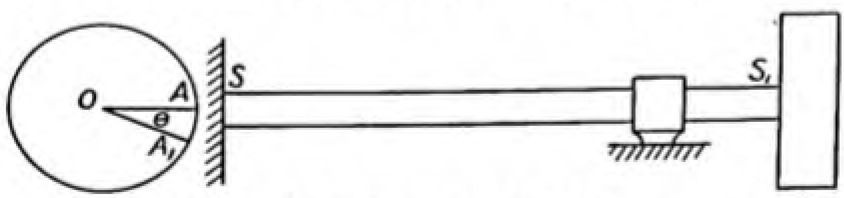


FIG. 41. TORSIONAL VIBRATIONS

angle of twist is θ, then $C = C_0\theta$, where C_0 is a constant which we call the couple per unit twist. The couple C acts in the opposite sense to that of the angular displacement and is, therefore, negative. From dynamics we have

$$\text{Couple} = \text{moment of inertia} \times \text{angular acceleration} \tag{VI.71}$$

i.e.
$$-C_0\theta = I\frac{d^2\theta}{dt^2}$$

or
$$\frac{d^2\theta}{dt^2} + n^2\theta = 0 \qquad . \quad . \quad . \quad \text{(VI.72)}$$

where
$$n^2 = \frac{C_0}{I}$$

This is the same equation as (VI.61) with θ replacing x, and the solution is
$$\theta = \theta_0 \sin(nt + \alpha) \qquad . \quad . \quad \text{(VI.73)}$$

The motion is thus simple harmonic motion in which each particle of the moving mass traces out an arc of a circle. The angular frequency of the natural vibrations is $n = \sqrt{\frac{C_0}{I}}$, which corresponds to $\frac{1}{2\pi}\sqrt{\frac{C_0}{I}}$ vibrations per sec.

The alternative procedure which uses the energy method is simple. We have,

Kinetic energy + Elastic energy = constant

i.e.
$$\tfrac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 + \tfrac{1}{2}C_0\theta^2 = \text{constant} \qquad . \quad . \quad \text{(VI.74)}$$

By differentiating both sides with respect to t,
$$I\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} + C_0\theta\frac{d\theta}{dt} = 0$$

Dividing through by $I\frac{d\theta}{dt}$, we obtain the equation of motion above.

If we wish to find the frequency only, we have

Maximum kinetic energy = maximum potential energy

i.e.
$$\tfrac{1}{2}In^2\theta_0{}^2 = \tfrac{1}{2}C_0\theta_0{}^2$$

whence
$$n^2 = \frac{C_0}{I}, \text{ as before}$$

The vibrating system in Fig. 42 consists of a uniform horizontal bar of weight w lb freely pivoted at O and carrying a load of W lb weight at P. The bar is supported in a horizontal position by a

vertical spring SS_1 fixed to the bar at S_1. Let $\overline{OS_1} = a$ ft, $\overline{OG} = b$ ft, where G is the centre of gravity of the bar, and $\overline{OP} = c$ ft, and let the spring constant be s lb/ft. We shall assume that the bar is set vibrating with initial amplitude θ_0 so small that $1 - \cos \theta_0$ can be ignored compared with θ_0. Let θ radians be the angle through which OP is displaced downwards at time t sec. Then, if I is the

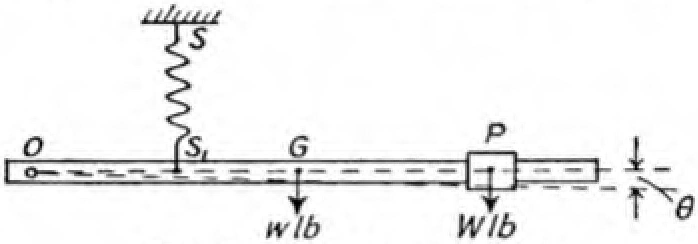


FIG. 42. VIBRATING SYSTEM

moment of inertia of the bar with its attached mass about the axis of rotation through O, the sum of the moments of the acting forces about O is equal to the product of the moment of inertia and the angular acceleration. The stretch of the spring is $a\theta$ ft, the force in it is $sa\theta$ lb, and the moment of this force about O is $sa^2\theta$ lb-ft. In addition to this there is the initial tension T_0 lb in the spring where, from the condition for equilibrium, $T_0 a = wb + Wc$. Thus, the turning moment about O of all the forces acting on the bar in its displaced position is given by

Turning moment about O

$$= - sa^2\theta - T_0 a \cos \theta + wb \cos \theta + Wc \cos \theta$$

and with the value of T_0 substituted,

$$\text{Turning moment} = - sa^2\theta \text{ lb-ft} \qquad \text{(VI.75)}$$

In forming the equation of motion in elastic vibrating systems with small amplitudes we can leave out of consideration the gravity forces provided that we also leave out the elastic forces in the equilibrium position.

The equation of motion is

Moment of forces about O

= moment of inertia about O $\times$ angular acceleration

i.e.

$$- sa^2\theta = I\frac{d^2\theta}{dt^2}$$

or

$$\frac{d^2\theta}{dt^2} + n^2\theta = 0 \qquad \text{(VI.76)}$$

where $n^2 = \frac{sa^2}{I}$. The angular frequency is $n = a\sqrt{\frac{s}{I}}$, and the frequency is $\frac{a}{2\pi}\sqrt{\frac{s}{I}}$ vibrations per sec.

The energy equation in its simplest form is

Maximum kinetic energy = maximum elastic energy

and with amplitude θ_0 and angular frequency n, this becomes

$$\tfrac{1}{2}I(n\theta_0)^2 = \tfrac{1}{2}sa^2\theta_0{}^2$$

from which $$n^2 = \frac{sa^2}{I}, \text{ as before.}$$

62. **Forced Vibrations.** We shall now assume that in addition to the spring force sx lb a periodic force $P_0 \sin(pt + q)$ lb downwards, where P_0, p, q are known constants, acts on the mass in Fig. 40 (*a*). The force in the direction of x increasing is $P_0 \sin(pt + q) - sx$, and the equation of motion is

$$P_0 \sin(pt + q) - sx = \frac{W}{g}\frac{d^2x}{dt^2}$$

which may be written

$$\left.\begin{aligned} &\frac{d^2x}{dt^2} + n^2x = \frac{P_0 g}{W}\sin(pt + q) \\ \text{or} \quad &\frac{d^2x}{dt^2} + n^2x = \frac{n^2P_0}{s}\sin(pt + q) \end{aligned}\right\} \quad . \quad . \quad \text{(VI.77)}$$

where $$n^2 = \frac{sg}{W}$$

To find a particular integral of (VI.77), we have

$$x = \frac{n^2P_0}{s}\cdot\frac{1}{D^2 + n^2}\sin(pt + q)$$

and by Rule VII, $$x = \frac{n^2P_0}{s(n^2 - p^2)}\sin(pt + q) \quad . \quad . \quad \text{(VI.78)}$$

The complete solution of (VI.77) is the sum of the values of x given by (VI.62) and (VI.78), but, as we know that the free vibrations will gradually die away, the motion after a time will be completely represented by (VI.78) alone.

If $p < n$, i.e. if the frequency of the forcing vibrations is less than that of the free vibrations, the displacement x and the force $P_0 \sin(pt + q)$ will be exactly in phase at all times, i.e. they will increase together, decrease together, attain their maximum and minimum values at the same time, and so on. If $p > n$, the forcing vibration $P_0 \sin(pt + q)$ and x are opposite in phase.

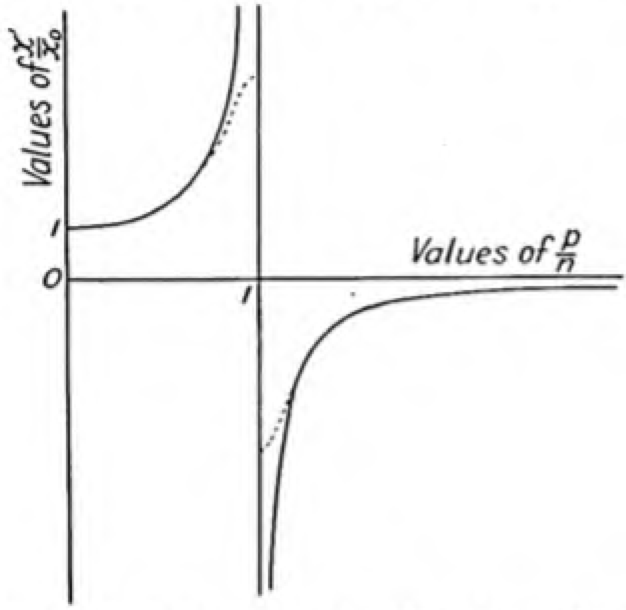


FIG. 43. AMPLITUDES OF FORCED UNDAMPED VIBRATIONS

Now $x_0 = \dfrac{P_0}{s}$ is the static deflection due to a constant force P_0 lb in the direction of motion, and (VI.78) may be written

$$\frac{x}{x_0} = \frac{1}{1 - \left(\frac{p}{n}\right)^2} \sin(pt + q) \qquad . \qquad . \quad \text{(VI.79)}$$

If x' represents the amplitude of the vibrations then $\dfrac{x'}{x_0} = \dfrac{1}{1 - \left(\frac{p}{n}\right)^2}$

Fig. 43 shows the graph of $\dfrac{x'}{x_0}$ plotted against $\dfrac{p}{n}$

When the frequency of the forcing vibrations is very small, i.e. when $\dfrac{p}{n}$ is nearly zero, so that $\dfrac{x'}{x_0} = 1$, the mass vibrates very slowly with amplitude equal to the static deflection under a load P_0. When

the frequency is very large, i.e. $\frac{p}{n}$ is large, the amplitude is nearly zero, as would be expected, since the period is so short that the mass has not time to respond. When $\frac{p}{n} = 1$, $\frac{x'}{x_0}$ becomes infinite, but, as we have assumed the amplitude to be small and have neglected damping forces, which increase with the amplitude, we are not entitled to use portions of the graph near $\frac{p}{n} = 1$. Whilst the amplitude x may not be infinitely large when $p = n$, it will evidently become large and may cause overstrain in the material and consequent breakdown in the elastic member. The condition in which the free and the forced frequencies are equal is known as the state of *resonance*. In the design of structures or machines resonance must be avoided.

To examine forced vibrations in the circuit of Fig. 40 (*b*) we assume that with the switch open its poles are connected to a source of alternating current of voltage $E_0 \sin (pt + q)$, and this opposes the sum of the voltages due to capacity and inductance. Equating the opposing voltages, we have

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E_0 \sin (pt + q)$$

i.e.

$$\frac{d^2q}{dt^2} + \frac{q}{LC} = \frac{E_0}{L} \sin (pt + q) \quad . \quad . \quad \text{(VI.80)}$$

By comparison with (VI.77) and (VI.78) we see that the particular integral of this is

$$q = \frac{E_0}{L(n^2 - p^2)} \sin (pt + q) \quad . \quad . \quad \text{(VI.81)}$$

where $n = \frac{1}{\sqrt{LC}}$. This represents the forced vibrations, the frequency of which is $\frac{p}{2\pi}$ per sec. The voltage E_C across the condenser is given by

$$E_C = \frac{q}{C} = \frac{E_0}{LC\left(\frac{1}{LC} - p^2\right)} \sin (pt + q)$$

i.e.

$$E_C = \frac{E_0}{1 - LCp^2} \sin (pt + q)$$

The resonant frequency is $\frac{p}{2\pi} = \frac{n}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$ cycles per sec.

Whilst the relations (VI.78) and (VI.81) indicate that the amplitude at resonance is large, they do not show how the amplitude increases. To see this, we must solve the equation (VI.77) or (VI.80) in the particular case $p = n$. By the method of Art. 58 the particular integral of (VI.77) is found to be

$$x = -\frac{P_0 g}{2nW} t \cos(nt + \alpha) \qquad . \qquad . \qquad \text{(VI.82)}$$

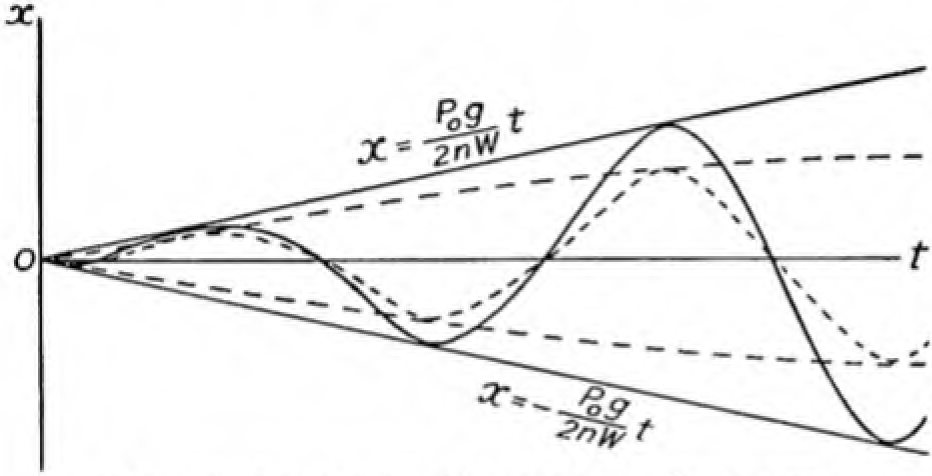


FIG. 44. INCREASING AMPLITUDE AT RESONANCE

Fig. 44 shows a sketch of the graph of this. The sloping lines are the graphs of $x = \pm \frac{P_0 g}{2nW} t$, and the full curve is the graph of (VI.82). It is seen from the equation as well as from the graph that the amplitude increases directly with the time. As, however, friction increases with the amplitude, damping occurs, and the dotted lines show the more likely limits of the amplitude, which ultimately becomes constant at values which may or may not be excessive, this depending upon the relative magnitudes of the forcing vibrations and the damping effects.

63. Damped Vibrations. Fig. 45 (*a*) and (*b*) show the vibrating systems of Fig. 40 (*a*) and (*b*) with the addition of frictional forces in the former and a resistance in the latter. In (*a*) the mass is shown displaced downwards through a distance of x ft from its equilibrium position OO. This mass is shown sliding between two vertical guides

between which and the vibrating mass, frictional forces of total amount $f\frac{dx}{dt}$ lb weight are supposed to act. f lb is the force when the speed $\frac{dx}{dt}$ ft/sec is unity. Putting this frictional force in the equation of motion, we have

$$sx + f\frac{dx}{dt} = -\frac{W}{g}\frac{d^2x}{dt^2}$$

i.e.

$$\frac{d^2x}{dt^2} + \frac{gf}{W}\frac{dx}{dt} + \frac{gs}{W}x = 0 \qquad \text{(VI.83)}$$

Substituting $n^2 = \frac{gs}{W}$, as before, and $2a = \frac{gf}{W}$,

$$\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + n^2x = 0 \qquad \text{(VI.84)}$$

n is the angular frequency of the free undamped vibrations of the system.

Now

$$D^2 + 2aD + n^2 = (D + a)^2 - (a^2 - n^2)$$
$$= (D + a - \sqrt{a^2 - n^2})(D + a + \sqrt{a^2 - n^2})$$

and there are three cases to consider—(1) $a^2 > n^2$, (2) $a^2 = n^2$, and (3) $a^2 < n^2$.

Case (1), $a^2 > n^2$. In this case the solution of (VI.84) is

$$x = Ce^{(-a + \sqrt{a^2 - n^2})t} + Ee^{(-a - \sqrt{a^2 - n^2})t} \qquad \text{(VI.85)}$$

which is the sum of two decay functions. C and E being arbitrary may be of the same sign or of opposite signs, and the value of x approaches zero for large values of t. Also, x is zero if

$$Ce^{\sqrt{a^2 - n^2}t} + Ee^{-\sqrt{a^2 - n^2}t} = 0$$

i.e. if

$$e^{2\sqrt{a^2 - n^2}t} = -\frac{E}{C} \qquad \text{(VI.86)}$$

The equation (VI.86) has no root if E and C have the same algebraic sign, and one root if they have opposite signs. Thus, if the

mass is set in motion, it will pass once at most through the equilibrium position, and then will swing slowly back to that position. This is the case of large frictional forces such as would arise if the mass were immersed in a viscous fluid.

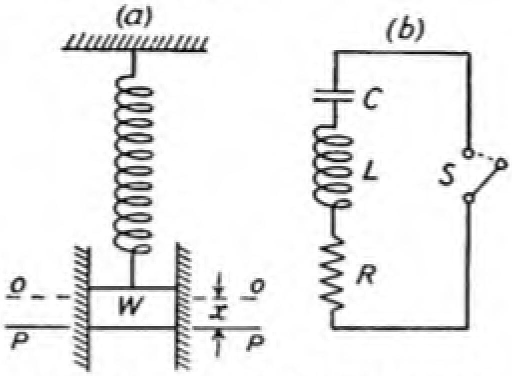


FIG. 45. DAMPED VIBRATIONS

Case (2), $a^2 = n^2$. In this case the two decay functions in (VI.85) become the same, i.e. e^{-at}, and (VI.85) is not the full solution.

From (VI.84), we have $(D^2 + 2aD + a^2)x = 0$, and by (VI.28), the solution is

$$x = (At + B)e^{-at} = (At + B)e^{-nt} \quad . \quad \text{(VI.87)}$$

Since $\underset{t\to\infty}{Lt.} \dfrac{At + B}{e^{nt}} = \underset{t\to\infty}{Lt.} \dfrac{\dfrac{d}{dt}(At + B)}{\dfrac{d}{dt}(e^{nt})} = \underset{t\to\infty}{Lt.} \dfrac{A}{ne^{nt}} = 0$ [see Art. 63, Vol. I], and the only root of $(At + B)e^{-nt} = 0$ is $t = -\dfrac{B}{A}$, we see that x may pass through the value zero only once at the most, i.e. if B and A have opposite signs, and that the mass will afterwards swing back towards OO. This is known as *critical damping* (the friction being just sufficient to prevent vibration) and also as *dead-beat* motion.

Case (3), $a^2 < n^2$. In this case $\sqrt{a^2 - n^2} = i\sqrt{n^2 - a^2}$, and (VI.85) becomes

$$\begin{aligned} x &= e^{-at}(Ce^{i\sqrt{n^2 - a^2}t} + Ee^{-i\sqrt{n^2 - a^2}t}) \\ &= e^{-at}[C\cos\sqrt{n^2 - a^2}t + E\cos\sqrt{n^2 - a^2}t \\ &\qquad + iC\sin\sqrt{n^2 - a^2}t - iE\sin\sqrt{n^2 - a^2}t] \end{aligned}$$

$$\left.\begin{aligned} &\text{i.e. } x = e^{-at}[F\cos\sqrt{n^2 - a^2}t + G\sin\sqrt{n^2 - a^2}t] \\ &\text{or } \ x = Re^{-at}\begin{matrix}\sin\\ \cos\end{matrix}(\sqrt{n^2 - a^2}t + \alpha) \end{aligned}\right\} \quad . \quad \text{(VI.88)}$$

where F and G, or R and α, are arbitrary constants, $a = \frac{gf}{2W}$, and $n = \sqrt{\frac{gs}{W}}$

From the latter of the two expressions in (VI.88) we see that the motion in this case may be looked upon as vibratory motion with amplitude Re^{-at} decreasing with time. Suppose that a particle is moving along the polar curve $r = Re^{-\frac{a}{\sqrt{n^2-a^2}}\theta}$ (the logarithmic spiral) so that the radius vector through the particle rotates at $\sqrt{n^2 - a^2}$ radians per sec; then the damped vibratory motion may be considered as the projection on the initial line $\theta = 0$ of the motion of the particle. A graph of this kind of motion is shown in Fig. 75, Vol. I, this being the graph of $y = 2{\cdot}2e^{-0{\cdot}1t} \sin (0{\cdot}3t + 0{\cdot}7)$. n is the angular frequency of the free vibrations, and $\sqrt{n^2 - a^2}$ is that of the natural damped vibrations. If x_n and x_{n+1} are the amplitudes of successive swings on the same side of OO, it is easy to show that $\frac{x_n}{x_{n+1}} = e^{\frac{2\pi a}{\sqrt{n^2-a^2}}}$. $\frac{2\pi a}{\sqrt{n^2 - a^2}}$, the hyperbolic logarithm of this expression, is called the *logarithmic decrement.*

For the electrical circuit of Fig. 45 (*b*), which includes a resistance of R ohms, the equation corresponding to (VI.63) is

$$L\frac{d^2q}{dt^2} - Ri + \frac{q}{C} = 0$$

or, since $i = -\frac{dq}{dt}$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

i.e.

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0 \quad . \quad . \quad . \quad \text{(VI.89)}$$

With $x = q$, $n^2 = \frac{1}{LC}$ and $a = \frac{R}{2L}$, (VI.89) is the same as (VI.84), and the above solutions apply to the electrical circuit as well as to the material system. Critical damping occurs when $a^2 = n^2$, i.e. when $\frac{R^2}{4L^2} = \frac{1}{LC}$ or $R = 2\sqrt{\frac{L}{C}}$. If R is less than this, the variations in q are given by

$$\left.\begin{aligned} q &= e^{-at}(F\cos\sqrt{n^2 - a^2}t + G\sin\sqrt{n^2 - a^2}t) \\ \text{or} \qquad q &= q_0e^{-at}\,{}^{\sin}_{\cos}\,(\sqrt{n^2 - a^2}t + \alpha) \end{aligned}\right\} \quad \text{(VI.90)}$$

64. Forced Vibrations Damped. Suppose that we introduce into the systems of Fig. 45 a periodic force $P_0 \sin pt$ lb acting on the mass in (*a*) and an alternating voltage $E_0 \sin pt$ in (*b*) acting across the poles of the switch, which is open.

In (*a*) the equation of motion is now

$$\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + n^2x = \frac{gP_0}{W}\sin pt \quad . \qquad . \quad \text{(VI.91)}$$

A particular integral of this is found by substituting $c = \dfrac{gP_0}{W}$ and $q = 0$ in (7) of Example 6, Art. 57.

This give
$$x = \frac{gP_0}{W}\cdot\frac{\sin(pt-\alpha)}{\sqrt{(n^2-p^2)^2+4a^2p^2}} \quad . \qquad . \quad \text{(VI.92)}$$

where
$$\alpha = \tan^{-1}\frac{2ap}{n^2-p^2}$$

The complete solution of (VI.91) is the sum of (VI.92) and the complementary function, but as the free vibrations are quickly damped out, (VI.92) represents the subsequent motion. Thus we see that the forced vibrations lag behind the forcing vibrations when $p < n$, the angle of lag increasing with p from zero when $p = 0$ to 90° when $p = n$. As p passes through the value n, the angle suddenly changes to an angle of lead of 90° which decreases as p increases further and approaches zero as p approaches infinity. When $p = n$, the amplitude remains finite, though, if a is small, it may be large. For the circuit in (*b*) the relations (VI.91) and (VI.92) apply with $\dfrac{E_0}{L}$ in place of $\dfrac{gP_0}{W}$ and the appropriate changes in the values of a and n. The nature of the forced vibrations is the same in both cases. In each case the amplitude varies inversely as

$$\sqrt{(n^2-p^2)^2+4a^2p^2}$$

Now

$$\begin{aligned}(n^2-p^2)^2+4a^2p^2 &= p^4 - 2p^2(n^2-2a^2) + (n^2-2a^2)^2 + 4a^2n^2 - 4a^4 \\ &= [p^2-(n^2-2a^2)]^2 + 4a^2(n^2-a^2)\end{aligned}$$

and the least value of this expression as p varies is $4a^2(n^2-a^2)$, which occurs when $p^2 = n^2 - 2a^2$. The amplitude is greatest, therefore, when $p^2 = n^2 - 2a^2$, i.e. at a lower frequency than for the natural undamped vibrations. The maximum amplitude is

$$\frac{gP_0}{2aW\sqrt{n^2-a^2}} \text{ in } (a), \text{ and } \frac{E_0}{2aL\sqrt{n^2-a^2}} \text{ in } (b)$$

65. **Homogeneous Linear Equations.** Equations of the type

$$\frac{d^n y}{dx^n} + f_1(x)\frac{d^{n-1}y}{dx^{n-1}} + f_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + f_n(x)y = X$$

where $f_1(x)$, $f_2(x)$, . . . , $f_n(x)$, X are functions of x, are, in general, beyond the scope of this book. We shall, however, deal here with equations which can be put in the form

$$x^n\frac{d^n y}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots$$
$$+ a_{n-1}x\frac{dy}{dx} + a_ny = X \quad \text{(VI.93)}$$

where a_1, a_2, etc. are constants. Such equations are known as *homogeneous linear equations.*

In the particular case $n = 2$, we have the equation

$$x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_2y = X \quad . \quad . \quad \text{(VI.94)}$$

To solve (VI.94) we introduce a new variable θ such that $x = e^\theta$

Then $\qquad \dfrac{d\theta}{dx} = \dfrac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{1}{x}\frac{dy}{d\theta}$$

whence $\qquad x\dfrac{dy}{dx} = \dfrac{dy}{d\theta}$

and $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{1}{x}\dfrac{dy}{d\theta}\right) = \dfrac{d}{d\theta}\left(\dfrac{1}{x}\dfrac{dy}{d\theta}\right)\dfrac{d\theta}{dx} = \left(\dfrac{1}{x}\dfrac{d^2y}{d\theta^2} - \dfrac{1}{x^2}\dfrac{dy}{d\theta}\cdot\dfrac{dx}{d\theta}\right)\dfrac{1}{x}$

whence $\qquad x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{d\theta^2} - \dfrac{dy}{d\theta}$

With these substitutions (VI.94) becomes

$$\frac{d^2y}{d\theta^2} + (a_1 - 1)\frac{dy}{d\theta} + a_2y = F(\theta) \quad . \quad . \quad \text{(VI.95)}$$

an equation of a type dealt with in Vol. I and in this chapter.

We may write the general case (VI.93) in the form

$$(x^nD^n + a_1x^{n-1}D^{n-1} + a_2x^{n-2}D^{n-2} + \ldots + a_{n-1}xD + a_n)y = X \quad \text{(VI.96)}$$

Substituting $x = e^{\theta}$, noting that $\dfrac{d\theta}{dx} = e^{-\theta}$, and writing D_θ for $\dfrac{d}{d\theta}$ we have

$$Dy = \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

i.e.
$$Dy = e^{-\theta} D_\theta y$$

Again
$$D^2 y = D(Dy) = e^{-\theta}\, D_\theta(e^{-\theta} D_\theta y),$$

and by Rule II, Art. 52, $D^2 y = e^{-2\theta}(D_\theta - 1)D_\theta y$

Similarly
$$D^3 y = D(D^2 y) = e^{-\theta} D_\theta[e^{-2\theta}(D_\theta - 1)D_\theta y],$$

and by Rule II
$$D^3 y = e^{-3\theta}(D_\theta - 2)\,(D_\theta - 1)D_\theta y$$

In this way we can show that

$$D^4 y = e^{-4\theta}(D_\theta - 3)\,(D_\theta - 2)\,(D_\theta - 1)D_\theta y$$

and so on.

Generally, for all integral positive values of n, we have

$$D^n y = e^{-n\theta} D_\theta(D_\theta - 1)\,(D_\theta - 2)\,.\;.\;.\,(D_\theta - n + 1)y$$

or by multiplying through by $e^{n\theta} = x^n$,

$$x^n \frac{d^n y}{dx^n} = D_\theta\,(D_\theta - 1)\,(D_\theta - 2)\,.\;.\;.\,(D_\theta - n + 1)y \qquad \text{(VI.97)}$$

If now we substitute this in the left-hand side of (VI.93) we obtain an equation of the type (VI.1) whose solution we have already discussed.

This method of solution by operators can be applied, of course, to (VI.94), which is a particular case of (VI.96).

EXAMPLE 1

Solve the equation $\dfrac{d^2u}{dr^2} - \dfrac{2u}{r^2} = 0$, which occurs in finding the strain in a thin sphere under internal pressure.

When multiplied through by r^2, the equation is homogeneous, thus

$$r^2 \frac{d^2u}{dr^2} - 2u = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

With $r = e^\theta$ and $D_\theta = \frac{d}{d\theta}$, the equation (1) becomes from (VI.97),

$$D_\theta\,(D_\theta - 1)u - 2u = 0$$

i.e.

$$(D_\theta^2 - D_\theta - 2)\,u = 0$$

so that
$$u = \frac{1}{(D_\theta - 2)\,(D_\theta + 1)}\,(0) = \tfrac{1}{3}\left[\frac{1}{D_\theta - 2} - \frac{1}{D_\theta + 1}\right](0)$$

Hence
$$u = \tfrac{1}{3}\left[\frac{1}{D_\theta - 2}\,e^{2\theta} \times 0 - \frac{1}{D_\theta + 1}\,e^{-\theta} \times 0\right]$$

$$= \tfrac{1}{3}\left[e^{2\theta}\frac{1}{D}\,(0) - e^{-\theta}\frac{1}{D}\,(0)\right]$$

$$= \tfrac{1}{3}[Ae^{2\theta} - Be^{-\theta}]$$

i.e. $\qquad u = Ce^{2\theta} + De^{-\theta}$, where C and D are arbitrary constants.

Writing r for e^θ, we have the required solution

$$u = Cr^2 + \frac{D}{r}$$

Another method of solving an equation such as (1) is to substitute $u = Ar^n$

This gives
$$An(n-1)r^n - 2Ar^n = 0$$

so that
$$n^2 - n - 2 = 0$$

This is the auxiliary equation and is a quadratic in n. The roots of this equation are $n = 2$ and $n = -1$, and the full solution of (1) is $u = Ar^2 + \frac{B}{r}$, which is equivalent to that already found.

EXAMPLE 2

Solve the equation

$$r\frac{d^2y}{dr^2} + \frac{dy}{dr} - \frac{y}{r} = -(1 + 3c)kr^2$$

and determine the constants of integration so that the solution satisfies the conditions $\frac{dy}{dr} + ckr^2 = 0$ when $r = a$ and $r = b$. (U.L.)

Multiplying through by r we have the homogeneous equation

$$r^2\frac{d^2y}{dr^2} + r\frac{dy}{dr} - y = -(1 + 3c)kr^3 \quad . \quad . \quad . \quad .(1)$$

Substituting $r = e^\theta$ and using (VI.97), we obtain

$$[D_\theta(D_\theta - 1) + D_\theta - 1]\,y = -(1 + 3c)ke^{3\theta}$$

i.e.

$$(D_\theta^2 - 1)y = -(1 + 3c)ke^{3\theta} \quad . \quad . \quad . \quad . \quad (2)$$

Hence $y = \frac{1}{D_\theta^2 - 1}(0) + \frac{1}{D_\theta^2 - 1}\{-(1 + 3c)ke^{3\theta}\}$

$$= \tfrac{1}{2}\left[\frac{1}{D_\theta - 1} - \frac{1}{D_\theta + 1}\right](0) - (1 + 3c)k \cdot \frac{1}{D_\theta^2 - 1}e^{3\theta}$$

$$= \tfrac{1}{2}\left[\frac{1}{D_\theta - 1}(e^\theta \times 0) - \frac{1}{D_\theta + 1}(e^{-\theta} \times 0)\right] - (1 + 3c)k \cdot \frac{1}{3^2 - 1}e^{3\theta}$$

$$= \tfrac{1}{2}\left[e^\theta \cdot \frac{1}{D_\theta}(0) - e^{-\theta} \cdot \frac{1}{D_\theta}(0)\right] - \frac{(1 + 3c)k}{8}e^{3\theta}$$

i.e. $\quad y = \tfrac{1}{2}[Ae^\theta - Be^{-\theta}] - \frac{(1 + 3c)k}{8}e^{3\theta}$

or $\quad y = Cr + \frac{E}{r} - \frac{(1 + 3c)k}{8}r^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$

To determine the constants C and E we have

$$\frac{dy}{dr} = C - \frac{E}{r^2} - \frac{3}{8}(1 + 3c)kr^2$$

and, since $\frac{dy}{dr} + ckr^2 = 0$, this gives

$$C - \frac{E}{r^2} - \left(\frac{3}{8} + \frac{1}{8}c\right)kr^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Since (4) is true when $r = a$ and $r = b$, then

$$C - \frac{E}{a^2} - \left(\frac{3}{8} + \frac{1}{8}c\right)ka^2 = 0$$

and

$$C - \frac{E}{b^2} - \left(\frac{3}{8} + \frac{1}{8}c\right)kb^2 = 0$$

Solving these simultaneous equations in C and E, we find

$$C = \frac{(a^2 + b^2)k}{8}(3 + c) \text{ and } E = \frac{a^2b^2k}{8}(3 + c)$$

The solution (3) then becomes

$$y = \frac{(a^2 + b^2)k}{8}(3 + c)r + \frac{a^2b^2k}{8r}(3 + c) - \frac{(1 + 3c)k}{8}r^3$$

66. **Equations of the First Order but not of the First Degree.** The operator D cannot be used in the solution of equations of this type as this would lead to confusion between D^ny^n meaning $\left(\frac{dy}{dx}\right)^n$ and D^ny^n meaning $\frac{d^n}{dx^n}(y^n)$. Such equations are of little practical

importance, and only in a few special cases can they be solved. The general equation of this type is

$$\phi(x, y, p) = 0 \quad . \quad . \quad . \quad \text{(VI.98)}$$

where $p \equiv \dfrac{dy}{dx}$

We give below examples of methods of solution, any general discussion of the subject being beyond our scope.

EXAMPLE 1

Solve $$p^2 = y + x \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We first solve for y and then differentiate with respect to x, putting p for $\dfrac{dy}{dx}$.

Thus $y = p^2 - x$ and, differentiating, $p = 2p\dfrac{dp}{dx} - 1$, i.e. $\dfrac{dp}{dx} = \dfrac{1 + p}{2p}$, the solution of which is

$$x = 2p - 2\log_e(p + 1) + c \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2), $$y = p^2 - 2p + 2\log_e(p + 1) - c \ . \quad . \quad . \quad . \quad (3)$$

We may look upon (2) and (3) as the solution of (1), giving x and y in terms of the parameter p. Otherwise, since p can be found readily in terms of x and y from the original equation, which is not usually the case, we substitute $p = \pm\sqrt{y + x}$ in (3), obtaining the solution

$$y = (y + x) \mp 2\sqrt{y + x} + 2\log_e(1 \pm \sqrt{y + x}) - c$$

or $$x + 2\log_e(1 \pm \sqrt{y + x}) = \pm 2\sqrt{y + x} + c$$

Alternatively, we can solve the given equation by expressing x in terms of y and p from (1) and differentiating with respect to y.

Thus, $x = p^2 - y$ and, differentiating, $\dfrac{1}{p} = 2p\dfrac{dp}{dy} - 1$, i.e. $\dfrac{dp}{dy} = \dfrac{1 + p}{2p^2}$, the solution of which is

$$y = p^2 - 2p + 2\log_e(p + 1) - c$$

which is the relation (3). The solution then follows as above.

EXAMPLE 2

Solve $p^2 + (2x - y)p - 2xy = 0$.

Here the method is to solve for p. Factorizing, we have

$$(p - y)(p + 2x) = 0$$

from which $\dfrac{dy}{dx} = y$ and $\dfrac{dy}{dx} = -2x$, the solutions of which are $y = ce^x$ and $y = c - x^2$.

Both these solutions are contained in

$$(y - ce^x)(y + x^2 - c) = 0$$

which is therefore the solution of the given equation.

EXAMPLE 3

Solve
$$y = x(p + p^2) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating with respect to x, we have

$$p = p + p^2 + x(1 + 2p)\frac{dp}{dx}$$

i.e.
$$\frac{2p + 1}{p^2}\,dp = -\frac{dx}{x}$$

the solution of which is $x = cp^{-2}e^{\frac{1}{p}}$ . . . . (2)

From (1) and (2),
$$y = c\left(\frac{1}{p} + 1\right)e^{\frac{1}{p}} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(2) and (3) give x and y in terms of the parameter p.

EXAMPLES VI

(1) If $D = \frac{d}{dx}$, show that $y = Ae^{-2x} + Be^{-3x}$ satisfies $D^2y + 5Dy + 6y = 0$

$y = (A + Bx)e^{-5x}$ satisfies $D^2y + 10Dy + 25y = 0$

and $y = Re^{-x}\sin(4x + q)$ satisfies $D^2y + 2Dy + 17y = 0$

(2) If $D = \frac{d}{dt}$, show that $y = \frac{a}{n^2 - p^2}\sin pt + C\sin(nt + q)$ satisfies $D^2y + n^2y = a\sin pt$.

(3) Write down the rules for interpreting the results of operating with the polynomial $F(D)$, where $D = \frac{d}{dx}$, on the functions $e^{\alpha x}$, $e^{\alpha x}X$, $\sin(px + q)$ or $\cos(px + q)$.

If $F(D) = D^2 + 6D + 12$, give the results of operating with $F(D)$ on $7e^{-2x}$, $5e^{4x}x^5$, $6e^{-x}\sin 7x$, and $8e^{-3x}\cos 3x$.

(4) Simplify $D^2s + 6Ds - 16s$ where $D = \frac{d}{dt}$ and $s = Ae^{2t} + Be^{-8t} + 18\sin 3t - 25\cos 3t$. Operate on $\sin 3x$ with $D^2 - 4D + 4$.

(5) Using operators integrate $\int e^{4x}\sin 2x dx$, $\int x^2e^{2x}dx$, $\int x^3e^{5x}dx$, and $\int e^{kt}(a\sin pt + b\cos pt)dt$.

(6) Write down the results of operating with $\frac{1}{D^2 + 6D + 12}$ on the functions $7e^{-2x}$, $5xe^{4x}$, $6e^{-x}\sin 7x$, and $8e^{-3x}\cos 3x$, where $D = \frac{d}{dx}$

(7) Simplify $D[e^{3x}\cos(2x + 3)]$, $D^6[e^{3x}\cos(2x + 3)]$, $D^{-6}[e^{3x}\cos(2x + 3)]$, $D^{-1}[e^{2x}\cos 3x]$, $D^{-3}[e^{2x}\cos 3x]$, $D^3[e^{2x}\cos 3x]$, and $\int(x + 2x^2)\cos 3x dx$.

(8) Solve $\frac{dy}{dx} = 2x - y$, $\frac{dy}{dx} = 6y + 7$, $\frac{dy}{dx} - 2y = 6\sin 3x$, $\frac{dy}{dx} + ay = b$, and $\frac{dy}{dx} + ay = b\sin(px + q)$.

(9) Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$, $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$, $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 12x = 0$, $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 0$, and $\frac{d^2y}{dt^2} + 2a\frac{dy}{dt} + a^2y = 0$.

(10) Using operators show that if $D \equiv \frac{d}{dx}$, $(D + a)^n y = 0$ has for solution $y = (A + Bx + Cx^2 + \ldots$ to n terms$)e^{-ax}$, and write down the solutions of $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$ and $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} - 8y = 0$.

(11) Solve $(D^2 + 6D + 13)y = 0$, $(D^3 + 2D^2 + 9D + 18)y = 0$, and $(D + 3)(D + 4)(D + 5)^2y = 0$, where $D \equiv \frac{d}{dx}$.

(12) Solve $\frac{d^4y}{dx^4} + 18\frac{d^2y}{dx^2} + 81y = 0$ and $\frac{d^4x}{dt^4} - 13\frac{d^2x}{dt^2} + 36x = 0$.

(13) Find the complete solution of $3\frac{d^2x}{dt^2} + 5\frac{dx}{dt} - 2x = 40e^{3t}$, and determine the constants of integration so that $x = 10$ and $\frac{dx}{dt} = 1$ when $t = 0$.

(14) Obtain the general solutions of the equations—

(i) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin 2x$

(ii) $\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + a^2y = x^2e^{-ax}$ (U.L.)

(15) Solve the equations—

(i) $(1 - x^2)\frac{dy}{dx} + 2xy = x - x^3$

(ii) $(3x - y)\frac{dy}{dx} = 2x$

(iii) $2x\frac{dy}{dx} = \frac{\sin x}{y} - y$ (U.L.)

(16) Solve the differential equations—

(i) $\frac{dy}{dx} + 2y\tan x = \sin x$

(ii) $(5y + 7x)\frac{dy}{dx} + 8y + 10x = 0$ (U.L.)

(17) Solve the following equations subject to the conditions that, when $x = 0$, $y = 2$ and $\frac{dy}{dx} = -1$

(*a*) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \sin x$

(*b*) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$ (U.L.)

(18) Solve $(D^2 + D + 1)y = -12\cos 2x - 8\sin 2x - 3\sin x$, where $D = \dfrac{d}{dx}$

(19) Solve $\dfrac{d^2y}{dt^2} + 10\dfrac{dy}{dt} + 25y = 12t^2e^{-5t}$, and determine the constants of integration so that, when $t = 0$, $y = \dfrac{dy}{dt} = 1$.

Also solve $\dfrac{d^3y}{dt^3} + 4\dfrac{dy}{dt} = 6\sin t$.

(20) Solve $(2D^2 - 7D - 4)x = t^2e^t$, where $D = \dfrac{d}{dt}$, and $(D^2 + 5D + 36)y = e^{-2x}\sin 5x$, where $D = \dfrac{d}{dx}$

(21) Solve

(i) $\dfrac{d^2y}{dx^2} + 16y = 4\sin 4x$

(ii) $\dfrac{d^2\theta}{dt^2} + 4\dfrac{d\theta}{dt} + 8\theta = 6\cos 2t$

(iii) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} - 6y = 3e^{-6x} + 4e^x$

(22) Integrate the differential equations

(i) $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{y^2}{x^2}$

(ii) $\dfrac{d^2y}{dx^2} = 4x + e^x$

(iii) $(3y^2 + 4x)dx + 2xy\,dy = 0$. (U.L.)

(23) Solve the differential equations

(i) $\dfrac{d^2y}{dx^2} + 9y = x^2 + x + 1$

(ii) $\dfrac{dy}{dx} - y\tan x = e^x$ (U.L.)

(24) (*a*) Find the solution of $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 17x = 5\sin 4t$ for which $x = 0$ when $t = 0$ and when $t = \dfrac{\pi}{8}$

(*b*) Find the general solution of $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = (x^2 + 5)e^{2x}$ (U.L.)

(25) Solve the equations

(i) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{-x}\sin 2x$

(ii) $\dfrac{d^2y}{dx^2} = 2(y^3 + y)$

under the conditions that $y = 0$ and $\dfrac{dy}{dx} = 1$ when $x = 0$. (U.L.)

(26) Solve the following differential equations—

(i) $\dfrac{dy}{dx} = 1 + y^2$, $y = 1$ when $x = 0$

(ii) $\dfrac{dy}{dx} + y \cot x = \sin 2x$, $y = 0$ when $x = \dfrac{\pi}{2}$

(iii) $(1 + x)\dfrac{dy}{dx} + \beta xy = \beta e^{-\beta x}$, $y = 0$ when $x = 0$ (U.L.)

(27) Solve any *two* of the following—

(i) $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = 6 \sin x$

(ii) $\dfrac{d^4y}{dx^4} - y = \cos x \cosh x$

(iii) $\dfrac{d^2y}{dx^2} + 2a\dfrac{dy}{dx} + a^2y = 12x^2e^{-ax}$

with the conditions that $y = 1$ and $\dfrac{dy}{dx} = 0$ when $x = 0$. (U.L.)

(28) (i) Solve the differential equation

$$(x^2 + x)\frac{dy}{dx} + y = 2x$$

(ii) Find a particular integral of the equation

$$\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = \sin 2x$$ (U.L.)

(29) Solve the equation

$$\frac{d^4y}{dx^4} - 16y = 15 \cos x$$

subject to the conditions that, when $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = 2$, and, when $x = \frac{1}{2}\pi$, $y = -1$ and $\dfrac{dy}{dx} = -1$, showing that the solution is purely periodic in x. (U.L.)

(30) Solve the equations

(i) $x\dfrac{dy}{dx} + 3y = 4x + 3$

(ii) $y\dfrac{dy}{dx} = y^2 + 4x$ (U.L.)

(31) Solve the differential equations

(i) $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{1}{x^2}$

(ii) $\dfrac{dy}{dx} = \dfrac{x - y + 2}{x + y - 2}$ (U.L.)

(32) Solve

(i) $\dfrac{dy}{dx} = 2x - 3y$

(ii) $\dfrac{dy}{dx} = \dfrac{-2x - y + 2}{x + 2y - 4}$

(33) (i) Solve the equation $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \cos x$

(ii) If $x = \cosh z$, prove that $(x^2 - 1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = \dfrac{d^2y}{dz^2}$

Solve the equation $(x^2 - 1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x$ (U.L.)

(34) Show that the solution of the equation

$$\frac{d^2y}{dt^2} + 2n\frac{dy}{dt} + n^2y = A\cos pt$$

for which y and $\dfrac{dy}{dt}$ both vanish when $t = 0$ can be written

$$y = \frac{A\{\cos(pt - \phi) - e^{-nt}(nt + \cos\phi)\}}{n^2 + p^2}$$

where $\tan\phi = \dfrac{2np}{n^2 - p^2}$ (U.L.)

(35) Integrate the differential equations

(i) $2\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} - 4y = e^{3x}$

(ii) $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$ (U.L.)

(36) Obtain the general solutions of the differential equations

(i) $\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} + 12y = 10e^{-4x}$

(ii) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = x^2$ (U.L.)

(37) Solve the differential equations

(i) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{-x}\cos x$

(ii) $\dfrac{dy}{dx} + y\sec^2 x = 3\sec^2 x \tan x$ (U.L.)

(38) Find the period of vertical vibrations of a mass of W lb weight at one end of a light flexible cantilever, l in. long, built in horizontally at the other end. Let E be the modulus of elasticity in lb/in.^{2}, b in. the breadth and d in. the depth. Find also the period of oscillation if the cantilever is placed in a vertical position.

(39) A straight uniform circular shaft of negligible mass carries a rotor of radius of gyration k ft at one end, the other end being fixed. If C_0 lb-ft is the couple to produce unit twist in the shaft and the weight of the rotor is W lb, find the frequency of torsional oscillations.

(40) If the fixed end of the shaft in the last example is freed and a rotor of weight W_1 lb and radius of gyration k_1 ft is mounted there, show that the frequency of torsional vibrations of the system is now $\dfrac{30}{\pi}\sqrt{\dfrac{C_0(Wk^2 + W_1k_1^2)}{WW_1k^2k_1^2}}$ per min.

(41) A load rests on a light horizontal beam. Show that the periodic time of free vertical vibrations of the load is the same as that of a simple pendulum whose length is equal to the static deflection produced by the load at its point of application.

(42) A mass of 161 lb weight is carried by the lower end of a light spiral spring the upper end of which is fixed. The spring stretches through 15·46 in. under the load. The upper end is then released and given a simple harmonic motion $0{\cdot}5 \sin t$ ft downwards, where t is the time in seconds. There is a damping resistance of $30\,v$ lb acting on the mass, where v is the speed in ft/sec at time t. Find the amplitude of the forced vibrations after the free vibrations have died away, and find also the angle of lag between the forcing and the forced vibrations.

(43) A spring of negligible mass which stretches 1 in. under a tension of 2 lb is fixed at one end and attached to a mass of M lb weight at the other. It is found that resonance occurs when an axial periodic force of $2 \cos 2t$ lb acts on the mass. Show that, when the free vibrations have died out, the forced vibrations are given by $x = ct \sin 2t$, and find the values of M and c.

(44) A spring supports a mass of 40 lb weight at its lower end. The stiffness is 20 lb per in. of stretch. Find the period of free vibrations with the upper end fixed, neglecting the mass of the spring. If the upper end is now made to vibrate axially so that its displacement at time t sec is $0{\cdot}3 \sin 7t$ ft downwards, neglecting damping and assuming the mass to start from rest when $t = 0$, find its displacement at time t sec. Assume $g = 32$.

(45) Assuming that in the last example a damping force of $10\,v$ lb acts on the mass, where v is the velocity of the mass in ft/sec, find the forced vibrations after the free vibrations have died away.

(46) Write down the equation of motion for a mass vibrating under the action of a restoring force proportional to the displacement and a small frictional resistance proportional to the speed, and show that the ratio of the amplitudes of two successive swings on the same side of the equilibrium position is constant.

(47) A 4-lb mass hangs at rest on a spring producing in the spring an extension of 1 ft. The upper end of the spring is now made to execute a vertical simple harmonic oscillation $x = \sin 4t$, x being measured vertically downwards in feet. If the mass is subject to a frictional resistance whose magnitude in pounds weight is one-quarter of its velocity in feet per sec, obtain the differential equation for the motion of the mass, and find the expression for its displacement at time t, when t is large. (U.L.)

(48) The motion of a mass vibrating along a straight path is given by

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 100y = 5 \sin 3t$$

y being the displacement in feet at time t sec. The term on the right is produced by a periodic force acting on the mass. If the mass weighs W lb, what is the

periodic force? What is the displacement t sec after the motion has begun but after the free vibrations have died out?

(49) A mass of W lb weight makes vertical vibrations when suspended from the lower end of a light spring which stretches a ft under a load aE lb, the upper end of the spring being fixed. Write down the energy equation and obtain from it the equation of motion. Show that the frequency of vibration is $\frac{30}{\pi}\sqrt{\frac{gE}{W}}$ per minute. Assuming that the spring is not light and weighs w lb and that the stretch at any point in it varies as the distance from the fixed end, include the energy of the spring in the energy equation, and show that in this case the frequency is $\frac{30}{\pi}\sqrt{\frac{gE}{W+\frac{1}{3}w}}$ per minute.

(50) Show that the frequency of free vibration in a closed electrical circuit with inductance L and capacity C in series is $\frac{30}{\pi\sqrt{LC}}$ per min.

(51) A condenser of capacity C farads, a resistance R ohms and an inductance L henrys are connected in series in a closed circuit. Show that, if i amperes is the current in the circuit at time t seconds when free vibrations are set up in the circuit,

$$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{CL}i=0$$

Show also that, if v is the voltage across the condenser at time t sec, v may be substituted for i in the equation. (1) Show that if $CR^2 > 4L$, i passes once at most through the value zero and that for large values of t, i becomes very small. (2) Show also that if $CR^2 = 4L$, i again passes once at most through the value zero and becomes very small as t becomes very large. What is the difference between cases (1) and (2)? (3) Show that if $CR^2 < 4L$ the solution of the equation is $i = Re^{-\frac{Rt}{2L}}\sin\left(\sqrt{\frac{1}{CL}-\frac{R^2}{4L^2}}\,t+\alpha\right)$ where R and α are arbitrary constants.

(52) An electrical circuit has inductance L henrys in series with capacity C farads. Find an expression for the forced vibrations if i amperes is the current in the circuit when a voltage $V_0 \sin \omega t$ is impressed on the circuit. Sketch a graph showing the amplitude i_0 of the current i plotted vertically against ω plotted horizontally and show on the graph the effect on the amplitude of a slight resistance in the circuit.

(53) Find the complete solution of $\frac{d^2y}{dt^2}+n^2y = a\sin(pt+q)$ where y ft is the displacement of a vibrating mass at time t sec. Show in a graph how the amplitude of the forced vibrations depends on p as p increases from zero to a value greater than n. Show on the graph by means of a dotted line the effect on the amplitude of a slight resistance.

(54) A pulley is eccentrically mounted on a shaft which turns with negligible friction in horizontal bearings. The mass of pulley and shaft is M, the centre of gravity G is distant h from the axis of the shaft, the radius of gyration about that axis is k, and the radius of the shaft is r. From a cord coiled round the shaft hangs a mass m, and in equilibrium the line joining G to the axis is at an angle α to the vertical. Show that $mr = Mh \sin \alpha$, and establish the equation of motion when this line makes an angle θ with the equilibrium position. Show that in

small oscillations about the equilibrium position the length of the equivalent simple pendulum is

$$(k^2/h + mr^2/Mh) \sec \alpha \qquad \text{(U.L. Chem. Eng.)}$$

(55) An alternating electromotive force $E \sin pt$ is applied at time $t = 0$ to a circuit. Obtain, in the usual notation, the equation

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = pE\cos pt$$

and hence obtain an expression for the current at time t, in the two cases (i) $CR^2 > 4L$, (ii) $CR^2 < 4L$. (U.L.)

(56) A condenser of capacity C and initial charge Q_0 is discharged through a resistance R and an inductance L in series. Prove that, if $R^2C < 4L$, the current at time t is

$$-Q_0e^{-ht}\left(k + \frac{h^2}{k}\right)\sin kt$$

where $-h + ik$ and $-h - ik$ are the roots of the equation

$$CLx^2 + CRx + 1 = 0 \qquad \text{(U.L.)}$$

(57) An uncharged condenser of capacity C is charged by applying an e.m.f. of $E \sin (t/\sqrt{LC})$ through leads of self-inductance L and negligible resistance. Prove that at time t the charge on one of the plates is

$$\tfrac{1}{2}EC\left[\sin\frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}\right]$$

If, in addition, there is a small resistance, in what respect is the mathematical form of the above result altered? (U.L.)

(58) A helical spring of such a strength that a force of p dynes extends it 1 cm hangs from a fixed support, and a mass of m grammes is attached to the lower end. From this mass another spring of the same strength hangs, and a mass of m grammes is attached to its lower end. Write down the differential equations for the vertical oscillations of each mass, and integrate the equations. The masses of the springs may be neglected. Discuss the nature of the motion of each mass. (U.L.)

[*Hint*. Suppose that the first spring is stretched x cm from its length in the equilibrium position, and the second spring y cm when the time is t sec. The force accelerating the first mass is $p(y - x)$ dynes, and that accelerating the second is $-py$ dynes. The acceleration of the first mass is $\frac{d^2x}{dt^2}$, and that of the second is $\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2}$. Hence, write down the equations of motion.]

(59) A spring which stretches an amount e under a force mk^2e is suspended from a support P and has a mass m at the lower end. At time $t = 0$ the mass is at rest in its equilibrium position at a point A below P. A vertical oscillation is now given to the support P such that, at any time t (> 0), its displacement below its initial position is $a \sin nt$. Show that the displacement x of the mass below A is given by the equation

$$\ddot{x} + k^2x = k^2a\sin nt$$

Hence show that, if $n \neq k$, the displacement is given by

$$x = \frac{ka}{k^2 - n^2}(k\sin nt - n\sin kt) \qquad \text{(U.L.)}$$

(60) A circuit contains an inductance L, a capacitance C, and a resistance R, and is acted upon by a periodic e.m.f. of amount $V = E\sin(pt + \varepsilon)$. The current i in the circuit satisfies the conditions

$$i = -\frac{dq}{dt}, \quad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V$$

Show that, if R is sufficiently small, the current consists of a damped harmonic term and a permanent harmonic I whose amplitude becomes large if

$$p = \frac{1}{(CL)^{\frac{1}{2}}} \text{ and that in this case, } I = \frac{E}{R}\cos\left\{\frac{t}{(CL)^{\frac{1}{2}}} + \varepsilon\right\} \qquad \text{(U.L.)}$$

(61) Solve the homogeneous equations

(i) $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 0$

(ii) $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = x^3$

(62) If $x = e^\theta$, show that $x\dfrac{dy}{dx} = \dfrac{dy}{d\theta}$ and $x^2\dfrac{d^2y}{dx^2} = \dfrac{d}{d\theta}\left(\dfrac{dy}{d\theta} - y\right)$

The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2\dfrac{d^2u}{dr^2} + r\dfrac{du}{dr} - u + kr^3 = 0$, where k is a constant depending on the velocity. Solve the equation and determine the constants of integration so that $u = 0$ when $r = 0$ and $r = a$; the disc being assumed to be contained within a rigid boundary. (U.L.)

(63) Find the complete solution of the equation

$$\frac{d^2y}{dr^2} + \frac{1}{r}\frac{dy}{dr} - \frac{y}{r^2} = f(r)$$

when (i) $f(r) = -4r$, and (ii) $f(r) = 3r^2 + \dfrac{4}{r^3}$

In case (i) determine the constants of integration so that $\dfrac{dy}{dr} + r^2 = 0$ when $r = a$ and when $r = b$.

(64) Find the general solutions of the equations

(i) $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + 2y = 5$

(ii) $x^2\dfrac{d^2y}{dx^2} + 6x\dfrac{dy}{dx} + 4y = \log_e x$

(iii) $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x\log_e x$

(iv) $x^3\dfrac{d^3y}{dx^3} + 3x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$

(65) Solve completely the equations

(i) $x^2\dfrac{d^2y}{dx^2} + y = 5x^4$

(ii) $x^4\dfrac{d^2y}{dx^2} + 3x^3\dfrac{dy}{dx} - 8x^2y = x^4 - 1$

given that, when $x = 1$, $\dfrac{dy}{dx} = \frac{1}{9}$ and $y = \frac{1}{18}$

(66) Show that there is a value of m for which $y = x^m$ is a solution of the equation

$$2x^3 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

and hence, or otherwise, obtain the complete solution of the equation. (U.L.)

(67) Show that the constant n may be chosen so that, by the substitution $y = x^n z$, the differential equation

$$x^2 \frac{d^2y}{dx^2} + 4x(x+1)\frac{dy}{dx} + (8x+2)y = \cos x$$

reduces to the form

$$\frac{d^2z}{dx^2} + a\frac{dz}{dx} + bz = \cos x$$

where a and b are constants.

Hence, or otherwise, solve the given equation. (U.L. Gen. Sc.)

(68) Solve the differential equations

(i) $\dfrac{dy}{dx} = \dfrac{x^2 + 2xy - y^2}{x^2 - 2xy - y^2}$

(ii) $2y\dfrac{d^2y}{dx^2} - 3\left(\dfrac{dy}{dx}\right)^2 - 4y^2 = 0$ (U.L.)

(69) The equation $\dfrac{d^2y}{dr^2} + \dfrac{1}{r}\dfrac{dy}{dr} - \dfrac{y}{r^2} = 0$ occurs in finding the strains in a thick cylinder under internal pressure. Solve this equation (i) by substituting $y = Ar^n$, and (ii) by substituting $r = e^\theta$.

Also give the complete solution of the equation

$$\frac{d^2y}{dr^2} + \frac{1}{r}\frac{dy}{dr} - \frac{y}{r^2} = 3r^2 + \frac{4}{r^3}$$

(70) (i) Solve the equation $\dfrac{d}{dt}\left(x\dfrac{dx}{dt}\right) = 6x$, subject to the conditions that x and $\dfrac{dx}{dt}$ both vanish when $t = 0$.

(ii) By means of the substitution $x = e^t$, or otherwise, find the general solution of the equation $x^2\dfrac{d^3y}{dx^3} + 3x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2 \log_e x$ (U.L.)

(71) Show that the equation $y = px + \dfrac{c}{p}$, where $p = \dfrac{dy}{dx}$ and c is a constant, is satisfied by $y = mx + \dfrac{c}{m}$, where m is a constant, and also by $y^2 = 4cx$. Show also that the latter solution gives the envelope of the family of straight lines represented by the former solution.

(72) (a) Solve $x\dfrac{dy}{dx} + y = x^3$, given that y is a minimum when $x = 1$

(b) Solve $\left(\dfrac{dy}{dx}\right)^2 + x(x-2)\dfrac{dy}{dx} = 2x^3$ (U.L.)

(73) Solve the equations

(i) $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 2y^2 = 0$

(ii) $\left(y + \frac{dy}{dx}\right)\frac{dy}{dx} = x(x + y)$

(74) (i) Solve the equation $x\frac{dy}{dx} = 1 + \left(\frac{dy}{dx}\right)^2$

(ii) Find that solution of the equation $\frac{d^2y}{dx^2} = y^3 - y$ for which $\frac{dy}{dx}$ is zero when $y = 1$ and is always positive, and $y = 2$ when $x = 0$. (U.L.)

(75) (i) By differentiating with respect to x, or otherwise, solve the differential equation

$$y = x(p + p^3),\text{ where } p = \frac{dy}{dx}$$

(ii) Find a particular integral of the equation

$$\frac{d^4y}{dx^4} - y = x\sin x \qquad \text{(U.L.)}$$

(76) The equation $\frac{d^2S}{d\theta^2} + 2b\frac{dS}{d\theta} - c^2S = 0$ occurs in an investigation of the plastic flow of solids. $S = 0$ when $\theta = 0$. Show (i) using operators and (ii) by means of the auxiliary equation that the solution is $S = 2S_0e^{-b\theta}\sinh\sqrt{b^2 + c^2}\theta$, b and c being constants.

(77) By means of complex algebra simplify the following—

$$(2 + 3D)(3 - 2D)\sin(2t + 1),\ \frac{2 + D}{2 - D}\cos(t + 2),\ \sqrt{5 + 3D}\sin 4t$$

and $$\sqrt{\frac{2 + D}{2 - D}}\sin(3t + 2)\text{ where } D \equiv \frac{d}{dt}$$

(78) Express as a function of t $\sqrt{\frac{6 + 0{\cdot}03D}{4 + 0{\cdot}018D}}E_m\sin 500t$, where $D \equiv \frac{d}{dt}$

CHAPTER VII

ORDINARY DIFFERENTIAL EQUATIONS II

67. Numerical Solutions. In many practical problems in which differential equations occur it is possible to proceed without using any of the foregoing methods of solution. In beam theory the equation $EI\dfrac{d^4y}{dx^4} =$ load per foot run is often solved by students with no knowledge of calculus by a double application of the funicular polygon, each application giving the solution of an equation of the type $\dfrac{d^2y}{dx^2} = X$, where X is a function of x. The use of a planimeter or of Simpson's rule or of any other approximate method to find the area A under a curve can give the solution of an equation of the type $\dfrac{dA}{dx} = y$, where (x, y) is a point on the curve. In the finding of the critical speeds of vibration of crank-shafts a numerical method is the most convenient. Further, a solution in the form of tabulated values of the variables is what is required in many cases rather than a functional relationship in the form of an equation. Consider the first order equation

$$\frac{dy}{dx} + 3y = 5 \quad . \quad . \quad . \quad \text{(VII.1)}$$

of which we are required to find values of y over the range $x = 0$ to $x = 0{\cdot}5$, having given that, when $x = 0$, $y = 1$.

We take a small increment of x, say $\Delta x = 0{\cdot}1$, and evaluate y when $x = 0 + 0{\cdot}1 = 0{\cdot}1$. To do this we assume that $\dfrac{\Delta y}{\Delta x} \approx \dfrac{dy}{dx}$. Substituting in (VII.1) and transposing,

$$\Delta y \approx (5 - 3y)\Delta x \quad . \quad . \quad . \quad \text{(VII.2)}$$

Putting $y = 1$ and $\Delta x = 0{\cdot}1$, we have $\Delta y = 2 \times 0{\cdot}1$. Thus we see that, when $x = 0{\cdot}1$, $y = 1 + 2 \times 0{\cdot}1 = 1{\cdot}2$.

Again taking $\Delta x = 0{\cdot}1$, $y = 1{\cdot}2$ and substituting in (VII.2), $\Delta y = (5 - 3 \times 1{\cdot}2) \times 0{\cdot}1 = 0{\cdot}14$ and $y = 1{\cdot}2 + 0{\cdot}14 = 1{\cdot}34$. These values are shown in Table II.

TABLE II

x	Δx	$\frac{dy}{dx} = 5 - 3y$	$\Delta y = \frac{dy}{dx}\Delta x$	y
0	0·1	2·0	0·2	1·0
0·1	0·1	1·4	0·14	1·2
0·2	0·1	0·98	0·098	1·34
0·3	0·1	0·686	0·0686	1·438
0·4	0·1	0·479	0·0479	1·507
0·5	0·1	—	—	1·555

In the above it is not essential that the values of Δx should be equal. It is worth while plotting the graph as the values are found. Where the curve is flat, the values of Δx may be larger than where it is more curved. The method has the advantage of simplicity, the solution being readily obtainable by anyone who can do simple arithmetic. On the other hand, the method is not very accurate and there is no way of testing the degree of accuracy. The errors are cumulative if the curve is entirely concave upwards or entirely concave downwards. The main source of error lies in the assumption that the gradient at (x, y) is the gradient of the chord joining that point to $(x + \Delta x, y + \Delta y)$.

The solution of (VII.1) is $y = \frac{1}{3}(5 - 2e^{-3x})$, and from this, when $x = 0{\cdot}5$, $y = 1{\cdot}518$ as against 1·555 in the table. Greater accuracy could be obtained by taking smaller values of Δx. The above is known as *Euler's Method.* In order to increase the accuracy we assume that the average gradient is half the sum of those of the tangents at the end points. As we do not know the gradient at $(x + \Delta x, y + \Delta y)$ we take an approximation to it, i.e. that found in Table II. Thus for the first interval we have—average gradient $= \frac{1}{2}(2{\cdot}0 + 1{\cdot}4) = 1{\cdot}7$. This is entered in Table III as mean gradient. From this $\Delta y = 1{\cdot}7\Delta x = 0{\cdot}17$, whence $y = 1{\cdot}0 + 0{\cdot}17 = 1{\cdot}17$. This is a more accurate value of y than the entry above it found as in Table II. Keeping to the first interval we repeat the calculations until two consecutive values of y agree, i.e. those at $y = 1{\cdot}1739$. This is then taken as the starting point for the next stage $x = 0{\cdot}1$ to $x = 0{\cdot}2$. For comparison we give values of $y = \frac{1}{3}(5 - 2e^{-3x})$ in the last column of Table III.

The closeness of the results to the correct values rather flatters the method. The errors are still cumulative when $\frac{d^2y}{dx^2}$ does not change

TABLE III

x	Δx	Gradient at (x, y) $\frac{dy}{dx}=5-3y$	Mean Gradient	Δy $=\Delta x$ $\times$ Mean Gradient	y	$y=\frac{5-2e^{-3x}}{3}$
0	0·1	2·0	—	0·2	1·0	
0·1		1·4	1·7	0·17	1·2	
0·1		1·49	1·745	0·1745	1·17	
0·1		1·4765	1·7382	0·17382	1·1745	
0·1		1·4786	1·7393	0·17393	1·1738	
0·1					1·1739	1·173
0·1	0·1	1·4783	—	0·14783	1·1739	
0·2		1·0349	1·2566	0·12566	1·3217	
0·2		1·1012	1·2898	0·12898	1·2996	
0·2		1·0913	1·2848	0·12848	1·3029	
0·2		1·0928	1·2856	0·12856	1·3024	
0·2					1·3025	1·301
0·2	0·1	1·0928	—	0·10928	1·3025	
0·3		0·7646	0·9287	0·09287	1·4118	
0·3		0·8138	0·9533	0·09533	1·3954	
0·3		0·8066	0·9497	0·09497	1·3978	
0·3		0·8075	0·9501	0·09501	1·3975	
0·3					1·3975	1·396
0·3	0·1	0·8075	—	0·08075	1·3975	
0·4		0·5651	0·6863	0·06863	1·4783	
0·4		0·6017	0·7046	0·07046	1·4661	
0·4		0·5960	0·7018	0·07018	1·4680	
0·4		0·5969	0·7022	0·07022	1·4677	
0·4					1·4677	1·466
0·4	0·1	0·5969	—	0·05969	1·4677	
0·5		0·4178	0·5073	0·05073	1·5274	
0·5		0·4448	0·5208	0·05208	1·5184	
0·5		0·4406	0·5188	0·05188	1·5198	
0·5		0·4412	0·5190	0·05190	1·5196	
0·5					1·5196	1·518

sign and increase as the steepness of the graph increases. This is known as the *modified* or *improved* Euler's method.

A more useful method is known as *Runge's method.* It must be remembered that we are trying to solve an equation of the type

$$\frac{dy}{dx}=f(x, y) \qquad . \quad . \quad . \quad \text{(VII.3)}$$

The expression on the right has values at all points of the xy-plane. We are attempting to find one only of the family of curves represented by (VII.3). In Fig. 46 we show part of this curve of which we are given the point A and the value of $\frac{dy}{dx}$ at A. MN is the projection on the axis of x of the straight line joining A to a point B on the curve. Let the co-ordinates of A be (x_0, y_0) and those of B $(x_0 + h, y_0 + k)$. $AB_0 \perp BN$. For a first approximation to the position of B we draw the tangent at A which meets BN at B_1. Since $\overline{B_0B} = k$, the first approximation to k, which we shall call k_1, is

$$k_1 = h \times \text{gradient at } A$$

i.e. $$k_1 = hf(x_0, y_0) \quad . \quad . \quad \text{(VII.4)}$$

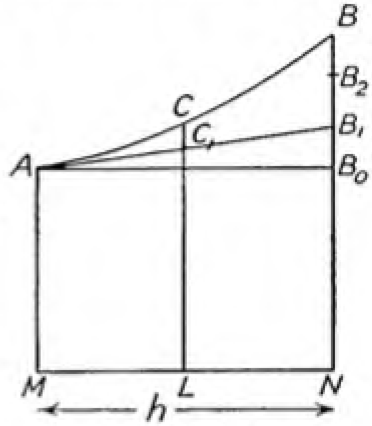


FIG. 46.
RUNGE'S METHOD

The gradient at B_1 of the curve of the family through B_1 is, by (VII.3),

$$f(x_0 + h, y_0 + k_1) \quad . \quad . \quad \text{(VII.5)}$$

This is a first approximation to the gradient at B. To find a closer approximation we assume the gradient at B_1 to be transferred to A, as this is nearer to the mean gradient, i.e. that of the chord AB, than is the gradient at A. Let $\overline{B_0B_2} = k_2$, where B_2 is the point in B_0B such that the gradient of the straight line AB_2 is equal to the gradient (VII.5). Then k_2 is a second approximation to k, and

$$k_2 = hf(x_0 + h, y_0 + k_1) \quad . \quad . \quad \text{(VII.6)}$$

The co-ordinates of B_2 are $(x_0 + h, y_0 + k_2)$, and by (VII.3) the gradient at B_2 of the curve of the family which passes through B_2 is

$$f(x_0 + h, y_0 + k_2) \quad . \quad . \quad \text{(VII.7)}$$

We assume this to be a close approximation to the gradient of the curve AB at B. Let the ordinate through the mid-point of MN cut the curve AB at C and the straight line AB_1 at C_1. We assume that the gradient at C_1 of the curve of the family which passes through C_1 is a sufficiently close approximation to the gradient of the curve AB at C. The co-ordinates of C_1 are $\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$, and the gradient at C_1 is by (VII.3),

$$f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \quad . \quad . \quad . \quad \text{(VII.8)}$$

The gradient at C is very approximately given by this. Thus we can find the gradient of the curve ACB at the points A, B and C with considerable accuracy. The value of k has still to be found. Integrating both sides of (VII.3) between the limits x_0 and $x_0 + h$, we have

$$\int_{y_0}^{y_0+k} dy = \int_{x_0}^{x_0+h} f(x, y)dx \quad . \quad . \quad \text{(VII.9)}$$

The integral on the left has the value k whilst that on the right is found by Simpson's Rule to be $\frac{h}{6}$ (first gradient $+ 4 \times$ middle gradient $+$ last gradient), the three gradients being those at A, C, and B respectively. We have then, if y_B is the ordinate at B,

$$y_B = y_A + \frac{h}{6}\Big[f(x_0, y_0) + 4f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$+ f(x_0 + h, y_0 + k_2)\Big] \quad . \quad \text{(VII.10)}$$

We shall apply (VII.10) to the solution of (VII.1).

EXAMPLE 1

$$\frac{dy}{dx} = 5 - 3y \text{ and } y = 1 \text{ when } x = 0$$

$$x_0 = 0,\ h = 0{\cdot}1,\ f(x_0, y_0) = f(0, 1) = 5 - 3 = 2,\ k_1 = 2 \times 0{\cdot}1 = 0{\cdot}2$$
$$f(x_0 + h, y_0 + k_1) = f(0{\cdot}1, 1{\cdot}2) = 5 - 3{\cdot}6 = 1{\cdot}4,\ k_2 = 1{\cdot}4 \times 0{\cdot}1 = 0{\cdot}14$$
$$f(x_0 + h, y_0 + k_2) = f(0{\cdot}1, 1{\cdot}14) = 5 - 3{\cdot}42 = 1{\cdot}58$$
$$f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = f(0{\cdot}05, 1{\cdot}1) = 5 - 3{\cdot}3 = 1{\cdot}7$$

Then $\quad y_B = y_A + \frac{h}{6}(2 + 4 \times 1{\cdot}7 + 1{\cdot}58)$

$$y_B = 1 + \frac{0{\cdot}1}{6} \times 10{\cdot}38 = 1{\cdot}173. \quad \text{When } x = 0{\cdot}1,\ y = 1{\cdot}173$$

This value found in one stage is practically the same as that found after five stages in Table III. This suggests that we may be able to increase h to $0{\cdot}2$. For the next stage then we have—

$$x_0 = 0{\cdot}1,\ y_0 = 1{\cdot}173,\ h = 0{\cdot}2,\ f(x_0, y_0) = 5 - 3{\cdot}519 = 1{\cdot}481,$$
$$k_1 = 1{\cdot}481 \times 0{\cdot}2 = 0{\cdot}296$$
$$f(x_0 + h, y_0 + k_1) = f(0{\cdot}3, 1{\cdot}469) = 5 - 4{\cdot}407 = 0{\cdot}593,$$
$$k_2 = 0{\cdot}593 \times 0{\cdot}2 = 0{\cdot}1186$$
$$f(x_0 + h, y_0 + k_2) = f(0{\cdot}3, 1{\cdot}292) = 5 - 3{\cdot}876 = 1{\cdot}124$$
$$f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = f(0{\cdot}2, 1{\cdot}321) = 5 - 3{\cdot}963 = 1{\cdot}037$$

As before, $y_B = y_A + \frac{h}{6}(1{\cdot}481 + 4 \times 1{\cdot}037 + 1{\cdot}124)$

$$y_B = 1{\cdot}173 + \frac{0{\cdot}2}{6} \times 6{\cdot}753 = 1{\cdot}398$$

When $\quad x = 0{\cdot}3, y = 1{\cdot}398$

By increasing h to 0·2 we have covered in one stage a range which required nine or ten stages in Table III. We shall leave the reader to carry on the calculations up to $x = 1$.

EXAMPLE 2

To start the solution of $\frac{dy}{dx} = \frac{y}{y + x}$ given that $x = 1$, when $y = 1$

Taking $h = 0{\cdot}1$ we have

$x = 1, y = 1, h = 0{\cdot}1, f(x_0, y_0) = \frac{1}{2} = 0{\cdot}5, k_1 = 0{\cdot}1 \times 0{\cdot}5 = 0{\cdot}05$

$$f(x_0 + h, y_0 + k_1) = f(1{\cdot}1, 1{\cdot}05) = \frac{1{\cdot}05}{2{\cdot}15} = 0{\cdot}4884\ .\ k_2 = 0{\cdot}0488$$

$$f(x_0 + h, y_0 + k_2) = f(1{\cdot}1, 1{\cdot}049) = \frac{1{\cdot}049}{2{\cdot}149} = 0{\cdot}4881$$

$$f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = f(1{\cdot}05, 1{\cdot}024) = \frac{1{\cdot}024}{2{\cdot}074} = 0{\cdot}4938$$

$$y_B = y_A + \frac{h}{6}(0{\cdot}5 + 4 \times 0{\cdot}4938 + 0{\cdot}4881) = 1 + \frac{0{\cdot}1}{6} \times 2{\cdot}9633 = 1{\cdot}049$$

Hence, when $x = 1{\cdot}1, y = 1{\cdot}049$

Again, taking $h = 0{\cdot}1$, we have

$$x_0 = 1{\cdot}1, y_0 = 1{\cdot}049, h = 0{\cdot}1, f(x_0, y_0) = \frac{1{\cdot}049}{2{\cdot}149} = 0{\cdot}4881,$$
$$k_1 = 0{\cdot}4881 \times 0{\cdot}1 = 0{\cdot}0488$$

$$f(x_0 + h, y_0 + k_1) = f(1{\cdot}2, 1{\cdot}098) = \frac{1{\cdot}098}{2{\cdot}298} = 0{\cdot}4779,$$
$$k_2 = 0{\cdot}1 \times 0{\cdot}4779 = 0{\cdot}0478$$

$$f(x_0 + h, y_0 + k_2) = f(1{\cdot}2, 1{\cdot}097) = \frac{1{\cdot}097}{2{\cdot}297} = 0{\cdot}4776$$

$$f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = f(1{\cdot}15, 1{\cdot}073) = \frac{1{\cdot}073}{2{\cdot}223} = 0{\cdot}4826$$

Therefore $y_B = y_A + \frac{0{\cdot}1}{6}(0{\cdot}4881 + 4 \times 0{\cdot}4826 + 0{\cdot}4776)$

or $\quad y_B = 1{\cdot}049 + 0{\cdot}048 = 1{\cdot}097$ is the value of y when $x = 1{\cdot}2$.

As the two increments of y are practically equal, the curve is nearly straight in this neighbourhood, and the value of the increment $\Delta x = h$ may be increased to 0·2 or 0·3. It is left to the reader to continue the calculations (Ex. 25, page 285).

Runge's method is usually given in the following form. Let x and y be known values of the variables, Δx a small increase in x and Δy the corresponding increase in y. The following quantities are calculated

$$\Delta' = f(x, y)\Delta x \quad . \quad . \quad . \quad . \quad \text{(VII.11)}$$

$$\Delta'' = f(x + \Delta x, y + \Delta')\Delta x \quad . \quad . \quad \text{(VII.12)}$$

$$\Delta''' = f(x + \Delta x, y + \Delta'')\Delta x \quad . \quad . \quad \text{(VII.13)}$$

$$\Delta^{(4)} = f(x + \tfrac{1}{2}\Delta x, y + \tfrac{1}{2}\Delta')\Delta x \quad . \quad . \quad \text{(VII.14)}$$

Then, as the reader should see from the above,

$$\Delta y = \tfrac{1}{6}(\Delta' + 4\Delta^{(4)} + \Delta''') \quad . \quad . \quad \text{(VII.15)}$$

which for convenience of calculation may be written

$$\Delta y = \Delta^{(4)} + \tfrac{1}{3}\{\tfrac{1}{2}(\Delta' + \Delta''') - \Delta^{(4)}\} \quad . \quad \text{(VII.16)}$$

We shall use the latter relation to carry out the first step of the solution of Example 1 above.

$$\begin{aligned}
\Delta' &= 0{\cdot}1f(0, 1) &&= 0{\cdot}2 \\
\Delta'' &= 0{\cdot}1f(0{\cdot}1, 1{\cdot}2) &&= 0{\cdot}14 \\
\Delta''' &= 0{\cdot}1f(0{\cdot}1, 1{\cdot}14) &&= 0{\cdot}158 \\
&\tfrac{1}{2}(\Delta' + \Delta''') &&= 0{\cdot}179 \\
\Delta^{(4)} &= 0{\cdot}1f(0{\cdot}05, 1{\cdot}1) &&= 0{\cdot}17 \\
&\tfrac{1}{3} \times \text{difference} &&= 0{\cdot}003 \\
\Delta y &= \text{sum} &&= 0{\cdot}173
\end{aligned}$$

and $\qquad y = 1{\cdot}173$ as above

68. **Picard's Method.** This method of solution is one of successive approximation by iteration. An approximate value of y, a constant, is substituted on the right-hand side of (VII.3). The equation is then integrated, giving y in terms of x as a second approximation. This is then substituted on the right-hand side and the equation again integrated. This process is repeated continuously until two successive approximations for y are alike within the required limits of accuracy and this value of y is the required approximate solution. We shall use the method to find a series solution of the equation

$$\frac{dy}{dx} = x^2 + y^2 \quad . \quad . \quad . \quad \text{(VII.17)}$$

having given that $y = 0$ when $x = 0$.

Integrating both sides of this with respect to x between $x = 0$ and $x = x$,

$$\int_0^x \frac{dy}{dx}\,dx = \int_0^x (x^2 + y^2)dx$$

i.e.

$$y - y_0 = \int_0^x (x^2 + y^2)dx$$

or

$$y = y_0 + \int_0^x (x^2 + y^2)dx\,. \qquad \text{(VII.18)}$$

where y_0 is the value of y when $x = 0$. In this case $y_0 = 0$. We approximate by assuming $y = y_0$ in the integrand.

Thus

$$y = 0 + \int_0^x (x^2 + 0)dx$$

$$= \frac{x^3}{3} \text{ is the first approximation to the solution.}$$

Substituting this for y on the right-hand side of (VII.18) we have as a second approximation

$$y = 0 + \int_0^x \left(x^2 + \frac{x^6}{9}\right) dx$$

$$y = \frac{x^3}{3} + \frac{x^7}{63}$$

The third approximation is

$$y = 0 + \int_0^x \left[x^2 + \left(\frac{x^3}{3} + \frac{x^7}{63}\right)^2\right] dx$$

or

$$y = \int_0^x \left(x^2 + \frac{x^6}{9} + \frac{2x^{10}}{189} + \frac{x^{14}}{63^2}\right) dx$$

$$= \frac{x^3}{3} + \frac{x^7}{63} + \frac{2x^{11}}{189 \times 11} + \frac{x^{15}}{15 \times 63^2} \qquad \text{(VII.19)}$$

Neglecting powers of x higher than the fifteenth, the fourth approximation is

$$y = \int_0^x \left[x^2 + \left(\frac{x^3}{3} + \frac{x^7}{63} + \frac{2x^{11}}{189 \times 11} + \frac{x^{15}}{15 \times 63^2}\right)^2\right] dx$$

$$= \int_0^x \left(x^2 + \frac{x^6}{9} + \frac{2x^{10}}{189} + \frac{39}{11 \times 63^2}\, x^{14}\right) dx$$

$$y = \frac{x^3}{3} + \frac{x^7}{63} + \frac{2}{2\,079}\, x^{11} + \frac{13x^{15}}{55 \times 63^2} + \ldots \quad \text{(VII.20)}$$

The first three terms on the right of (VII.19) and (VII.20) agree, and

$$y = \frac{x^3}{3} + \frac{x^7}{63} + \frac{2}{2\,079}\, x^{11} \quad \text{(VII.21)}$$

is an approximate solution of (VII.17) for values of x such that $|x| < 1$. Values of y over the range $x = 1$ to $x = 2$ are best found by Runge's Method.

EXAMPLE

Find a solution over the range $x = 0$ to $x = 1$ of

$$\frac{dy}{dx} = y + e^x \quad (1)$$

having given $y = 0$ when $x = 0$.

We have
$$y = y_0 + \int_0^x (y + e^x)\, dx$$

and putting $y_0 = 0$ and $y = y_0 = 0$ on the right the first approximation is

$$y = \int_0^x e^x\, dx = e^x - 1$$

Second approximation,
$$y = \int_0^x (2e^x - 1)\, dx$$
$$= 2e^x - x - 2$$

Third approximation,
$$y = \int_0^x (3e^x - x - 2)\, dx = 3e^x - \frac{x^2}{2} - 2x - 3$$

Fourth approximation,
$$y = \int_0^x \left(4e^x - \frac{x^2}{2} - 2x - 3\right) dx$$
$$= 4e^x - \frac{x^3}{6} - x^2 - 3x - 4$$

Fifth approximation, $$y = \int_0^x \left(5e^x - \frac{x^3}{6} - x^2 - 3x - 4\right) dx$$

$$= 5e^x - \frac{x^4}{24} - \frac{x^3}{3} - \frac{3x^2}{2} - 4x - 5$$

Sixth approximation, $$y = \int_0^x \left(6e^x - \frac{x^4}{24} - \frac{x^3}{3} - \frac{3x^2}{2} - 4x - 5\right) dx$$

i.e. $$y = 6e^x - \frac{x^5}{120} - \frac{x^4}{12} - \frac{x^3}{2} - 2x^2 - 5x - 6 \quad . \quad . \quad (2)$$

When $x = 1$ the successive approximations calculated from the above are 1·718, 2·437, 2·655, 2·706, 2·716, and 2·717 and since the fifth and sixth of these are practically the same (2) gives the approximate relation between x and y over the given range.

The actual solution of (1) is $y = xe^x$ and when $x = 1$, $y = 2{\cdot}718$. The approximate solution is in error by less than $\frac{1}{2\,000}$ over the whole range. By obtaining more terms of the series we can find values of y for values of $x > 1$. The calculations become very laborious but can be made easier by taking a new origin at $x = 1$, $y = 2{\cdot}717$. We shall write e for 2·717 in order to reduce the space taken by the expressions. If (X, Y) are the new co-ordinates we have $X = x - 1$, $Y = y - e$ and $\frac{dY}{dX} = \frac{dy}{dx}$.

From (1) $$\frac{dY}{dX} = Y + e(1 + e^X) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let Y_1, Y_2, Y_3, etc., be the successive approximations to the true value of Y. As before, putting $Y = Y_0 = 0$ on the right-hand side and integrating

$$Y_1 = \int_0^X e(1 + e^X)\, dX$$

$$Y_1 = e\Big[X + e^X\Big]_0^X = e(e^X + X - 1)$$

$$Y_2 = e\int_0^X (X + 2e^X)dX = e(2e^X + \tfrac{1}{2}X^2 - 2)$$

$$Y_3 = e\int_0^X (3e^X + \tfrac{1}{2}X^2 - 1)dX = e(3e^X + \tfrac{1}{6}X^3 - X - 3)$$

$$Y_4 = e\int_0^X (4e^X + \tfrac{1}{6}X^3 - X - 2)dX = e(4e^X + \tfrac{1}{24}X^4 - \tfrac{1}{2}X^2 - 2X - 4)$$

$$Y_5 = e\int_0^X (5e^X + \tfrac{1}{24}X^4 - \tfrac{1}{2}X^2 - 2X - 3)dX$$
$$= e(5e^X + \tfrac{1}{120}X^5 - \tfrac{1}{6}X^3 - X^2 - 3X - 5)$$

$$Y_6 = e\int_0^X \left(6e^X + \frac{X^5}{120} - \frac{X^3}{6} - X^2 - 3X - 4\right)dX$$
$$= e\left(6e^X + \frac{X^6}{720} - \frac{X^4}{24} - \frac{X^3}{3} - \frac{3X^2}{2} - 4X - 6\right) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Putting $X = 1$ we have $Y_5 = Y_6 = 12{\cdot}04$. When $X = 1$, i.e. $x = 2$, $y = 2{\cdot}717 + 12{\cdot}04 = 14{\cdot}76$. This is a close approximation to the correct value of y for when $x = 2$, $y = 2e^2 = 14{\cdot}78$. Using the series (2) and (4) we can calculate values of y over the range $x = 0$ to $x = 2$.

69. **Solution in Series.** Many differential equations do not permit of a general solution in terms of ordinary known functions. Some of these can be solved in terms of an infinite convergent series of powers of the independent variable. We shall first find the series solution of an equation whose solution we know in terms of known functions.

Consider
$$\frac{d^2y}{dx^2} = m^2y \qquad \text{(VII.22)}$$

The solution is
$$\left.\begin{aligned} y &= Ae^{mx} + Be^{-mx} \\ \text{or alternatively,} \quad y &= C\cosh mx + D\sinh mx \end{aligned}\right\} \qquad \text{(VII.23)}$$

Assume the solution to be
$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots \qquad \text{(VII.24)}$$

Then
$$\frac{d^2y}{dx^2} = 2{\cdot}1a_2 + 3{\cdot}2a_3x + 4{\cdot}3a_4x^2 + 5{\cdot}4a_5x^3 + \ldots + (n+2)(n+1)a_{n+2}x^n + \ldots$$

and substituting in (VII.22), we have the identity
$$2{\cdot}1a_2 + 3{\cdot}2a_3x + 4{\cdot}3a_4x^2 + \ldots + (n+2)(n+1)a_{n+2}x^n + \ldots \equiv m^2a_0 + m^2a_1x + m^2a_2x^2 + \ldots + m^2a_nx^n + \ldots$$

Equating coefficients of corresponding powers of x, we obtain
$$2{\cdot}1a_2 = m^2a_0, \text{ i.e. } a_2 = \frac{m^2}{2{\cdot}1}a_0$$
$$3{\cdot}2a_3 = m^2a_1, \text{ i.e. } a_3 = \frac{m^2}{3{\cdot}2}a_1$$

. . . .

Generally, from a comparison of powers of x^n,
$$(n+2)(n+1)a_{n+2} = m^2a_n, \text{ i.e. } a_{n+2} = \frac{m^2}{(n+2)(n+1)}a_n \qquad \text{(VII.25)}$$

From (VII.25) we see that any term in (VII.24) may be obtained from the next but one preceding it by multiplying by m^2x^2 and dividing by $(n + 2)(n + 1)$, where n is the index of the power of x in the earlier term.

Substituting these values in (VII.24), we find

$$y = a_0 + a_1x + m^2\left(\frac{a_0x^2}{2!} + \frac{a_1x^3}{3!}\right) + m^4\left(\frac{a_0x^4}{4!} + \frac{a_1x^5}{5!}\right) + \dots$$

i.e. $$y = a_0\left(1 + \frac{m^2x^2}{2!} + \frac{m^4x^4}{4!} + \frac{m^6x^6}{6!} + \dots\right)$$

$$+ \frac{a_1}{m}\left(mx + \frac{m^3x^3}{3!} + \frac{m^5x^5}{5!} + \frac{m^7x^7}{7!} + \dots\right) \quad \text{(VII.26)}$$

(VII.26) is equivalent to (VII.23), for the series in the brackets are those for cosh mx and sinh mx respectively, and a_0 and $\frac{a_1}{m}$ are arbitrary constants.

The relation (VII.25) is known as a *recurrence* relation.

An alternative method is to write the series (VII.24) in the form of Maclaurin's series given in Vol. I, thus

$$y = f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0)$$

$$+ \frac{x^4}{4!}f^{(4)}(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots \quad \text{(VII.27)}$$

where $f^{(n)}(0)$ is the value of $\frac{d^ny}{dx^n}$ when $x = 0$, and $f(0)$ is the value of y when $x = 0$.

We shall assume that the values of $f(0)$ and $f'(0)$ are known.

From (VII.22) by successive differentiation we obtain

$$\frac{d^3y}{dx^3} = m^2\frac{dy}{dx}, \frac{d^4y}{dx^4} = m^2\frac{d^2y}{dx^2}, \frac{d^5y}{dx^5} = m^2\frac{d^3y}{dx^3}, \text{ etc.}$$

so that with $x = 0$ in (VII.22) and in these, $f''(0) = m^2f(0)$, $f'''(0) = m^2f'(0)$, $f^{(4)}(0) = m^2f''(0)$, $f^{(5)}(0) = m^2f'''(0)$, and so on.

Substituting these values in (VII.27)

$$y = f(0) + xf'(0) + \frac{x^2}{\underline{|2}}\, m^2 f(0) + \frac{x^3}{\underline{|3}}\, m^2 f'(0) + \frac{x^4}{\underline{|4}}\, m^4 f(0) + \frac{x^5}{\underline{|5}}\, m^4 f'(0) + \ldots$$

$$= f(0)\left[1 + \frac{m^2x^2}{\underline{|2}} + \frac{m^4x^4}{\underline{|4}} + \ldots\right] + \frac{f'(0)}{m}\left[mx + \frac{m^3x^3}{\underline{|3}} + \frac{m^5x^5}{\underline{|5}} + \ldots\right]$$

i.e. $\quad y = C\cosh mx + D\sinh mx$

where $C = f(0)$ and $D = \dfrac{f'(0)}{m}$ are arbitrary constants. If $f(0)$ and $f'(0)$ are known, then C and D are known, and the equation represents one of the family of expressions which satisfy (VII.22).

EXAMPLE 1

To solve $$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \text{ by series} \quad . \quad . \quad . \qquad \text{(VII.28)}$$

Let $$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots$$

Then $$x\frac{dy}{dx} = a_1x + 2a_2x^2 + 3a_3x^3 + \ldots + na_nx^n + \ldots$$

and $$\frac{d^2y}{dx^2} = 2.1a_2 + 3.2a_3x + 4.3a_4x^2 + 5.4a_5x^3 + \ldots + (n+2)(n+1)a_{n+2}x^n + \ldots$$

Substituting these values in (VII.28),

$$(a_0 + 2.1a_2) + (2a_1 + 3.2a_3)x + (3a_2 + 4.3a_4)x^2 + (4a_3 + 5.4a_5)x^3 + \ldots + [(n+1)a_n + (n+2)(n+1)a_{n+2}]x^n + \ldots \equiv 0$$

Since this is an identity, the coefficient of each power of x must vanish.

Hence, $$a_0 + 2.1a_2 = 0, \quad \text{i.e. } a_2 = -\tfrac{1}{2}a_0$$

$$2a_1 + 3.2a_3 = 0, \quad \text{i.e. } a_3 = -\frac{2a_1}{3.2} = -\tfrac{1}{3}a_1$$

$$3a_2 + 4.3a_4 = 0, \quad \text{i.e. } a_4 = -\frac{3a_2}{4.3} = -\tfrac{1}{4}a_2$$

and so on.

Generally, $$(n+1)a_n + (n+2)(n+1)a_{n+2} = 0, \text{ i.e. } a_{n+2} = -\frac{1}{n+2}a_n \qquad \text{(VII.29)}$$

which is the recurrence relation. From this formula we see that to find the coefficient of any term we change the sign of the one two terms before and divide by the index of the power of x in the later term. The series is then

$$y = a_0 + a_1x - \tfrac{1}{2}a_0x^2 - \tfrac{1}{3}a_1x^3 + \frac{1}{2\,.\,4}\,a_0x^4 + \frac{1}{3\,.\,5}\,a_1x^5 - \frac{1}{2\,.\,4\,.\,6}\,a_0x^6 - \frac{1}{3\,.\,5\,.\,7}\,a_1x^7 + \ldots$$

i.e.
$$y = a_0\left(1 - \tfrac{1}{2}x^2 + \frac{1}{2\,.\,4}\,x^4 - \frac{1}{2\,.\,4\,.\,6}\,x^6 + \ldots\right) + a_1\left(x - \tfrac{1}{3}x^3 + \frac{1}{3\,.\,5}\,x^5 - \frac{1}{3\,.\,5\,.\,7}\,x^7 + \ldots\right) \quad \text{(VII.30)}$$

in which a_0 and a_1 are arbitrary constants. The series in the first bracket is the expansion of $e^{-\frac{1}{2}x^2}$.

We shall solve (VII.28) by the use of Maclaurin's series.

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Differentiating successively,

$$\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

$$\frac{d^4y}{dx^4} + x\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} = 0$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{d^ny}{dx^n} + x\frac{d^{n-1}y}{dx^{n-1}} + (n-1)\frac{d^{n-2}y}{dx^{n-2}} = 0$$

Putting $x = 0$ in these and using the notation of (VII.27)—
$f''(0) = -f(0)$; $f'''(0) = -2f'(0)$; $f^{(4)}(0) = -3f''(0) = 3f(0)$; $f^{(5)}(0) = -4f'''(0) = 8f'(0)$; $f^{(6)}(0) = -5f^{(4)}(0) = -15f(0)$; and so on.

Substituting in (VII.27)

$$y = f(0) + xf'(0) - \frac{x^2}{2}f(0) - \frac{x^3}{3}f'(0) + \frac{x^4}{2\,.\,4}f(0) + \frac{x^5}{3\,.\,5}f'(0) + \ldots$$

which is the same as (VII.30) with $f(0)$ in place of a_0 and $f'(0)$ in place of a_1

EXAMPLE 2

To solve $$\frac{d^2y}{dx^2} + xy = 0 \text{ by series} \quad . \quad . \quad . \quad \text{(VII.31)}$$

Assuming the relation (VII.24) and differentiating twice with respect to x, we have

$$\frac{d^2y}{dx^2} = 2 \,.\, 1a_2 + 3 \,.\, 2a_3x + 4 \,.\, 3a_4x^2 + 5 \,.\, 4a_5x^3 + \ldots + (n+2)(n+1)a_{n+2}x^n + \ldots$$

Also $$xy = a_0x + a_1x^2 + a_2x^3 + \ldots + a_{n-1}x^n + \ldots$$

Substituting in (VII.31),

$$2 \,.\, 1a_2 + (3 \,.\, 2a_3 + a_0)x + (4 \,.\, 3a_4 + a_1)x^2 + (5 \,.\, 4a_5 + a_2)x^3 + \ldots + [(n+2)(n+1)a_{n+2} + a_{n-1}]x^n + \ldots \equiv 0$$

Equating the coefficients of the powers of x to zero,

$$a_2 = 0; \quad 3 \,.\, 2a_3 = -a_0, \quad \text{i.e. } a_3 = -\frac{a_0}{3 \,.\, 2};$$

$4 \,.\, 3a_4 = -a_1$, i.e. $a_4 = -\dfrac{a_1}{4 \,.\, 3}$; $5 \,.\, 4a_5 + a_2 = 0$, i.e. $a_5 = -\dfrac{a_2}{5 \,.\, 4} = 0$; and so on.

Generally $(n+2)(n+1)a_{n+2} = -a_{n-1}$, which is put into the more convenient form

$$n(n-1)a_n = -a_{n-3} \quad . \quad . \quad . \quad . \quad \text{(VII.32)}$$

by writing n for $n+2$.

Thus, from the above values and from the recurrence relation (VII.32), we have

$$a_6 = -\frac{a_3}{6 \,.\, 5} = \frac{4a_0}{\underline{|6}}; \quad a_7 = -\frac{a_4}{7 \,.\, 6} = \frac{5 \,.\, 2a_1}{\underline{|7}}; \quad a_8 = -\frac{a_5}{8 \,.\, 7} = \frac{a_2}{8 \,.\, 7 \,.\, 5 \,.\, 4} = 0;$$

$$a_9 = -\frac{a_6}{9 \,.\, 8} = -\frac{7 \,.\, 4a_0}{\underline{|9}}, \text{ etc.}$$

Substituting in (VI.24), we obtain as the series expansion for y,

$$y = a_0\left(1 - \frac{x^3}{\underline{|3}} + \frac{4x^6}{\underline{|6}} - \frac{7 \,.\, 4x^9}{\underline{|9}} + \frac{10 \,.\, 7 \,.\, 4x^{12}}{\underline{|12}} - \frac{13 \,.\, 10 \,.\, 7 \,.\, 4x^{15}}{\underline{|15}} + \ldots\right)$$
$$+ a_1\left(x - \frac{2x^4}{\underline{|4}} + \frac{5 \,.\, 2x^7}{\underline{|7}} - \frac{8 \,.\, 5 \,.\, 2x^{10}}{\underline{|10}} + \frac{11 \,.\, 8 \,.\, 5 \,.\, 2x^{13}}{\underline{|13}} - \ldots\right) \quad . \quad \text{(VII.33)}$$

EXAMPLE 3

A strut of uniform section, length l in., weight per inch run w lb, moment of inertia of cross-section I in.^{4}, stands in a vertical position with its lower end fixed. Find the greatest value of l so that the strut will just buckle. (See *Applied Elasticity* by Dr. John Prescott.)

Let y in. be the displacement from the upright position of a point x in. below the top end. It is easy to prove that $\dfrac{dM}{dy} + wx = 0$ where M is the bending

moment at that point. But from beam theory $EI\dfrac{d^2y}{dx^2} = M$ where E is Young's Modulus in lb/in.^{2}

We have then
$$EI\frac{d^3y}{dx^3} = \frac{dM}{dx} = \frac{dM}{dy}\cdot\frac{dy}{dx}$$
$$= -wpx$$

where $p = \dfrac{dy}{dx}$. Dividing through by EI, writing c for $\dfrac{w}{EI}$ and transposing,

$$\frac{d^3y}{dx^3} + cpx = 0$$

or
$$\frac{d^2p}{dx^2} + cpx = 0 \qquad (1)$$

Assume that p can be expanded in the form
$$p = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots \qquad (2)$$

then $\dfrac{d^2p}{dx^2} = 2\,.\,1a_2 + 3\,.\,2a_3x + 4\,.\,3a_4x^2 + 5\,.\,4a_5x^3 + 6\,.\,5a_6x^4 + 7\,.\,6a_7x^5 + \ldots$

Substituting these in (1),
$$2\,.\,1a_2 + (3\,.\,2a_3 + ca_0)x + (4\,.\,3a_4 + ca_1)x^2 + (5\,.\,4a_5 + ca_2)x^3$$
$$+ (6\,.\,5a_6 + ca_3)x^4 + (7\,.\,6a_7 + ca_4)x^5 + \ldots \equiv 0 \qquad (3)$$

As this is an identity, the coefficients of the various powers of x must be zero since no such powers occur on the right-hand side; hence

$$a_2 = 0,\ a_3 = -\frac{c}{3\,.\,2}a_0,\ a_4 = -\frac{c}{4\,.\,3}a_1,\ a_5 = 0,\ a_6 = -\frac{c}{6\,.\,5}a_3 = \frac{c^2}{6\,.\,5\,.\,3\,.\,2}a_0,$$

$$a_7 = -\frac{c}{7\,.\,6}a_4 = \frac{c^2}{7\,.\,6\,.\,4\,.\,3}a_1,\ a_8 = -\frac{ca_5}{8\,.\,7} = 0,\ a_9 = -\frac{ca_6}{9\,.\,8}$$
$$= -\frac{c^3}{9\,.\,8\,.\,6\,.\,5\,.\,3\,.\,2}a_0, \text{ etc.}$$

Substituting these in (2),
$$p = a_0 + a_1x - \frac{c}{3\,.\,2}a_0x^3 - \frac{c}{4\,.\,3}a_1x^4 + \frac{c^2}{6\,.\,5\,.\,3\,.\,2}a_0x^6 + \frac{c^2}{7\,.\,6\,.\,4\,.\,3}a_1x^7$$
$$- \frac{c^3}{9\,.\,8\,.\,6\,.\,5\,.\,3\,.\,2}a_0x^9 - \frac{c^3}{10\,.\,9\,.\,7\,.\,6\,.\,4\,.\,3}a_1x^{10} + \ldots$$

$$p = a_0\left(1 - \frac{cx^3}{3\,.\,2} + \frac{c^2x^6}{6\,.\,5\,.\,3\,.\,2} - \frac{c^3x^9}{9\,.\,8\,.\,6\,.\,5\,.\,3\,.\,2} + \frac{c^4x^{12}}{12\,.\,11\,.\,9\,.\,8\,.\,6\,.\,5\,.\,3\,.\,2} - \ldots\right)$$

$$+ a_1\left(x - \frac{cx^4}{4\,.\,3} + \frac{c^2x^7}{7\,.\,6\,.\,4\,.\,3} - \frac{c^3x^{10}}{10\,.\,9\,.\,7\,.\,6\,.\,4\,.\,3} + \frac{c^4x^{13}}{13\,.\,12\,.\,10\,.\,9\,.\,7\,.\,6\,.\,4\,.\,3} - \ldots\right) \qquad (4)$$

Now $\frac{d^2y}{dx^2} = 0$ when $x = 0$, because $M = EI\frac{d^2y}{dx^2} = 0$ there, and therefore $\frac{dp}{dx} = 0$ when $x = 0$. Differentiating both sides of (4) and putting in these values we see that $a_1 = 0$ and (4) becomes

$$p = a_0\left(1 - \frac{cx^3}{3\,.\,2} + \frac{c^2x^6}{6\,.\,5\,.\,3\,.\,2} - \frac{c^3x^9}{9\,.\,8\,.\,6\,.\,5\,.\,3\,.\,2} + \frac{c^4x^{12}}{12\,.\,11\,.\,9\,.\,8\,.\,6\,.\,5\,.\,3\,.\,2} - \ldots\right) \quad . \quad . \quad . \quad (5)$$

When $x = l$, $\frac{dy}{dx} = p = 0$. Substituting in (5) and writing X for $\frac{cl^3}{6}$

$$a_0(1 - X + \frac{1}{5}X^2 - \frac{1}{60}X^3 + \frac{1}{1\,320}X^4 - \ldots) = 0$$

This means that either $a_0 = 0$, in which case the strut does not buckle, or buckling takes place and the quantity in brackets is zero. Thus the least value of l at which buckling occurs is given by $l = \sqrt[3]{\frac{6X_1}{c}}$, where X_1 is the least of the infinite number of roots of

$$1 - X + \frac{1}{5}X^2 - \frac{1}{60}X^3 + \frac{1}{1\,320}X^4 - \ldots = 0 \quad . \quad . \quad (6)$$

To find the least root we only need retain the first few terms on the left and solve the equation by some approximate method. We shall retain the first four terms and solve the equation by the method of iteration of roots, Vol. I.

If X lies between 0 and 1 the even terms in (6) are each less than the preceding term so that the left-hand side is positive and there is no root between $X = 0$ and $X = 1$. We shall assume $X = 1{\cdot}2$ as a first approximation. Writing the equation in the form

$$X_1 = 1 + \frac{1}{5}X^2 - \frac{1}{60}X^3$$

and putting $X = 1{\cdot}2$ on the right we find for a second approximation

$$X_1 = 1 + 0{\cdot}2880 - 0{\cdot}0288 = 1{\cdot}2592, \text{ say } X_1 = 1{\cdot}26$$

Again putting $X = 1{\cdot}26$ on the right the third approximation is

$$X_1 = 1 + 0{\cdot}3175 - 0{\cdot}0333 = 1{\cdot}284, \text{ say } X_1 = 1{\cdot}29$$

Putting $X = 1{\cdot}29$ on the right the fourth approximation is

$$X_1 = 1 + 0{\cdot}3328 - 0{\cdot}0358 = 1{\cdot}297, \text{ say } X_1 = 1{\cdot}30$$

The fifth approximation is

$$X_1 = 1 + 0{\cdot}3380 - 0{\cdot}0366 = 1{\cdot}301$$

and putting $X = 1{\cdot}301$ on the right we have for the sixth approximation

$$X_1 = 1 + 0{\cdot}3386 - 0{\cdot}0367 = 1{\cdot}302$$

The least root is therefore $X_1 = 1{\cdot}302$ from which we find

$$l = \sqrt[3]{\frac{7{\cdot}812}{c}}$$

or

$$l = \sqrt[3]{\frac{7{\cdot}81\ EI}{w}} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Retaining five terms of the series (6) gives 7·84 instead of 7·81. (7) gives the greatest length which will just not buckle. For a solid steel rod of 2 in. diameter l is about 50 ft.

The reader must understand that the solution in series is not always possible, and, even when it is possible, that the series may diverge and prove useless. When the series converges, it may do so slowly and lead to very tedious calculation. When a series solution is found, the series should be tested for convergence by the methods of Art. 6 in Vol. I. Some of the above series represent known functions and the test is not needed. We shall test the series in (VII.33). The r^{th} term of the series in the first pair of brackets is

$$u_r = (-1)^{r-1}\frac{1\,.\,4\,.\,7\,.\,10\,.\,.\,.\,.\,(3r-8)\,(3r-5)}{\underline{|3r-3}}x^{3r-3}$$

and $\dfrac{u_{r+1}}{u_r} = -\dfrac{x^3}{3r(3r-1)}$

which tends to zero as $r \to \infty$. The series is, therefore, convergent for all values of x. For the series in the second pair of brackets

$$u_r = (-1)^{r-1}\frac{2\,.\,5\,.\,8\,.\,.\,.\,.\,.\,(3r-4)}{\underline{|3r-2}}\,x^{3r-2}$$

and $$\frac{u_{r+1}}{u_r} = -\frac{1}{3r(3r+1)}\,x^3$$

which tends to zero as $r \to \infty$, so that this series is also convergent for all values of x. Thus (VII.33) is a satisfactory solution of the differential equation (VII.31), and the series converges fairly rapidly.

The above recurrence relations are of the first order, i.e. they involve only two coefficients, and from them it is easy to obtain successive coefficients and to see the law connecting these. If the recurrence relation, or *difference equation* as it is also called, is of the third, or higher, order it is not always possible to find a general expression for the coefficient and, even if it is possible, it may be so difficult as not to be worth while. In practical cases of this kind it will often be sufficient to obtain just as many terms of the series as

will give a solution with the required degree of accuracy, and to dispense with the test for convergence. A result so obtained must be used cautiously. As a final example we take an equation which gives rise to a second order difference equation.

EXAMPLE 4

To find a series solution of $\dfrac{d^2y}{dx^2} + (x - 1)y = 0$ . . . . (VII.34)

Assuming the expansion

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots + a_nx^n + \ldots \qquad \text{(VII.35)}$$

we have
$$\frac{d^2y}{dx^2} = 2\,.\,1a_2 + 3\,.\,2a_3x + 4\,.\,3a_4x^2 + 5\,.\,4a_5x^3 + \ldots + (n+2)(n+1)a_{n+2}x^n + \ldots$$

and
$$(x-1)y = -a_0 + (a_0 - a_1)x + (a_1 - a_2)x^2 + (a_2 - a_3)x^3 + \ldots + (a_{n-1} - a_n)x^n + \ldots$$

Adding these last two relations and equating to zero, we have

$$(2\,.\,1a_2 - a_0) + (3\,.\,2a_3 + a_0 - a_1)x + (4\,.\,3a_4 + a_1 - a_2)x^2 + \ldots + [(n+2)(n+1)a_{n+2} + a_{n-1} - a_n]x^n + \ldots \equiv 0$$

Equating the coefficients of the powers of x in this identity to zero, we obtain

$$2\,.\,1a_2 - a_0 = 0, \quad \text{i.e. } a_2 = \frac{a_0}{\underline{|2}}$$

$$3\,.\,2a_3 + a_0 - a_1 = 0, \quad \text{i.e. } a_3 = \frac{a_1 - a_0}{\underline{|3}}$$

$$4\,.\,3a_4 + a_1 - a_2 = 0, \quad \text{i.e. } a_4 = \frac{a_2 - a_1}{4\,.\,3} = \frac{a_0 - 2a_1}{\underline{|4}}$$

$$5\,.\,4a_5 + a_2 - a_3 = 0, \quad \text{i.e. } a_5 = \frac{a_3 - a_2}{5\,.\,4} = \frac{a_1 - 4a_0}{\underline{|5}}$$

$$6\,.\,5a_6 + a_3 - a_4 = 0, \quad \text{i.e. } a_6 = \frac{a_4 - a_3}{6\,.\,5} = \frac{5a_0 - 6a_1}{\underline{|6}}$$

$$7\,.\,6a_7 + a_4 - a_5 = 0, \quad \text{i.e. } a_7 = \frac{a_5 - a_4}{7\,.\,6} = \frac{11a_1 - 9a_0}{\underline{|7}}$$

$$8\,.\,7a_8 + a_5 - a_6 = 0, \quad \text{i.e. } a_8 = \frac{a_6 - a_5}{8\,.\,7} = \frac{29a_0 - 12a_1}{\underline{|8}}, \text{ and so on.}$$

Generally,
$$(n+2)(n+1)a_{n+2} = a_n - a_{n-1} \qquad \text{(VII.36)}$$

This is the recurrence formula. The general form of the nth term is not easily recognizable. Substituting the above values of the coefficients in (VII.35), we have

$$y = a_0\left(1 + \frac{x^2}{\underline{|2}} - \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} - \frac{4x^5}{\underline{|5}} + \frac{5x^6}{\underline{|6}} - \frac{9x^7}{\underline{|7}} + \frac{29x^8}{\underline{|8}} + \ldots\right)$$

$$+ a_1\left(x + \frac{x^3}{\underline{|3}} - \frac{2x^4}{\underline{|4}} + \frac{x^5}{\underline{|5}} - \frac{6x^6}{\underline{|6}} + \frac{11x^7}{\underline{|7}} - \frac{12x^8}{\underline{|8}} + \ldots\right) \quad . \quad \text{(VII.37)}$$

in which a_0 and a_1 are arbitrary constants. In both these series the ratio of any coefficient to the preceding one appears to be less than unity and to be continually decreasing as we proceed along the series, but it would not be wise to assume that, because of this, the series is convergent for values of x whose absolute values are less than 1. It can be proved that equations of the type

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_2(x)y = 0 \quad . \quad . \quad \text{(VII.38)}$$

in which $f_1(x)$ and $f_2(x)$ are power series in x (or polynomials), can be solved in a series form, and that the series is convergent over the same range of convergence as that in which both power series are convergent. We can assume, therefore, that both series in (VII.37) are convergent for all values of x.

In the above we have assumed that the arbitrary constants a_0 and a_1 are the values of y and $\frac{dy}{dx}$ respectively when $x = 0$. If, however, the values of y and $\frac{dy}{dx}$ are given for some other value of x, say $x = a$, the constants a_0 and a_1 would not be easy to determine. In such a case it is advisable to assume an infinite series for y in the form

$$y = f(x) = a_0 + a_1(x - a) + a_2(x - a)^2 + a_3(x - a)^3 + \ldots$$
$$+ a_n(x - a)^n + \ldots \quad \text{(VII.39)}$$

In this, a_0 is the value of $y = f(x)$ when $x = a$, i.e. $a_0 = f(a)$, and similarly, $a_1 = f'(a)$, which is the value of $\frac{dy}{dx} = f'(x)$ when $x = a$. By differentiating successively and putting $x = a$ in the resulting series we find that

$$a_2 = \frac{f''(a)}{\underline{|2}},\quad a_3 = \frac{f'''(a)}{\underline{|3}},\quad a_4 = \frac{f^{(4)}(a)}{\underline{|4}},\ \ldots,\quad \text{and generally}$$

$$a_n = \frac{f^{(n)}(a)}{\underline{|n}}$$

Thus, the series (VII.39) takes the form

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\underline{|2}} f''(a) + \frac{(x-a)^3}{\underline{|3}} f'''(a)$$

$$+ \ldots + \frac{(x-a)^n}{\underline{|n}} f^{(n)}(a) + \ldots \qquad \text{(VII.40)}$$

This is an alternative form of Taylor's Series [see Art. 58, Vol. I].

EXAMPLE 5

Obtain an expression for the time of oscillation of the simple pendulum of Ex. 4, p. 388, Vol. I, when the oscillations are not so small that $\sin\theta$ may be assumed to be sensibly equivalent to θ.

The equation of motion was

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Multiplying through by $2\frac{d\theta}{dt}$ we have

$$2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + 2\frac{g}{l}\sin\theta\frac{d\theta}{dt} = 0$$

and integrating this with respect to t,

$$\left(\frac{d\theta}{dt}\right)^2 - \frac{2g}{l}\cos\theta = -\frac{2g}{l}\cos\alpha$$

the right-hand side being the constant of integration.

Hence,
$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}(\cos\theta - \cos\alpha) \quad . \quad . \quad . \quad . \quad (2)$$

Now $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$ and $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$. Hence,

$$\frac{d\theta}{dt} = \sqrt{\frac{4g}{l}\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)} \quad . \quad . \quad . \quad . \quad (3)$$

or
$$dt = \frac{1}{2}\sqrt{\frac{l}{g}}\frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}$$

and if t is the time of a quarter of an oscillation,

$$t = \frac{1}{2}\sqrt{\frac{l}{g}}\int_0^\alpha \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} \quad . \quad . \quad . \quad . \quad (4)$$

The upper limit $\theta = \alpha$ is found by putting $\frac{d\theta}{dt} = 0$ in (2), when we obtain $\cos\theta = \cos\alpha$ or $\theta = \alpha$. The time T of a complete oscillation is $4t$, so that

$$T = 2\sqrt{\frac{l}{g}}\int_0^\alpha \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} \quad . \quad . \quad . \quad . \quad (5)$$

In order to carry out the integration we introduce a new variable ϕ, putting $\sin\frac{\alpha}{2}\sin\phi = \sin\frac{\theta}{2}$. By differentiation, since α is constant,

$$\sin\frac{\alpha}{2}\cos\phi\, d\phi = \frac{1}{2}\cos\frac{\theta}{2}\, d\theta$$

Hence, $$d\theta = \frac{2\sin\frac{\alpha}{2}\cos\phi}{\cos\frac{\theta}{2}}\, d\phi = \frac{2\sin\frac{\alpha}{2}\cos\phi}{\sqrt{1 - \sin^2\frac{\alpha}{2}\sin^2\phi}}\, d\phi$$

Also $$\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}} = \sin\frac{\alpha}{2}\sqrt{1 - \sin^2\phi} = \sin\frac{\alpha}{2}\cos\phi$$

When $\theta = 0$, $\phi = 0$, and when $\theta = \alpha$, $\sin\phi = 1$ and $\phi = \frac{\pi}{2}$. Substituting these in (5), we obtain

$$T = 2\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{2d\phi}{\sqrt{1 - k^2\sin^2\phi}} \quad . \quad . \quad . \quad . \quad (6)$$

where $k = \sin\frac{\alpha}{2}$. The integral in (6) is known as an "elliptic integral." Its value may be found from tables of elliptic integrals, which give the values of the integral for different values of k. We shall, however, find a series from which the value of the integral, and therefore that of T, may be found with any required degree of accuracy when the value of $k = \sin\frac{\alpha}{2}$ is known with sufficient accuracy. Expanding by the binomial theorem, we have, since $k\sin\phi$ is less than 1,

$$\begin{aligned}\frac{1}{\sqrt{1 - k^2\sin^2\phi}} &= (1 - k^2\sin^2\phi)^{-\frac{1}{2}} \\ &= 1 + \tfrac{1}{2}k^2\sin^2\phi + \frac{1\,.\,3}{2^2\,\underline{|2}}k^4\sin^4\phi + \frac{1\,.\,3\,.\,5}{2^3\,\underline{|3}}k^6\sin^6\phi \\ &\quad + \frac{1\,.\,3\,.\,5\,.\,7}{2^4\,\underline{|4}}k^8\sin^8\phi + \frac{1\,.\,3\,.\,5\,.\,7\,.\,9}{2^5\,\underline{|5}}k^{10}\sin^{10}\phi + \dots\end{aligned}$$

Hence, using the reduction formula (Art. 54),

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\phi}}\,d\phi = \frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2k^4+\left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2k^6+\ldots\right]$$

and from (6),

$$T = 2\pi\sqrt{\frac{l}{g}}\left[1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2k^4+\left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2k^6+\ldots\right] \quad . \quad (7)$$

The amplitude of the swing is $\theta = \alpha$ radians. Also

$$k = \sin\frac{\alpha}{2} = \frac{\alpha}{2} - \frac{\alpha^3}{48}$$

approximately if α is sufficiently small (Art. 57). Hence, to the second order of small quantities,

$$T = 2\pi\sqrt{\frac{l}{g}}\left[1+\frac{1}{4}\frac{\alpha^2}{4}\right] = \left(1+\frac{\alpha^2}{16}\right)\times 2\pi\sqrt{\frac{l}{g}}$$

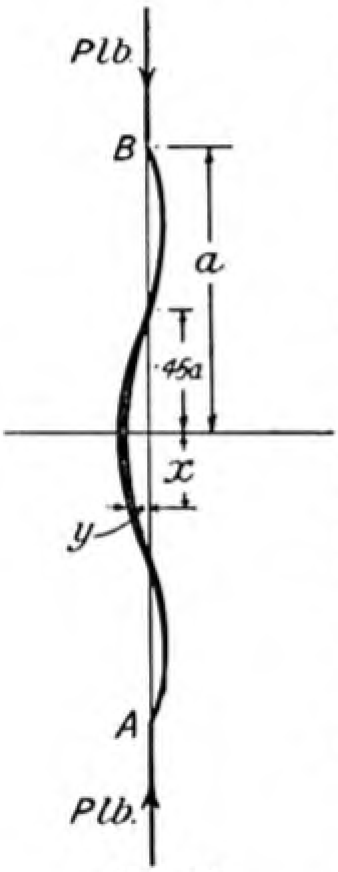


FIG. 47

EXAMPLE 6

Strut with Variable Cross-section. Assume the equation

$$EI\frac{d^2y}{dx^2} = -Py$$

for the deflection of a strut pinned at both ends and acted on by a thrust P. Suppose $2a$ is the length of the strut and let x be measured from the middle of the strut. If E is constant, and the cross-section varies so that

$$I = K\left(1-\frac{x^2}{a^2}\right)$$

show that the differential equation has a solution of the form

$$y = A\left(1+p\frac{x^2}{a^2}+q\frac{x^4}{a^4}\right)$$

provided that P has a particular value. Find this value of P and also the values of p and q. (U.L.)

If we write $m^2 = \dfrac{P}{EK}$, the given equation reduces to

$$\left(1-\frac{x^2}{a^2}\right)\frac{d^2y}{dx^2} = -m^2y \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Assume that

$$y = A\left(1+p\frac{x^2}{a^2}+q\frac{x^4}{a^4}\right) \quad . \quad . \quad . \quad (2)$$

is a solution of (1). Then, substituting in (1) we have

$$A\left(1-\frac{x^2}{a^2}\right)\left(\frac{2p}{a^2}+\frac{12qx^2}{a^4}\right) = -m^2A\left(1+\frac{px^2}{a^2}+\frac{qx^4}{a^4}\right)$$

or $$\frac{2p}{a^2} + \frac{x^2}{a^4}(12q - 2p) - \frac{12qx^4}{a^6} = -m^2 - \frac{pm^2}{a^2}x^2 - \frac{qm^2}{a^4}x^4$$

Equating coefficients of x^4, x^2, and x^0, we have respectively

$$\frac{12q}{a^6} = \frac{qm^2}{a^4} \qquad \text{or} \qquad m^2 = \frac{12}{a^2}$$

$$\frac{12q - 2p}{a^4} = -\frac{pm^2}{a^2} \qquad \text{or} \quad 12q - 2p = -12p$$

$$\frac{2p}{a^2} = -m^2 \qquad \text{or} \qquad m^2 = -\frac{2p}{a^2}$$

From these $p = -6$, $q = 5$, and $m^2 = \frac{12}{a^2}$. Thus, $y = A\left(1 - \frac{6x^2}{a^2} + \frac{5x^4}{a^4}\right)$ is a solution of (1) if $m^2 = \frac{12}{a^2}$, i.e. if $\frac{Pa^2}{EK} = 12$ or $P = \frac{12EK}{a^2}$

This value of P is only one of an infinite number of possible values of the buckling load, and it is not actually the value of the least buckling load, for as $y = A\left(1 - \frac{6x^2}{a^2} + \frac{5x^4}{a^4}\right) = A\left(1 - \frac{5x^2}{a^2}\right)\left(1 - \frac{x^2}{a^2}\right)$ we see that $y = 0$ when $x = \pm a$ and when $x = \pm \frac{\sqrt{5}}{5}a = \pm 0{\cdot}45a$ approximately. Thus the strut bends to the form shown in Fig. 47, when the thrust $P = \frac{12EK}{a^2}$ where K is the moment of inertia of the central section.

The above equations are also satisfied by $q = 0$, $a^2m^2 = 2$, $p = -1$. These values make $y = A\left(1 - \frac{x^2}{a^2}\right)$ and give the least buckling load $P = \frac{2EK}{a^2}$

70. Simultaneous Differential Equations. In problems relating to two or more quantities which are functions of a single independent variable, we sometimes have two or more differential equations involving some or all of the variables and differential coefficients of the dependent variables with reference to the independent variable. Such equations are known as ordinary *simultaneous* differential equations. We shall solve three examples only.

EXAMPLE 1

Two numerically large parties of armed men are fighting under the following conditions. A red army containing R individuals is fighting a blue army of B individuals. The armies are equally open to attack and are equally skilful in attack and defence, as well as being equally well equipped. Assuming that $R = 10\,000$ when $B = 5\,000$, find the number of men left in the red army when the blue army has been reduced to 2 500.

We shall assume that the rate at which either army is decreasing is proportional to the size of the other army. It might appear that the rate of decrease of either army would also be proportional to the number of men in it; this is not so if we assume that each shot fired is aimed at only one individual, that no shot hits

more than one individual, and that during the fight the combatants are always close enough together for each man to fire his shot as soon as he has completed loading. The above assumption gives us the two equations

$$\frac{dR}{dt} = -kB \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{dB}{dt} = -kR \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These are simultaneous equations, involving the two unknowns R and B and their differential coefficients with respect to t. Differentiating (1), we have

$$\frac{d^2R}{dt^2} = -k\frac{dB}{dt}$$

and substituting from (2),

$$\frac{d^2R}{dt^2} = k^2R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The solution of this is

$$R = Ce^{kt} + De^{-kt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where C and D are arbitrary constants. Substituting in (1) we have

$$Cke^{kt} - Dke^{-kt} = -kB$$

or

$$B = -Ce^{kt} + De^{-kt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Since $R = 10\,000$ when $t = 0$, $10\,000 = C + D$

Also $B = 5\,000$ when $t = 0$, $\therefore\ 5\,000 = -C + D$

From these $D = 7\,500$ and $C = 2\,500$, whence

$$R = 2\,500e^{kt} + 7\,500e^{-kt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$B = -2\,500e^{kt} + 7\,500e^{-kt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

When $B = 2\,500$,

$$2\,500 = -2\,500e^{kt} + 7\,500e^{-kt}$$

i.e.

$$e^{kt} - 3e^{-kt} + 1 = 0$$

or putting $x = e^{kt}$,

$$x - \frac{3}{x} + 1 = 0$$

or

$$x^2 + x - 3 = 0$$

giving the roots

$$x = \frac{-1 \pm \sqrt{13}}{2} = 1{\cdot}303$$

the lower sign being inadmissible, since e^{kt} cannot be negative. Substituting this value of e^{kt} in (6),

$$R = 2\,500 \times 1{\cdot}303 + \frac{7\,500}{1{\cdot}303}$$

$$= 9\,014$$

The red army thus destroys 2 500 men of the blue army whilst losing less than 1 000 of its own men.

This example is suitable for numerical solution. Assume the unit of time to be such that $k\Delta t = 0{\cdot}04$ and tabulate values.

R	B	$\Delta R = - Bk\Delta t$	$\Delta B = - Rk\Delta t$
10 000	5 000	− 200	− 400
9 800	4 600	− 184	− 392
9 616	4 208	− 168	− 385
9 448	3 823	− 153	− 378
9 295	3 445	− 138	− 372
9 157	3 073	− 123	− 366
9 034	2 707	− 108	− 361
8 926	2 346	—	—

By proportion when $B = 2\,500$, $R = 8\,926 + 108 \times \frac{154}{361} = 8\,972$. A closer approximation to the correct value 9 014 would be obtained by reducing the value of $k\Delta t$. We shall show how to improve the accuracy of the values of R and B in the second and third lines in the above table by using Euler's improved method, R_M and B_M are the averages of the values of R and B respectively at the beginning and end of the interval.

kt	$k\Delta t$	R	B	R_M	B_M	$\Delta R = - B_M k\Delta t$	$\Delta B = - R_M k\Delta t$
0	0·04	10 000	5 000			− 200	− 400
0·04		9 800	4 600	9 900	4 800	− 192	− 396
0·04		9 808	4 604	9 904	4 802	− 192·08	− 396·16
0·04		9 807·92	4 603·84	9 903·96	4 801·92	− 192·08	− 396·16
0·04	0·04	9 807·92	4 603·84			− 184·15	− 392·32
0·04		9 623·77	4 211·52	9 715·85	4 407·68	− 176·31	− 388·63
0·04		9 631·61	4 215·21	9 719·77	4 409·53	− 176·38	− 388·79
0·04		9 631·54	4 215·05	9 719·73	4 409·45	− 176·38	− 388·79
0·04	0·04	9 631·54	4 215·05				

The corrected pairs of values are $R = 9\,808$, $B = 4\,604$ and $R = 9\,632$ and $B = 4\,215$. By continuing the tabulation a more accurate solution of the problem may be obtained.

EXAMPLE 2

Solve the simultaneous equations

$$\frac{dx}{dt} + 3x - y = \sin 2t$$

$$\frac{dy}{dt} - x + 5y = 0$$

and adjust the constants of integration so that x and y are both zero when $t = 0$. (U.L.)

Write the equations thus—

$$(D + 3)x - y = \sin 2t \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$(D + 5)y - x = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Eliminate one of the variables x or y just as in algebra. Multiplying through by $(D + 3)$ in (2), and adding to (1),

$$(D + 5)(D + 3)y - y = \sin 2t$$

i.e.
$$(D^2 + 8D + 14)y = \sin 2t \quad . \quad . \quad . \quad . \quad (3)$$

an ordinary differential equation. Let the solution of (3) be $y = u + v$ as before. Then

$$\frac{d^2u}{dt^2} + 8\frac{du}{dt} + 14u = 0 \quad . \quad . \quad . \quad . \quad (4)$$

and
$$v = \frac{\sin 2t}{D^2 + 8D + 14} \quad . \quad . \quad . \quad . \quad (5)$$

Substituting $u = Ae^{kt}$ in (4), the auxiliary equation is

$$k^2 + 8k + 14 = 0$$

or
$$k = -4 \pm \sqrt{2}$$

from which
$$u = Ae^{(-4+\sqrt{2})t} + Be^{(-4-\sqrt{2})t}$$

From (5),
$$v = \frac{\sin 2t}{4i^2 + 16i + 14} = \frac{10 - 16i}{10^2 + 16^2}\sin 2t$$

or
$$v = \frac{1}{\sqrt{356}}\sin\left(2t - \tan^{-1}\tfrac{16}{10}\right)$$

$$= \frac{1}{\sqrt{356}}\sin(2t - 1{\cdot}012)$$

Hence, $y = Ae^{(-4+\sqrt{2})t} + Be^{(-4-\sqrt{2})t} + \dfrac{1}{\sqrt{356}}\sin(2t - 1{\cdot}012) \quad . \quad (6)$

We can find x by eliminating y from (1) and (2) or by substitution in (2). We shall adopt the latter method. Thus,

$$\frac{dy}{dt} = (-4 + \sqrt{2})\,Ae^{(-4+\sqrt{2})t} - B(4 + \sqrt{2})e^{(-4-\sqrt{2})t}$$

$$+ \frac{2}{\sqrt{356}}\cos(2t - 1{\cdot}012)$$

Hence, substituting for y and $\frac{dy}{dt}$ in (2),

$$x = \frac{dy}{dt} + 5y$$

$$= Ae^{(-4+\sqrt{2})t}(1 + \sqrt{2}) + Be^{(-4-\sqrt{2})t}(1 - \sqrt{2})$$

$$+ \frac{1}{\sqrt{356}}\{5 \sin(2t - 1{\cdot}012) + 2 \cos(2t - 1{\cdot}012)\} \qquad (7)$$

(6) and (7) give the values of x and y. Since both x and $y = 0$ when $t = 0$, the two equations give us

$$0 = A + B - \frac{1}{\sqrt{356}} \sin(1{\cdot}012)$$

and $$0 = 2{\cdot}414A - 0{\cdot}414B + \frac{1}{\sqrt{356}}(2 \cos 1{\cdot}012 - 5 \sin 1{\cdot}012)$$

Putting the values of sin 1·012 and cos 1·012 in these we have

$$A + B = 0{\cdot}0449 \qquad (8)$$

and $$2{\cdot}414A - 0{\cdot}414B = 0{\cdot}169 \qquad (9)$$

From (8) and (9) we find that $A = 0{\cdot}0663$, $B = -0{\cdot}0214$.

The solution of (1) and (2) is then given by

$$x = 0{\cdot}160e^{-2{\cdot}59t} + 0{\cdot}0089e^{-5{\cdot}41t} + 0{\cdot}265 \sin(2t - 1{\cdot}012) + 0{\cdot}106 \cos(2t - 1{\cdot}012)$$

and $$y = 0{\cdot}066e^{-2{\cdot}59t} - 0{\cdot}0214e^{-5{\cdot}41t} + 0{\cdot}053 \sin(2t - 1{\cdot}012)$$

EXAMPLE 3

Solve the simultaneous equations

$$\frac{dx}{dt} + 6\frac{dy}{dt} + x = 0, \; -3\frac{dx}{dt} + 2\frac{dy}{dt} + 2y = 0$$

being given that $x = 0$ and $y = a$ when $t = 0$. (U.L.)

Writing D for $\frac{d}{dt}$ the equations become

$$(D + 1)x + 6Dy = 0 \qquad (1)$$

$$-3Dx + 2(D + 1)y = 0 \qquad (2)$$

Multiplying through by $3D$ in (1) and by $(D + 1)$ in (2), and adding, we have

$$\{18D^2 + 2(D + 1)^2\}y = 0$$

which reduces to

$$(10D^2 + 2D + 1)y = 0 \qquad (3)$$

the auxiliary equation of which is, if $y = Ae^{kt}$,

$$10k^2 + 2k + 1 = 0$$

from which $$k = -0{\cdot}1 \pm 0{\cdot}3i$$

The solution of (3) is

$$y = Ae^{(-0\cdot1+0\cdot3i)t} + Be^{(-0\cdot1-0\cdot3i)t}$$

from which, by methods previously explained,

$$y = e^{-0\cdot1t}(C\cos 0\cdot3t + E\sin 0\cdot3t) \qquad . \quad . \quad . \quad (4)$$

where C and E are arbitrary constants. From this

$$Dy = e^{-0\cdot1t}(\overline{0\cdot3E - 0\cdot1C}\cos 0\cdot3t - \overline{0\cdot1E + 0\cdot3C}\sin 0\cdot3t)$$

and $$(D+1)y = e^{-0\cdot1t}(\overline{0\cdot3E + 0\cdot9C}\cos 0\cdot3t + \overline{0\cdot9E - 0\cdot3C}\sin 0\cdot3t)$$

Hence, substituting in (2), we obtain

$$\frac{dx}{dt} = Dx = \frac{2}{3}e^{-0\cdot1t}(L\cos 0\cdot3t + M\sin 0\cdot3t) \qquad . \quad . \quad (5)$$

where $$L = 0\cdot3E + 0\cdot9C,\ M = 0\cdot9E - 0\cdot3C$$

Integrating (5),

$$x = \frac{2}{3}\int e^{-0\cdot1t}(L\cos 0\cdot3t + M\sin 0\cdot3t)dt$$

Integrating by parts and simplifying,

$$x = \frac{20}{3}e^{-0\cdot1t}\{(0\cdot3L - 0\cdot1M)\sin 0\cdot3t - (0\cdot3M + 0\cdot1L)\cos 0\cdot3t\}$$

or $$x = \frac{20}{3}e^{-0\cdot1t}\{0\cdot3C\sin 0\cdot3t - 0\cdot3E\cos 0\cdot3t\}$$

or $$x = 2e^{-0\cdot1t}(C\sin 0\cdot3t - E\cos 0\cdot3t) \qquad . \quad . \quad . \quad . \quad . \quad (6)$$

(4) and (6) are the solutions of (1) and (2).

Putting $x = 0$, $t = 0$, in (6), we have

$$0 = -2E \text{ or } E = 0$$

Hence, the solutions are

$$x = 2Ce^{-0\cdot1t}\sin 0\cdot3t$$

and $$y = Ce^{-0\cdot1t}\cos 0\cdot3t$$

Again, putting $y = a$, $t = 0$,

$$a = C$$

Hence,

$$x = 2ae^{-0\cdot1t}\sin 0\cdot3t$$
$$y = ae^{-0\cdot1t}\cos 0\cdot3t$$

are the values x and y which satisfy the simultaneous equations and the given conditions.

71. **Vibrating Systems with more than One Degree of Freedom.** So far we have considered vibrating systems with only *one degree of freedom*, i.e. in which the state of the system at any instant of time may be fully specified in terms of a single variable which is a function of the time. Suppose we suspend a series of masses and springs so that each mass is supported by a vertical spring the upper end of

which is attached to the mass above it, the first spring having its upper end fixed. If there are n such masses, then in order to state the positions of all the masses at a given instant of time, n variables must be given, these being usually the displacements of the masses from their equilibrium positions. Because there are n variables, each depending on the time, we say that the system has *n degrees of freedom*. Where a vibrating body is continuous, as in the case of a vibrating string or wire, or a flexible shaft or beam, the displacement of a particle at any distance x ft, say, along the wire, etc. is a function of the time t sec. Since the number of such particles is infinite, the positions of the particles at any instant require in general an infinite number of variables for their definition, and we say that the system has an *infinite number of degrees of freedom*. We treat of several such systems in Chapter VI.

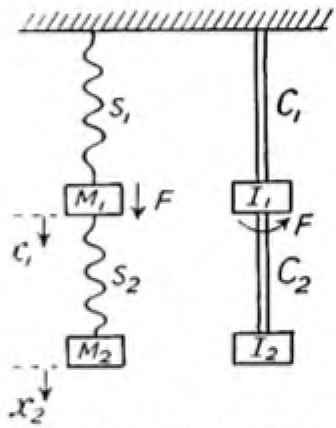


FIG. 48. TWO DEGREES OF FREEDOM

Consider the system of two masses M_1, M_2 (Fig. 48). The mass M_1 engineers' units of mass is suspended from a spring s_1 of stiffness s_1 lb per foot. A second spring of stiffness s_2 lb per foot suspended from M_1 carries at its lower end a mass M_2 engineers' units. Suppose that the system is making small vertical vibrations, and let x_1 ft and x_2 ft be the respective downward displacements of M_1 and M_2 from their positions of rest. Suppose also that a periodic disturbing force $F = F_0 \sin pt$ lb weight acts on the mass M_1, F_0 and p being constants. The elastic restoring force acting on M_1 is $s_1x_1 - s_2(x_2 - x_1)$ lb, and that on M_2 is $s_2(x_2 - x_1)$ lb. The equations of motion are, since

$$\text{Force} = \text{mass} \times \text{acceleration}$$

for mass M_1, $$F - (s_1x_1 - s_2\overline{x_2 - x_1}) = M_1\frac{d^2x_1}{dt^2}$$

and for mass M_2, $$- s_2(x_2 - x_1) = M_2\frac{d^2x_2}{dt^2}$$

Writing D for $\dfrac{d}{dt}$ and re-arranging the terms

$$(M_1D^2 + \overline{s_1 + s_2})x_1 - s_2x_2 = F \quad . \quad . \quad (\text{VII.41})$$

and $$(M_2D^2 + s_2)x_2 - s_2x_1 = 0 \quad . \quad . \quad (\text{VII.42})$$

Eliminating x_1, we have

$$[(M_1D^2 + \overline{s_1 + s_2})(M_2D^2 + s_2) - s_2^2]x_2 = Fs_2$$

i.e. $[M_1M_2D^4 + (M_2\overline{s_1 + s_2} + M_1s_2)D^2 + s_1s_2]x_2 = F_0s_2 \sin pt$ (VII.43)

Let $s_1/M_1 = n_1^2$, $s_2/M_2 = n_2^2$, $M_2 = \mu M_1$ and $F_0/s_1 = a_0$. n_1 and n_2 radians per sec are the angular frequencies of the natural vibrations of the spring-mass systems M_1, s_1 and M_2, s_2 respectively, Art. 59, and a_0 is the static deflection of the upper spring due to a steady load of F_0 lb weight. Dividing through (VII.43) by M_1M_2 and making the above substitutions, we have

$$[D^4 + D^2\{n_1^2 + n_2^2(1 + \mu)\} + n_1^2n_2^2]x_2 = n_1^2n_2^2a_0 \sin pt$$

i.e. $$[(D^2 + n_1^2)(D^2 + n_2^2) + \mu D^2n_2^2]x_2 = n_1^2n_2^2a_0 \sin pt \quad \text{(VII.44)}$$

To find the free vibrations of the system we put $F = 0$, i.e. $a_0 = 0$; then

$$[(D^2 + n_1^2)(D^2 + n_2^2) + \mu D^2n_2^2]x_2 = 0 \quad . \text{(VII.45)}$$

This may be written in the form

$$(D^2 + \omega_1^2)(D^2 + \omega_2^2)x_2 = 0 \quad . \quad . \text{(VII.46)}$$

Comparing (VII.45) and (VII.46), we have

$$\omega_1^2\omega_2^2 = n_1^2n_2^2$$

and $$\omega_1^2 + \omega_2^2 = n_1^2 + n_2^2 + \mu n_2^2$$

Hence, $$\omega_1^2 - 2\omega_1\omega_2 + \omega_2^2 = n_1^2 + n_2^2 + \mu n_2^2 - 2n_1n_2$$

i.e. $$(\omega_1 - \omega_2)^2 = (n_1 - n_2)^2 + \mu n_2^2$$

which shows that the difference between ω_1 and ω_2 is greater than that between n_1 and n_2. But, since $\omega_1\omega_2 = n_1n_2$, this means that the larger of the values ω_1 and ω_2 is greater than, and the smaller of these values is less than, both n_1 and n_2. We shall assume that $\omega_2 > \omega_1$. Then ω_2 is greater than, and ω_1 less than, both n_1 and n_2. To solve (VII.46), we have

$$x_2 = \frac{1}{(D^2 + \omega_1^2)(D^2 + \omega_2^2)} \cdot 0 \quad . \quad . \quad . \quad . \text{(VII.47)}$$

$$= \left[\frac{A}{D + i\omega_1} + \frac{B}{D - i\omega_1} + \frac{C}{D + i\omega_2} + \frac{E}{D - i\omega_2}\right] 0$$

where $i = \sqrt{-1}$. Thus,

$$x_2 = \frac{A}{D + i\omega_1} 0 \times e^{-i\omega_1 t} + \frac{B}{D - i\omega_1} 0 \times e^{i\omega_1 t}$$

$$+ \frac{C}{D + i\omega_2} 0 \times e^{-i\omega_2 t} + \frac{E}{D - i\omega_2} 0 \times e^{i\omega_2 t}$$

$$= (Ae^{-i\omega_1 t} + Be^{i\omega_1 t} + Ce^{-i\omega_2 t} + Ee^{i\omega_2 t}) \frac{1}{D} \cdot 0$$

$$= F(Ae^{-i\omega_1 t} + Be^{i\omega_1 t} + Ce^{-i\omega_2 t} + Ee^{i\omega_2 t})$$

where $A . F$, $B . F$, $C . F$, and $E . F$ are arbitrary constants. Simplifying this as in Example 7, Art. 55,

$$x_2 = A_1 \sin(\omega_1 t + \alpha_1) + A_2 \sin(\omega_2 t + \alpha_2) \qquad \text{(VII.48)}$$

where A_1, A_2, α_1 and α_2 are arbitrary constants. Substituting in (VII.42), we have

$$x_1 = A_1 \left(1 - \frac{\omega_1^2}{n_2^2}\right) \sin(\omega_1 t + \alpha_1) + A_2 \left(1 - \frac{\omega_2^2}{n_2^2}\right) \sin(\omega_2 t + \alpha_2) \qquad \text{(VII.49)}$$

These two relations (VII.48) and (VII.49) represent the natural vibrations of the system. If $A_2 = 0$, they give

$$x_1 = A_1 \left(1 - \frac{\omega_1^2}{n_2^2}\right) \sin(\omega_1 t + \alpha_1) \text{ and } x_2 = A_1 \sin(\omega_1 t + \alpha_1) \qquad \text{(VII.50)}$$

These are simple harmonic motions which are in phase, since $\omega_1 < n_2$. In this case the motion is known as the *first normal mode of motion*. If $A_1 = 0$, we have

$$x_1 = A_2 \left(1 - \frac{\omega_2^2}{n_2^2}\right) \sin(\omega_2 t + \alpha_2) \text{ and } x_2 = A_2 \sin(\omega_2 t + \alpha_2) \qquad \text{(VII.51)}$$

These are simple harmonic motions which are out of phase, since $\omega_2 > n_2$, and the masses always move in opposite senses. This is the *second normal mode of motion*. In general, neither A_1 nor A_2 will be zero, and the natural motion of each mass will be a mixture of its two modes. If ω_1 and ω_2 are nearly equal, which means that μ is small and n_1 is nearly equal to n_2, the difference in phase of the two terms in (VII.48) and (VII.49) will gradually increase so that there

will be considerable intervals of time during which they will be alternately nearly in phase and nearly opposed in phase, thus producing *beats*.

To find the forced vibrations due to the periodic force $F = F_0 \sin pt$ we have from (VII.44)

$$x_2 = \frac{a_0 n_1{}^2 n_2{}^2}{(D^2 + n_1{}^2)(D^2 + n_2{}^2) + \mu D^2 n_2{}^2} \sin pt$$

i.e.
$$x_2 = \frac{a_0 \sin pt}{\left(1 - \frac{p^2}{n_1{}^2}\right)\left(1 - \frac{p^2}{n_2{}^2}\right) - \mu \frac{p^2}{n_1{}^2}} \quad . \quad . \quad (\text{VII.52})$$

and from (VII.42)

$$x_1 = \left(1 + \frac{D^2}{n_2{}^2}\right) x_2$$

i.e.
$$x_1 = \frac{a_0 \left(1 - \frac{p^2}{n_2{}^2}\right) \sin pt}{\left(1 - \frac{p^2}{n_1{}^2}\right)\left(1 - \frac{p^2}{n_2{}^2}\right) - \frac{\mu p^2}{n_1{}^2}} \quad . \quad . \quad . \quad (\text{VII.53})$$

The ratio of the amplitude of the forced vibrations of M_1 to the static deflection a_0 produced by a force F_0 on M_1 is called the *dynamic multiplier* or *magnification factor*, i.e.

$$\text{Dynamic multiplier} = \frac{1 - \frac{p^2}{n_2{}^2}}{\left(1 - \frac{p^2}{n_1{}^2}\right)\left(1 - \frac{p^2}{n_2{}^2}\right) - \frac{\mu p^2}{n_1{}^2}} \quad . \quad (\text{VII.54})$$

The values of p which make the denominator zero, and the amplitude of M_1 infinite, are $p = \omega_1$ and $p = \omega_2$. This follows by comparison with (VII.45) and (VII.46). If we put $-p^2$ for D^2 in these and equate the left-hand sides we have

$$(\omega_1{}^2 - p^2)(\omega_2{}^2 - p^2) \equiv (n_1{}^2 - p^2)(n_2{}^2 - p^2) - \mu p^2 n_2{}^2$$

i.e.
$$\frac{(\omega_1{}^2 - p^2)(\omega_2{}^2 - p^2)}{n_1{}^2 n_2{}^2} \equiv \left(1 - \frac{p^2}{n_1{}^2}\right)\left(1 - \frac{p^2}{n_2{}^2}\right) - \mu \frac{p^2}{n_1{}^2}$$

ω_1 and ω_2 are the critical angular frequencies of the system. If the frequency of the disturbing force coincides with that of a normal

mode the amplitude of vibration of M_1 will be theoretically infinite or, practically, very large. Now consider the case in which the forcing frequency coincides with that of the lower system M_2, s_2, which is $30n_2/\pi$ per minute. Putting n_2 for p in (VII.53) we find that $x_1 = 0$, and from (VII.52)

$$x_2 = -\frac{a_0 n_1^2}{\mu n_2^2} \sin n_2 t$$

These results show that M_1 remains at rest whilst M_2 executes simple harmonic motion of amplitude

$$\frac{a_0 n_1^2}{\mu n_2^2} = \frac{F_0 M_1 M_2 s_1}{s_1 s_2 M_1 M_2} = \frac{F_0}{s_2}$$

The maximum tension in the lower spring is $s_2 \times$ amplitude $= F_0$. Thus we see that, if the frequency of the lower system is equal to that of the disturbing force, the upper mass remains stationary whilst the lower mass vibrates with such an amplitude that the tension in the lower spring balances exactly the disturbing force F. This is the principle of the *tuned vibration damper* which is an auxiliary vibration system fitted to an existing system subject to forced vibrations. The former, being tuned to the forcing frequency, will, if properly designed, neutralize the effects of the periodic force on the existing system.

72. **Torsional Vibrations.** The system shown on the right in Fig. 48 has two rotors or discs I_1 and I_2 of moments of inertia I_1 and I_2 engineers' units respectively about their axes. These are mounted on shafts C_1 and C_2 of stiffness C_1 and C_2 lb-ft respectively. The upper end of C_1 is fixed, the lower end of C_2 carries I_2 and both shafts are attached to the upper disc. The upper disc is acted upon by a couple $F = F_0 \sin pt$ lb-ft about its axis. For a rotating body the acting couple is the product of its moment of inertia and its angular acceleration. If the system is vibrating and θ_1 and θ_2 are the angular displacements of the discs at time t sec the equations of motion are

$$F - (C_1\theta_1 - C_2\,\overline{\theta_2 - \theta_1}) = I_1 \frac{d^2\theta_1}{dt^2} \qquad \text{(VII.55)}$$

and

$$-C_2(\theta_2 - \theta_1) = I_2 \frac{d^2\theta_1}{dt^2} \qquad \text{(VII.56)}$$

Allowing for the difference in notation these equations are identical with the equations of motion obtained for masses M_1 and M_2 in the last article, and the reader is left to complete the solution.

Alternative to the above we shall show another method of finding the frequencies of the normal modes of vibration and the amplitudes of the forced vibration. Assuming that the discs vibrate with simple harmonic motion of angular frequency ω then, if θ_1 and θ_2 are the amplitudes of vibration, the angular accelerations of the discs are $-\omega^2\theta_1$ and $-\omega^2\theta_2$. At the maximum displacement the restoring couples in the shafts are $C_1\theta_1$ and $C_2(\theta_2-\theta_1)$. Equating the couple to the mass-acceleration couple we have

for disc I_2 $$C_2(\theta_2-\theta_1)=\omega^2 I_2\theta_2$$

and for discs I_1 and I_2 $$C_1\theta_1=\omega^2(I_1\theta_1+I_2\theta_2)$$

Re-arranging these, $$\left(\frac{\omega^2 I_2}{C_2}-1\right)\theta_2+\theta_1=0 \qquad \text{(VII.57)}$$

and $$\frac{\omega^2 I_2}{C_1}\theta_2+\left(\frac{\omega^2 I_1}{C_1}-1\right)\theta_1=0 \qquad \text{(VII.58)}$$

Eliminating θ_1 and θ_2 from these we have

$$\begin{vmatrix} \dfrac{\omega^2 I_2}{C_2}-1 & 1 \\ \dfrac{\omega^2 I_2}{C_1} & \dfrac{\omega^2 I_1}{C_1}-1 \end{vmatrix}=0 \qquad \text{(VII.59)}$$

This is a quadratic in ω^2 whose roots ω_1^2, ω_2^2 are the squares of the angular frequencies of the normal modes of vibration. For the forced vibrations the couple $F=F_0\sin pt$ adds the term $\dfrac{F_0}{C_1}$ to the right-hand side of (VII.58). The equations are now

$$\left(\frac{p^2 I_2}{C_2}-1\right)\theta_2+\theta_1=0 \qquad \text{(VII.60)}$$

and $$\frac{p^2 I_2}{C_1}\theta_2+\left(\frac{p^2 I_1}{C_1}-1\right)\theta_1=\frac{F_0}{C_1} \qquad \text{(VII.61)}$$

p takes the place of ω because the period of forced vibrations is that of the forcing couple. From these two equations the amplitudes θ_1, θ_2 of the forced vibrations are found.

73. **Coupled Electrical Circuits with Inductance and Resistance and Negligible Capacity.** Fig 49 shows two coupled electrical circuits with inductance and resistance in series. The inductances are L_1 and L_2 and the mutual inductance is M, all in henrys. The

resistances are R_1 ohms and R_2 ohms. At time $t = 0$ there is no current in either circuit, and the switch S is then closed impressing a voltage E volts on the left-hand circuit. If E remains constant, the currents i_1 amp and i_2 amp in the circuit at time t sec can be found as follows.

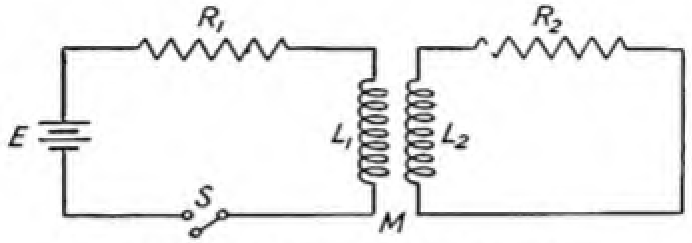


FIG. 49. TRANSIENT CURRENT

Since the applied voltage must be equal to the sum of the back e.m.f. due to the inductances and the voltage drop across the resistance, we have

for circuit (1), $$E = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} + R_1 i_1$$

and for circuit (2), $$0 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} + R_2 i_2$$

Putting D for $\frac{d}{dt}$ and re-arranging

$$(L_1 D + R_1) i_1 + M D i_2 = E \qquad \text{(VII.62)}$$

and $$M D i_1 + (L_2 D + R_2) i_2 = 0 \qquad \text{(VII.63)}$$

Eliminating i_2

$$[(L_1 D + R_1)(L_2 D + R_2) - M^2 D^2] i_1 = (L_2 D + R_2) E = R_2 E$$

i.e. $$[(L_1 L_2 - M^2) D^2 + (L_1 R_2 + L_2 R_1) D + R_1 R_2] i_1 = R_2 E \qquad \text{(VII.64)}$$

$L_1 L_2$ is usually greater than M^2, and $(L_1 R_2 + L_2 R_1)^2$ is easily shown to be greater than $4R_1 R_2 (L_1 L_2 - M^2)$. Thus the expression in square brackets in (VII.64) can be factorized, and (VII.64) may be written in the form

$$(L_1 L_2 - M^2)(D + \alpha)(D + \beta) i_1 = R_2 E$$

where α and β are real positive numbers.

From this $$i_1 = \frac{1}{(D+\alpha)(D+\beta)} \cdot \frac{R_2E}{L_1L_2 - M^2} + \frac{1}{(D+\alpha)(D+\beta)} \cdot 0$$

$$= \frac{R_2E}{\alpha\beta(L_1L_2 - M^2)} + Ae^{-\alpha t} + Be^{-\beta t}$$

Now $$\alpha\beta = \frac{R_1R_2}{L_1L_2 - M^2}$$

so that $$i_1 = \frac{E}{R_1} + Ae^{-\alpha t} + Be^{-\beta t} \quad . \quad . \quad . \text{(VII.65)}$$

From (VII.63) $$i_2 = -\frac{MD}{L_2D + R_2}\, i_1$$

$$= -\frac{MD}{L_2D + R_2} \cdot \frac{E}{R_1} - \frac{MD}{L_2D + R_2}(Ae^{-\alpha t} + Be^{-\beta t})$$

$$= -\frac{MD}{R_2}\left(1 + \frac{L_2}{R_2}D\right)^{-1} \frac{E}{R_1} + \frac{M\alpha A}{R_2 - L_2\alpha} e^{-\alpha t} + \frac{M\beta B}{R_2 - L_2\beta} e^{-\beta t}$$

Since E is constant, the first term on the right is zero

so that $$i_2 = \frac{M\alpha A}{R_2 - L_2\alpha} e^{-\alpha t} + \frac{M\beta B}{R_2 - L_2\beta} e^{-\beta t} \quad . \text{(VII.66)}$$

To determine the values of A and B, we have $i_1 = i_2 = 0$ when $t = 0$. Substituting in (VII.65) and (VII.66),

$$A + B = -\frac{E}{R_1}$$

and $$\frac{M\alpha A}{R_2 - L_2\alpha} + \frac{M\beta B}{R_2 - L_2\beta} = 0$$

Solving these equations simultaneously, we obtain

$$A = \frac{E\beta(R_2 - L_2\alpha)}{R_1R_2(\alpha - \beta)} \text{ and } B = -\frac{E\alpha(R_2 - L_2\beta)}{R_1R_2(\alpha - \beta)}$$

(VII.65) and (VII.66) then become

$$i_1 = \frac{E}{R_1} + \frac{E}{R_1R_2(\alpha - \beta)}[\beta(R_2 - L_2\alpha)e^{-\alpha t} - \alpha(R_2 - L_2\beta)e^{-\beta t}] \quad \text{(VII.67)}$$

and

$$i_2 = \frac{M\alpha\beta E}{R_1R_2(\alpha - \beta)}(e^{-\alpha t} - e^{-\beta t}) \quad . \quad . \quad \text{(VII.68)}$$

(VII.67) and (VII.68) give i_1 and i_2 in terms of t, the circuit constants, and the roots α and β of the quadratic equation

$$(L_1L_2 - M^2)x^2 - (L_1R_2 + L_2R_1)x + R_1R_2 = 0 \quad \text{(VII.69)}$$

Consider the case in which $E = 1$, $L_1 = L_2 = 2$, $M = 1$, and $R_1 = R_2 = 3$. Substituting in (VII.69), we have

$$3x^2 - 12x + 9 = 0$$

i.e.

$$x^2 - 4x + 3 = 0$$

from which $\alpha = 3$ and $\beta = 1$.

Substituting in (VII.67) and (VII.68) and simplifying

$$i_1 = \tfrac{1}{3} - \tfrac{1}{6}(e^{-t} + e^{-3t}) \quad . \quad . \quad . \quad \text{(VII.70)}$$

and

$$i_2 = -\tfrac{1}{6}(e^{-t} - e^{-3t}) \quad . \quad . \quad . \quad \text{(VII.71)}$$

We shall use the numerical method to solve the original equations in which the above values of the circuit constants are substituted, i.e. we shall obtain by the numerical method values of i_1 and i_2 which approximately satisfy the equations

$$2\frac{di_1}{dt} + \frac{di_2}{dt} + 3i_1 = 1$$

and

$$2\frac{di_2}{dt} + \frac{di_1}{dt} + 3i_2 = 0$$

It is convenient to solve these for $\frac{di_1}{dt}$ and $\frac{di_2}{dt}$. Thus we have

$$\frac{di_1}{dt} = \tfrac{2}{3} + i_2 - 2i_1 \text{ and } \frac{di_2}{dt} = i_1 - 2i_2 - \tfrac{1}{3}$$

We draw up the following table, putting in the original values $i_1 = i_2 = 0$.

TABLE IV

t	Δt	$\Delta i_1 = (0{\cdot}66667 + i_2 - 2i_1)\Delta t$	$\Delta i_2 = -(0{\cdot}33333 + 2i_2 - i_1)\Delta t$	i_1	i_2
0	—	—	—	0	0
0·05	0·05	0·03333	− 0·01667	0·03333	− 0·01667
0·10	0·05	0·02917	− 0·01333	0·06250	− 0·03000
0·15	0·05	0·02558	− 0·01054	0·08808	− 0·04054
0·20	0·05	0·02250	− 0·00821	0·11058	− 0·04875
0·30	0·10	0·03969	− 0·01253	0·15027	− 0·06128
0·40	0·10	0·03049	− 0·00605	0·18076	− 0·06733
0·50	0·10	0·02378	− 0·00179	0·20454	− 0·06912
0·60	0·10	0·01885	+ 0·00945	0·22339	− 0·05967

These values of i_1 and i_2 are only roughly correct, and we have not carried them very far. The application of Euler's improved method would give more accurate results after much laborious calculation. Fig. 50 shows the graphs of i_1 and i_2 plotted on a time

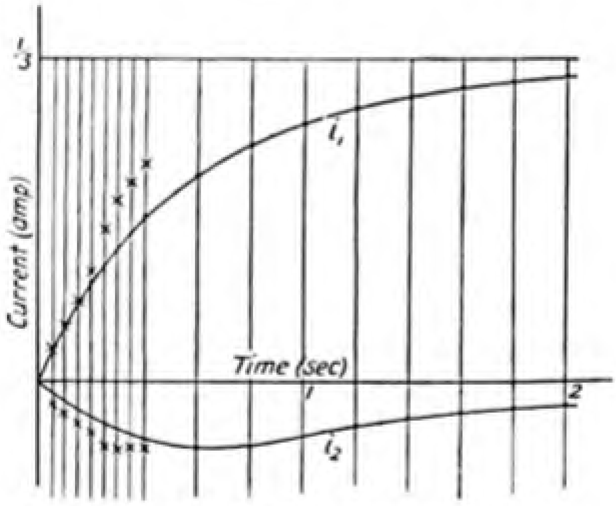


FIG. 50. TRANSIENTS

base from the equations (VII.70) and (VII.71), with crosses marking the values found by tabulation. After a short time i_1 and i_2 assume the values $\frac{1}{3}$ and 0 respectively. The decay functions in (VII.67) and (VII.68) diminish rapidly and for this reason are known as *transients*.

If the battery of voltage E is replaced by a source of alternating current of voltage $E = E_0 \sin \omega t$, the arrangement of Fig. 48

becomes a transformer in which we assume there is no capacity. The equations (VII.62) and (VII.63) now become

$$(L_1 D + R_1)i_1 + MDi_2 = E_0 \sin \omega t \quad . \quad \text{(VII.72)}$$

and

$$MDi_1 + (L_2 D + R_2)i_2 = 0 \quad . \quad . \quad . \quad \text{(VII.73)}$$

Eliminating i_2 as before,

$$[(L_1L_2 - M^2)D^2 + (L_1R_2 + L_2R_1)D + R_1R_2]i_1$$
$$= (L_2D + R_2)E_0 \sin \omega t = E_0\sqrt{R_2^2 + \omega^2L_2^2} \sin (\omega t + \phi) \quad \text{(VII.74)}$$

where

$$\phi = \tan^{-1} \frac{\omega L_2}{R_2}$$

The particular integral of this differential equation is

$$i_1 = \frac{E_0\sqrt{R_2^2 + \omega^2L_2^2}}{(L_1L_2 - M^2)D^2 + (L_1R_2 + L_2R_1)D + R_1R_2} \sin (\omega t + \phi)$$

$$= \frac{E_0\sqrt{R_2^2 + \omega^2L_2^2}}{[R_1R_2 - \omega^2(L_1L_2 - M^2)] + (L_1R_2 + L_2R_1)D} \sin (\omega t + \phi)$$

Hence

$$i_1 = E_0 \sqrt{\frac{R_2^2 + \omega^2L_2^2}{\{R_1R_2 - \omega^2(L_1L_2 - M^2)\}^2 + \omega^2(L_1R_2 + L_2R_1)^2}}$$
$$\times \sin (\omega t + \phi - \psi) \quad . \quad . \quad \text{(VII.75)}$$

where

$$\psi = \tan^{-1} \frac{\omega(L_1R_2 + L_2R_1)}{R_1R_2 - \omega^2(L_1L_2 - M^2)}$$

Eliminating i_1 from (VII.72) and (VII.73), we have

$$[(L_1L_2 - M^2)D^2 + (L_1R_2 + L_2R_1)D + R_1R_2]i_2$$
$$= - ME_0D \sin \omega t$$
$$= ME_0\omega \sin \left(\omega t - \frac{\pi}{2}\right) \quad . \quad . \quad . \quad \text{(VII.76)}$$

The particular integral of this equation is obviously obtained from the expression just derived for i_1 by replacing $\omega t + \phi$ by $\omega t - \frac{\pi}{2}$ and $\sqrt{R_2^2 + \omega^2L_2^2}$ by $M\omega$.

Hence,

$$i_2 = \frac{\omega ME_0}{\sqrt{\{R_1R_2 - \omega^2(L_1L_2 - M^2)\}^2 + \omega^2(L_1R_2 + L_2R_1)^2}}$$
$$\times \sin \left(\omega t - \frac{\pi}{2} - \psi\right) \quad . \quad . \quad . \quad \text{(VII.77)}$$

ψ having the value already given.

(VII.75) and (VII.77) give the currents in the two circuits after the transient effects caused by switching on the alternating current have died out. The transient terms are found by solving (VII.74) and (VII.76) when $E_0 = 0$, i.e. they are the complementary functions of the above differential equations, and they are given in (VII.65) and (VII.66).

74. Coupled Electrical Circuits with Inductance and Capacity and of Negligible Resistance. In Fig. 51 we show two coupled circuits

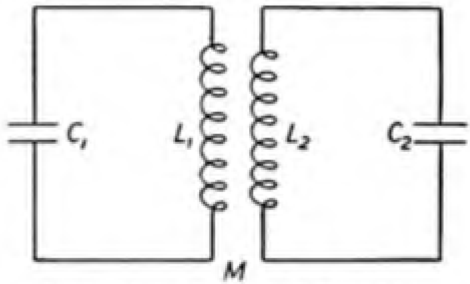


FIG. 51. COUPLED CIRCUITS

with inductances L_1 and L_2 henrys and capacities C_1 and C_2 farads respectively. M henrys is the mutual inductance. We assume the circuits to be in a state of electrical vibration. We saw in Art. 59 that, if the circuits are not coupled, the angular frequencies of free vibrations are respectively $n_1 = \dfrac{1}{\sqrt{L_1C_1}}$ and $n_2 = \dfrac{1}{\sqrt{L_2C_2}}$. When the circuits are coupled, the equations for the circuits are

$$L_1\frac{di_1}{dt} + M\frac{di_2}{dt} + \frac{1}{C_1}\int i_1\,dt = 0 \quad . \quad . \quad \text{(VII.78)}$$

$$\text{and} \quad L_2\frac{di_2}{dt} + M\frac{di_1}{dt} + \frac{1}{C_2}\int i_2\,dt = 0 \quad . \quad . \quad \text{(VII.79)}$$

Writing D for $\dfrac{d}{dt}$, $\dfrac{1}{D}$ for $\int \ldots dt$, clearing of fractions, and re-arranging, we have

$$(1 + L_1C_1D^2)i_1 + MC_1D^2i_2 = 0$$

$$\text{and} \qquad MC_2D^2i_1 + (1 + L_2C_2D^2)i_2 = 0$$

In these put $M = k_1L_1 = k_2L_2$, $L_1C_1 = \frac{1}{n_1^2}$, and $L_2C_2 = \frac{1}{n_2^2}$, and clear of fractions.

Then
$$(D^2 + n_1^2)i_1 + k_1D^2i_2 = 0 \qquad \text{(VII.80)}$$

and
$$k_2D^2i_1 + (D^2 + n_2^2)i_2 = 0 \qquad \text{(VII.81)}$$

Eliminating i_2 from these equations, we have

$$[(D^2 + n_1^2)(D^2 + n_2^2) - k_1k_2D^4]i_1 = 0$$

i.e.
$$[(1 - k_1k_2)D^4 + (n_1^2 + n_2^2)D^2 + n_1^2n_2^2]i_1 = 0 \quad \text{(VII.82)}$$

Since M^2 is less than L_1L_2, then k_1k_2 is less than unity. Now consider the expression in square brackets in (VII.82) as a quadratic expression in D^2. Since $(n_1^2 + n_2^2)^2 - 4n_1^2n_2^2(1 - k_1k_2) = (n_1^2 - n_2^2)^2 + 4k_1k_2n_1^2n_2^2$, which is essentially positive, and the coefficients of D^4, D^2 and D^0 are all positive, the expression will split into the factors $(1 - k_1k_2)(D^2 + \omega_1^2)(D^2 + \omega_2^2)$, where ω_1 and ω_2 are real and unequal. Substituting these factors for the expression in square brackets and dividing through by $1 - k_1k_2$, we have

$$(D^2 + \omega_1^2)(D^2 + \omega_2^2)i_1 = 0 \qquad \text{(VII.83)}$$

By (VII.46) and (VII.48) the solution of this equation is

$$i_1 = A_1 \sin(\omega_1t + \alpha_1) + A_2 \sin(\omega_2t + \alpha_2) \qquad \text{(VII.84)}$$

Substituting for i_1 in (VII.80), we obtain

$$i_2 = -A_1\frac{D^2 + n_1^2}{k_1D^2}\sin(\omega_1t + \alpha_1) - A_2\frac{D^2 + n_1^2}{k_1D^2}\sin(\omega_2t + \alpha_2)$$

i.e.
$$i_2 = \frac{A_1(n_1^2 - \omega_1^2)}{k_1\omega_1^2}\sin(\omega_1t + \alpha_1) + \frac{A_2(n_1^2 - \omega_2^2)}{k_1\omega_2^2}\sin(\omega_2t + \alpha_2) \qquad \text{(VII.85)}$$

These last two equations give i_1 and i_2 each as the sum of two simple harmonic terms. As in Art. 71, we see that there are two normal modes of vibration, whose angular frequencies are ω_1 and ω_2 respectively, and that, in general, the vibrations are made up of a mixture of these two modes. Now consider the expressions $D^4 + (n_1^2 + n_2^2)D^2 + n_1^2n_2^2$ and $(1 - k_1k_2)D^4 + (n_1^2 + n_2^2)D^2 + n_1^2n_2^2$, the former of which is zero when $D^2 = -n_1^2$ and

$D^2 = -n_2^2$ and the latter when $D^2 = -\omega_1^2$ and $D^2 = -\omega_2^2$. For real values of D^2 the former expression is greater than the latter by $k_1k_2D^4$, and, as the former is negative for values of D^2 between $-n_1^2$ and $-n_2^2$, so also is the latter. Further, the latter is positive when $D^2 = 0$ and when D^2 is a large negative number. It follows that the values of ω_1^2 and ω_2^2 must lie outside of the range n_1^2 to n_2^2, one on each side of this range. We see then that the frequencies of the normal modes are such that one is greater than and the other less than both the frequencies of the separate uncoupled circuits. Forced vibrations are dealt with by the method of Art. 73.

75. **Numerical Solution of Second Order Equations.** The general equation of the second order is

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \quad . \quad . \quad . \quad \text{(VII.86)}$$

With $$\frac{dy}{dx} = z, \text{ this becomes } f\left(x, y, z, \frac{dz}{dx}\right) = 0 \quad . \quad \text{(VII.87)}$$

The simultaneous equations (VII.87) are equivalent to (VII.86), and their solution is that of the second order equation.

TABLE V

Δx	x	y	z	z_m	$\frac{dz}{dx} = -(2xz+y)$	$\left(\frac{dz}{dx}\right)_m$	$\Delta y = z_m\Delta x$	$\Delta z = \left(\frac{dz}{dx}\right)_m \Delta x$
0·1	0	0·5	0·1	—	− 0·5	—	0·01	− 0·05
	0·1	0·51	0·05	0·075	− 0·52	− 0·51	0·0075	− 0·051
	0·1	0·5075	0·049	0·0745	− 0·5173	− 0·50865	0·00745	− 0·05087
	0·1	0·50745	0·04913	0·07457	− 0·5173	− 0·50865	0·00746	− 0·05087
•	0·1	0·50746	0·04913					
0·1	0·1	0·50746	0·04913	—	− 0·5173	—	0·00491	− 0·05173
	0·2	0·51237	− 0·00260	0·02327	− 0·5113	− 0·5143	0·00233	− 0·05143
	0·2	0·50979	− 0·00230	0·02342	− 0·5089	− 0·5131	0·00234	− 0·05131
	0·2	0·50980	− 0·00218	0·02348	− 0·5089	− 0·5131	0·00235	− 0·05131
•	0·2	0·50981	− 0·00218					
0·1	0·2	0·50981	− 0·00218	—	− 0·5089	—	− 0·000218	− 0·05089
	0·3	0·50959	− 0·05307	− 0·02763	− 0·4777	− 0·4933	− 0·00276	− 0·04933
	0·3	0·50705	− 0·05151	− 0·02684	− 0·4761	− 0·4925	− 0·00268	− 0·04925
•	0·3	0·50713	− 0·05143	− 0·02681				
0·1	0·3	0·50713	− 0·05143	—	− 0·4763	—	− 0·00514	− 0·04763
	0·4	0·50199	− 0·09906	− 0·07525	− 0·4227	− 0·4495	− 0·00753	− 0·04495
	0·4	0·49960	− 0·09638	− 0·07391	− 0·4225	− 0·4494	− 0·00739	− 0·04494
•	0·4	0·49974	− 0·09637	− 0·07390				
0·1	0·4	0·49974	− 0·09637	—	− 0·4225	—	− 0·00964	− 0·04225
	0·5	0·49010	− 0·13862	− 0·11750	− 0·3515	− 0·3870	− 0·01175	− 0·03870
	0·5	0·48799	− 0·13507	− 0·11572	− 0·3529	− 0·3877	− 0·01157	− 0·03877
	0·5	0·48817	− 0·13514	− 0·11576	− 0·3530	− 0·3876	− 0·01158	− 0·03876
•	0·5	0·48816	− 0·13513					

EXAMPLE

Solve $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$ for values of x from 0 to 0·5, having given that, when $x = 0$, $y = 0·5$ and $\frac{dy}{dx} = 0·1$. The simultaneous equations are $\frac{dy}{dx} = z$ and $\frac{dz}{dx} + 2xz + y = 0$. From these, $\Delta y = z\Delta x$ and $\Delta z = -(2xz + y)\Delta x$.

Using Euler's improved method, we draw up Table V.

The most accurate pairs of values of x and y are those marked by asterisks. In Ex. VII, No. 26, the reader is asked to find the value of y when $x = 0·6$. He should work this example. For a fuller treatment of this subject see *Numerical Studies in Differential Equations* by Levy and Baggot.

EXAMPLES VII

(1) Solve $\frac{dy}{dx} = x^2 + \frac{1}{2}y$ by Euler's method to obtain values of y for values of x from $x = 1$ to $x = 1·5$ in steps of 0·1, having given that $y = 2$ when $x = 1$.

(2) Use Euler's modified method in Ex. (1). Obtain a correct solution of the equation and compare the values of y when $x = 1·5$.

(3) Find values of y for $x = 1·1, 1·2, 1·3$, etc. up to 2 if $\frac{dy}{dx} = 2 + \sqrt{xy}$ and $x = 1$ when $y = 1$. Give values obtained by Euler's method and by Euler's modified method.

(4) Use Runge's method in the last Example, taking 0·2 as the interval between values of x.

(5) Use Runge's method to find y when $x = 1·2, 1·4, 1·6, 1·8$ and 2, if $\frac{dy}{dx} = x^2 + \frac{1}{2}y$ and $y = 2$ when $x = 1$, as in Ex. (1).

(6) Solve (1) by Euler's method and (2) by Euler's modified method, $\frac{dy}{dx} = \sqrt{y} + x$ from $x = 2$ to $x = 3$, having given that $y = 4$ when $x = 2$. Take steps of 0·1 in x.

(7) Solve exactly $\frac{dy}{dx} = 1 + \frac{y}{x}$, having given that $y = 2$ when $x = 1$. From the solution tabulate values of x from $x = 1$ to $x = 2$ in steps of 0·1. Obtain an approximate solution over this range by Euler's method and compare your solutions.

(8) Use Euler's modified method in the last example, increasing x by 0·5 at each step. Can we justify the use of steps of this magnitude?

(9) Using Picard's method, find an expression for y in a series of powers of x which satisfies the equation in Ex. (1), subject to the condition $y = 1$ when $x = 0$, and which will give three-figure accuracy over the range $x = 0$ to $x = 1$.

(10) Find, by Picard's method, y as a series of powers of x which will give three-figure accuracy between $x = 0$ and $x = 1$ if $\frac{dy}{dx} = 1 - xy$ and $y = 0$ when $x = 0$. Give the values of y when $x = \frac{1}{2}$ and when $x = 1$.

(11) Carry on the solution of (10) to cover the range $x = 1$ to $x = 1·5$. You will find it convenient to change the origin to the point (1, 0). Find y when $x = 1·25$ and when $x = 1·5$. Sketch the graph of y against x over the range $x = 0$ to $x = 1·5$.

(12) Using Runge's method, find y in Ex. (10) for values of x from 0 to 1 in steps of 0·2.

(13) Find a solution of $\frac{d^2y}{dx^2} = \frac{y}{x}$ in the form of an infinite power series. Do this (*a*) by assuming a series expansion as in (VII.24) and (*b*) by the use of Maclaurin's or Taylor's series.

(14) If $y^{(n)}$ is the nth differential coefficient of y with respect to x and $\frac{d^2y}{dx^2} = xy$ show that $y^{(n)} = xy^{(n-2)} + (n-2)y^{(n-3)}$. Using Maclaurin's series, obtain a series solution of the given differential equation.

(15) Write down the complete solution of $\frac{d^2y}{dx^2} - n^2y = 0$. Obtain a series solution of the equation and show that the solutions are equivalent.

(16) Repeat the working in Ex. (15) for the equation $\frac{d^2y}{dx^2} + n^2y = 0$.

(17) Find y as a series of powers of x if $\frac{d^2y}{dx^2} + (x+1)y = 0$. What is the recurrence formula if the $(n+1)$th term in the series is a_nx^n?

(18) Obtain a solution of the equation in Ex. (17) in the form of a series of powers of $x + 1$, having given that, when $x = -1$, $y = \frac{dy}{dx} = 1$.

(19) The equation $\frac{d^4y}{dx^4} = m^4y$ occurs in connection with whirling of shafts and is solved in Ex. 2, p. 401, Vol. I. Obtain a solution in the form of an infinite series of powers of x and show that the two solutions are equivalent.

(20) Solve in a power series $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$. Show that if the term in x^n is a_nx^n, the recurrence formula is $a_n = -\frac{1}{n^2}a_{n-2}$

(21) Solve in a power series $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 4y = 0$, and find the formula of recurrence.

(22) Show that the solution of $\frac{dy}{dx} = \frac{ny}{1+x}$ in a series of powers of x is

$$y = a_0\left[1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots\right]$$

Integrate the equation in finite form and show that the solutions are equivalent.

(23) Show that the series solution of $(1-x^2)\frac{dy}{dx} - xy = 1$ is

$$y = x + \frac{2}{3}x^3 + \frac{2.4}{3.5}x^5 + \frac{2.4.6}{3.5.7}x^7 + \ldots + a_0\left(1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \ldots\right)$$

(24) Using Runge's method continue the solution of $\frac{dy}{dx} + 3y = 5$ in Ex. 1, Art. 67, to cover the range $x = 0$ to $x = 1$, and compare your results with those found by direct integration of the differential equation.

(25) Repeat the working of Ex. (24) for the case of Ex. 2, Art. 67, over the range $x = 1$ to $x = 2$.

(26) Taking the last line in Table V in Art. 75 to be correct, find an approximately correct value of y when $x = 0{\cdot}6$.

(27) (i) Verify that $y = (1 - x)^{-\frac{1}{3}}$ satisfies the equation

$$9x(1 - x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$$

(ii) Find the general solution of this equation in series of ascending powers of x and identify one series with the solution in (i). (U.L.)

(28) Solve the differential equations—

$$\frac{dx}{dt} - 7x + \tfrac{1}{3}y = 0$$

$$\frac{dy}{dt} - 6x - 5y = 0$$

(29) Solve—

$$2\frac{dx}{dt} + 6x - y = 2\sin 2t$$

$$\frac{dy}{dt} - 2x + 5y = 0$$

(30) Solve—

$$\frac{d^2x}{dt^2} + 9x - 9y = 0$$

$$\frac{d^2y}{dt^2} + 18y = 0$$

(31) Solve—

$$\frac{d^2x}{dt^2} - 3x + 2y = 0$$

$$\frac{d^2y}{dt^2} - 2x + y = 0$$

(32) The rate of increase of y with respect to x is $4z$, and that of z is $3y$. If y is 1 000 and z is 500 when $x = 0$, find the values of x and y when z is 1 000.

(33) Solve the simultaneous differential equations

$$\frac{dx}{dt} + 2x + y = 0$$

$$\frac{dy}{dt} + x + 2y = 0$$

subject to the conditions that $x = 1$ and $y = 0$ when $t = 0$. (U.L.)

(34) Solve the simultaneous differential equations

$$3\frac{dx}{dt} + 3x + 2y = e^t, \quad 4x - 3\frac{dy}{dt} + 3y = 0 \qquad \text{(U.L.)}$$

(35) Solve the simultaneous differential equations

$$\frac{dx}{dt} + 3x - 2y = 1$$

$$\frac{dy}{dt} - 2x + 3y = e^t$$

given that, when $t = 0$, $x = y = 0$. (U.L.)

(36) Solve the simultaneous equations

$$\frac{dx}{dt} + 5x - 2y = t$$

$$\frac{dy}{dt} + 2x + y = 0$$

being given that $x = 0$ and $y = 0$ when $t = 0$. (U.L.)

(37) Solve the equations

(i) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 2e^x \cos 2x$

(ii) $\dfrac{dy}{dx} + y - z = \dfrac{dz}{dx} + z - y = e^x$ (U.L.)

(38) Solve the simultaneous differential equations

$$\frac{d^2x}{dt^2} + x + y = 0$$

$$4\frac{d^2y}{dt^2} - x = 0$$

subject to the conditions that, when $t = 0$, $x = 2a$, $y = -a$, $\dfrac{dx}{dt} = 2b$, $\dfrac{dy}{dt} = -b$, and show that the solution is then purely periodic. (U.L.)

(39) Solve the simultaneous differential equations

$$\frac{d^2y}{dx^2} + 15y + 3z + 30 = 0$$

$$\frac{d^2z}{dx^2} + 2y + 10z + 4 = 0 \qquad \text{(U.L.)}$$

(40) Solve the simultaneous differential equations

$$\frac{d^2y}{dt^2} + 4(3y + z) = 0$$

$$\frac{d^2z}{dt^2} + 3(6y + 11z) = 0$$

subject to the conditions that $y = 0$, $z = 27$, $\dfrac{dy}{dt} = 0 = \dfrac{dz}{dt}$ when $t = 0$. (U.L.)

(41) The co-ordinates (x, y) of a particle moving in a plane satisfy the differential equations

$$\frac{d^2x}{dt^2} + a\frac{dy}{dt} = b$$

and

$$\frac{d^2y}{dt^2} - a\frac{dx}{dt} = 0$$

where a and b are constants and t denotes time.

Given that, when $t = 0$, x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ all vanish, show that the path of the particle is a cycloid whose generating circle has radius $\frac{b}{a^2}$

(42) Solve the equations

$$6\frac{d^2x}{dt^2} = 2(y - x) = -3\frac{d^2y}{dt^2}$$

given that, when $t = 0$, $\frac{dx}{dt} = V$ and x, y, and $\frac{dy}{dt}$ all vanish.

Find the smallest value of t for which $(y - x)$ again vanishes, and the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ then. (U.L.)

(43) Solve the simultaneous differential equations

$$\frac{d^2x}{dt^2} + x = \frac{dy}{dt}$$

$$4\frac{dx}{dt} + 2x = \frac{dy}{dt} + 2y$$

If x, y are the cartesian co-ordinates of a point in a plane, and if, when $t = 0$, $x = 0$, $y = 1$, $\frac{dx}{dt} = 2$, prove that the point lies on the parabola $(5x - 2y)^2 = 4(y - 2x)$. (U.L.)

(44) The lengths of two simple pendulums are l_1 ft and l_2 ft and the weights of their bobs are w_1 lb and w_2 lb respectively. The second pendulum is attached to the bob w_1, and the coupled system then makes small oscillations in one vertical plane. Show that, if at any time t sec the inclinations of the first and the second pendulums to the vertical are θ_1 radn and θ_2 radn respectively, the equations of motions of the bobs are

$$\frac{w_1}{g}l_1\frac{d^2\theta_1}{dt^2} = -w_1\theta_1 + w_2(\theta_2 - \theta_1)$$

and

$$\frac{w_2}{g}\left(l_1\frac{d^2\theta_1}{dt^2} + l_2\frac{d^2\theta_2}{dt^2}\right) = -w_2\theta_2$$

Show further that, if $k = \frac{w_2}{w_1}$ and $D \equiv \frac{d}{dt}$, these equations can be expressed as

$$[l_1 D^2 + g(1 + k)]\theta_1 - gk\theta_2 = 0$$

and
$$l_1 D^2\theta_1 + (l_2 D^2 + g)\theta_2 = 0$$

and that the differential equation resulting from the elimination of θ_2 between these last two equations is

$$[D^4 + (1 + k)(n_1^2 + n_2^2)D^2 + (1 + k)n_1^2 n_2^2]\theta_1 = 0$$

where $\frac{2\pi}{n_1}$ and $\frac{2\pi}{n_2}$ are the respective periods of oscillation of the uncoupled pendulums.

Treating this equation as a quadratic in D^2, show that its roots are real, unequal, and negative, i.e. that the equation can be written in the form

$$(D^2 + \omega_1^2)(D^2 + \omega_2^2)\theta_1 = 0$$

and that ω_1^2 and ω_2^2 lie outside the range n_1^2 to n_2^2

Deduce the solutions

$$\theta_1 = A_1 \sin(\omega_1 t + \alpha_1) + A_2 \sin(\omega_2 t + \alpha_2)$$

and $$\theta_2 = \frac{n_2^2\omega_1^2}{n_1^2(n_2^2 - \omega_1^2)} A_1 \sin(\omega_1 t + \alpha_1) + \frac{n_2^2\omega_2^2}{n_1^2(n_2^2 - \omega_2^2)} A_2 \sin(\omega_2 t + \alpha_2)$$

CHAPTER VIII

PARTIAL DIFFERENTIAL EQUATIONS

76. **Partial Differential Equations.** A *partial differential equation* is a differential equation containing partial differential coefficients. An ordinary differential equation can be formed by eliminating arbitrary constants from a relation between two variables, such as $f(x, y) = 0$ or $y = \phi(x)$, and, in general, the order of the differential equation is equal to the number of arbitrary constants eliminated. A partial differential equation, on the other hand, can be formed by eliminating, not arbitrary constants, but arbitrary functions from a relation involving three or more variables—provided that such elimination is possible. In many engineering problems a dependent variable is connected implicitly or explicitly with two or more independent variables. If, for instance, a beam supported in a horizontal position is vibrating so that every point in it moves in a vertical line with amplitude varying from point to point, the displacement y of a point distant x from one end at time t is a function of both x and t. This relationship may be expressed in the explicit form $y = f(x, t)$ or in the implicit form $\phi(x, y, t) = 0$. x and t vary independently and they are the independent variables; y is the dependent variable. If we wish to find the shape of the beam at a particular instant of time t_0, we substitute t_0 for t in the relation and proceed as if y is a function of x alone, so that any differential coefficients are found on the assumption $t =$ constant, i.e. they are partial differential coefficients $\frac{\partial y}{\partial x}, \frac{\partial^2 y}{\partial x^2}$. Similarly, if we wish to examine the motion at any particular point $x = x_0$, we substitute this value for x in the given relation and treat y as a function of t alone. In this case the differential coefficients which occur will be the partial differential coefficients $\frac{\partial y}{\partial t}, \frac{\partial^2 y}{\partial t^2}$. Both sets of differential coefficients will enter into the analysis of the motion of the beam.

Partial differential equations arise, therefore, in cases where a dependent variable is a function of two or more independent variables. Consider the following functions—

(1) $$z = f(ax + by) \quad . \quad . \quad . \quad . \quad (\text{VIII}.1)$$

$$\frac{\partial z}{\partial x} = \frac{df(ax + by)}{d(ax + by)} \times \frac{\partial(ax + by)}{\partial x}$$

Here we have first differentiated $f(ax + by)$ with respect to $ax + by$, and then multiplied by $\dfrac{\partial(ax + by)}{\partial x}$. Since the first differentiation is of the type $\dfrac{df(s)}{ds}$, where $s = ax + by$, the differentiation is ordinary and not partial differentiation. We represent $\dfrac{df(s)}{ds}$ by $f'(s)$, i.e. $f'(ax + by)$, and hence

$$\frac{\partial z}{\partial x} = f'(ax + by) \times a$$

i.e.
$$\frac{\partial z}{\partial x} = af'(ax + by)$$

Similarly
$$\frac{\partial z}{\partial y} = bf'(ax + by)$$

Eliminating $f'(ax + by)$, we obtain the partial differential equation

$$\left.\begin{aligned} b\frac{\partial z}{\partial x} &= a\frac{\partial z}{\partial y} \\ \text{or} \quad b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(VIII.2)}$$

(2)
$$z = f(ax^2 + by^2) \quad . \quad . \quad . \quad \text{(VIII.3)}$$

Writing f' for $f'(ax^2 + by^2)$, we have $\dfrac{\partial z}{\partial x} = 2axf'$ and $\dfrac{\partial z}{\partial y} = 2\,byf'$. Eliminating f', we obtain the differential equation

$$by\frac{\partial z}{\partial x} - ax\frac{\partial z}{\partial y} = 0 \quad . \quad . \quad . \quad \text{(VIII.4)}$$

(3)
$$z = f\left(\frac{y}{x}\right) \quad . \quad . \quad . \quad . \quad \text{(VIII.5)}$$

Writing f' for $f'\left(\dfrac{y}{x}\right)$, we have $\dfrac{\partial z}{\partial x} = -\dfrac{y}{x^2}f'$ and $\dfrac{\partial z}{\partial y} = \dfrac{1}{x}f'$.

Eliminating f', we obtain the differential equation

$$x\frac{\partial z}{\partial y} = -\frac{x^2}{y}\frac{\partial z}{\partial x}$$

i.e.
$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0 \quad . \quad . \quad . \quad \text{(VIII.6)}$$

The reader should memorize the above results in the modified forms given in Table VI, and he should convince himself that the alternative forms of each solution are equivalent.

TABLE VI

First Order Partial Differential Equation	Solution
$a\dfrac{\partial z}{\partial x}+b\dfrac{\partial z}{\partial y}=0$	$z=f(bx-ay), f\left(x-\dfrac{a}{b}y\right), f(ay-bx), f\left(y-\dfrac{b}{a}x\right)$
$ay\dfrac{\partial z}{\partial x}+bx\dfrac{\partial z}{\partial y}=0$	$z=f(bx^2-ay^2), f\left(x^2-\dfrac{a}{b}y^2\right), f(ay^2-bx^2), f\left(y^2-\dfrac{b}{a}x^2\right)$
$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=0$	$z=f\left(\dfrac{y}{x}\right)$ or $f\left(\dfrac{x}{y}\right)$

We have used throughout the same symbol f to represent "function." Because we have written $f\left(\dfrac{y}{x}\right)$ and $f\left(\dfrac{x}{y}\right)$ as equivalent, we do not imply that any function of $\dfrac{y}{x}$ is the same function of $\dfrac{x}{y}$ (that is manifestly incorrect). f is used in every instance simply to denote an arbitrary function whose form is unknown.

The *order* of a partial differential equation is the same as that of the highest differential coefficient in it. The above are first order equations, and their solutions each involve an arbitrary function. Now consider the relations (4) and (5) below, each of which contains two arbitrary functions.

(4) $$z=f_1(x)+f_2(y) \qquad \text{(VIII.7)}$$

We have $\dfrac{\partial z}{\partial x}=f_1'(x)$, and on differentiating this with respect to y,

$$\frac{\partial^2 z}{\partial y\,\partial x}=0 \qquad \text{(VIII.8)}$$

which is a second order equation.

Note that differentiation of (VIII.7) first with respect to y and then with respect to x gives the same result (VIII.8).

(5) $$z=f_1(x+ct)+f_2(x-ct) \qquad \text{(VIII.9)}$$

Here $$\frac{\partial z}{\partial x} = f_1'(x + ct) + f_2'(x - ct)$$

and $$\frac{\partial z}{\partial t} = cf_1'(x + ct) - cf_2'(x - ct)$$

From these we can find expressions for $f_1'(x + ct)$ and $f_2'(x - ct)$, but we cannot yet eliminate these expressions from the above relations.

Differentiating again, we have

$$\frac{\partial^2 z}{\partial x^2} = f_1''(x + ct) + f_2''(x - ct)$$

and $$\frac{\partial^2 z}{\partial t^2} = c^2 f_1''(x + ct) + c^2 f_2''(x - ct)$$

The elimination is now simple, giving

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2} \quad . \quad . \quad . \text{ (VIII.10)}$$

In (4) and (5) we have eliminated in each case two arbitrary functions and have obtained second order partial differential equations.

77. Solution of Partial Differential Equations. The solutions of some of these equations may be found by inspection or from a knowledge of the above results. Thus the solution of

(1) $$6\frac{\partial u}{\partial x} + 11\frac{\partial u}{\partial y} = 0$$

is readily seen to be $$u = f(11x - 6y)$$

By inspection or by reference to Table VI, we find that the solution of

(2) $$6y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$$

is $$z = f(x^2 - 6y^2)$$

(3) The solution of

$$ax\frac{\partial z}{\partial x} + by\frac{\partial z}{\partial x} = 0 \quad . \quad . \quad . \text{ (VIII.11)}$$

is more difficult. Comparing it with the third line of Table VI, we might be inclined to try as a possible solution $z = f\left(\frac{ax}{by}\right)$, but, as this is a function of $\frac{x}{y}$, we see from the table that the corresponding equation is $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$. Instead we try the solution $z = f\left(\frac{x^b}{y^a}\right)$, for we know that partial differentiation will produce coefficients a and b. Thus, writing f' for the first derivative of the function with respect to $\frac{x^b}{y^a}$, we have

$\frac{\partial z}{\partial x} = \frac{bx^{b-1}}{y^a}f'$ and $\frac{\partial z}{\partial y} = -\frac{ax^b}{y^{a+1}}f'$, and eliminating f' we obtain

$$\frac{y^a}{bx^{b-1}}\frac{\partial z}{\partial x} + \frac{y^{a+1}}{ax^b}\frac{\partial z}{\partial y} = 0$$

i.e.
$$ax\frac{\partial z}{\partial x} + by\frac{\partial z}{\partial y} = 0$$

The solution is, therefore,

$$z = f\left(\frac{x^b}{y^a}\right) \quad . \quad . \quad . \quad . \text{(VIII.12)}$$

Note that $z = f\left(\frac{y^a}{x^b}\right)$ is an equivalent solution.

(4) Consider the equation

$$\frac{\partial^2 z}{\partial x\,\partial y} = 0 \quad . \quad . \quad . \quad . \text{(VIII.13)}$$

The complete solution will contain two arbitrary functions. Integrating both sides of (VIII.12) with respect to y, keeping x constant, we have

$$\frac{\partial z}{\partial x} = \phi(x) \quad . \quad . \quad . \quad . \text{(VIII.14)}$$

the arbitrary function $\phi(x)$ being the constant of integration. Now integrating both sides of (VIII.14) with respect to x, keeping y constant, we have

$$z = \int\phi(x)\,dx + f_2(y)$$

$f_2(y)$ being the constant of integration. $\int\phi(x)dx$, being the integral

of a function of x, is itself such a function $f_1(x)$, say. The solution of (VIII.13) is, therefore,

$$z = f_1(x) + f_2(y) \qquad \text{. (VIII.15)}$$

where $f_1(x)$ and $f_1(y)$ are arbitrary functions of x and y respectively.

(5) Now consider the equation

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2} \qquad \text{. (VIII.16)}$$

This is an equation of great importance in the study of vibrations. The solution can be written down by inspection and the reader should attempt this. We shall change the independent variables from t and x to r and s by means of the relations $r = x + ct$ and $s = x - ct$. Since z is now a function of r and s, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial z}{\partial s} \cdot \frac{\partial s}{\partial x} \quad \text{(see p. 153, Vol. I)}$$

But $\dfrac{\partial r}{\partial x} = \dfrac{\partial s}{\partial x} = 1$, so that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r} + \frac{\partial z}{\partial s}$$

Thus, we may look upon the operators $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial r} + \dfrac{\partial}{\partial s}$ as equivalent.

Differentiating again partially with respect to x, we have

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial r} + \frac{\partial z}{\partial s}\right)$$

$$= \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right)\left(\frac{\partial z}{\partial r} + \frac{\partial z}{\partial s}\right)$$

i.e.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial r^2} + 2\frac{\partial^2 z}{\partial r\, \partial s} + \frac{\partial^2 z}{\partial s^2} \qquad \text{. (VIII.17)}$$

Similarly

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial s} \cdot \frac{\partial s}{\partial t}$$

and, since

$$\frac{\partial r}{\partial t} = c = -\frac{\partial s}{\partial t}, \text{ then}$$

$$\frac{\partial z}{\partial t} = c\left(\frac{\partial z}{\partial r} - \frac{\partial z}{\partial s}\right)$$

In this case we may look upon the operators $\dfrac{\partial}{\partial t}$ and $c\left(\dfrac{\partial}{\partial r} - \dfrac{\partial}{\partial s}\right)$ as equivalent.

Differentiating again partially with respect to t, we have

$$\frac{\partial^2 z}{\partial t^2} = c\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right) c\left(\frac{\partial z}{\partial r} - \frac{\partial z}{\partial s}\right)$$

i.e.

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} - 2\frac{\partial^2 z}{\partial r\,\partial s} + \frac{\partial^2 z}{\partial s^2}\right) \quad . \quad . \text{(VIII.18)}$$

Substituting in (VIII.16) from (VIII.17) and (VIII.18), we have

$$c^2\left(\frac{\partial^2 z}{\partial r^2} - 2\frac{\partial^2 z}{\partial r\,\partial s} + \frac{\partial^2 z}{\partial s^2}\right) = c^2\left(\frac{\partial^2 z}{\partial r^2} + 2\frac{\partial^2 z}{\partial r\,\partial s} + \frac{\partial^2 z}{\partial s^2}\right)$$

which reduces to

$$\frac{\partial^2 z}{\partial r\,\partial s} = 0$$

the solution of which is

$$z = f_1(r) + f_2(s) \qquad \text{(see (VIII.15))}$$

and on substituting the original variables, this gives

$$z = f_1(x + ct) + f_2(x - ct) \quad . \quad . \text{(VIII.19)}$$

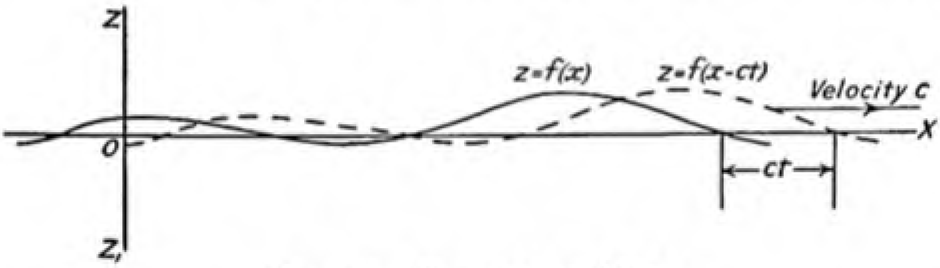


FIG. 52. TRAVELLING WAVE

Let us consider the function $z = f(x - ct)$ in which z and x are rectangular co-ordinates (Fig. 52) and t is time. If $t = 0$, $z = f(x)$. Let the full line in the figure be the graph of $z = f(x)$. At the instant t_1 seconds later, i.e. at time t_1, $z = f(x - ct_1)$. As x varies whilst t remains constant at $t = t_1$, the values of x will be greater by ct_1 than the corresponding values of $x - ct_1$. Values of $z = f(x)$ will correspond to values of $z = f(x + ct_1 - ct_1)$ for given values of x, and any ordinate of $z = f(x)$ will correspond to the ordinate of $z = f(x - ct)$ whose abscissa is greater by ct. Thus, the graph of $z = f(x - ct)$ shown dotted in the figure is an exact copy of the graph of $z = f(x)$ displaced to the right through a distance ct. Now suppose t to increase continuously from $t = 0$. The dotted curve will

emerge from identity with the full curve at $t = 0$, and, as t increases uniformly, will move to the right with uniform speed c. Thus, $z = f(x - ct)$ represents a wave travelling with velocity c along the axis of x in the sense of x increasing. In the same way we see that $z = f(x + ct)$ represents a wave of the shape of $z = f(x)$ travelling with velocity c along the axis of x in the sense of x decreasing. The solution (VIII.19) above represents the sum of two travelling waves, the first of the shape $z = f_1(x)$ travelling to the left and the second of the shape $z = f_2(x)$ travelling to the right, both with wave velocity c. These waves are known as *progressive waves* or *travelling waves.*

Under certain conditions the two waves will combine to form a single stationary wave.

If, for example

$$f_1(x + ct) = a \sin n(x + ct)$$

and $$f_2(x - ct) = a \sin n(x - ct)$$

then $$z = a[\sin n(x + ct) + \sin n(x - ct)]$$

i.e. $$z = 2a \sin nx \cos nct \qquad \text{(VIII.20)}$$

which we can look upon as a sine wave $z = A \sin nx$ whose amplitude $A = 2a \cos nct$ varies harmonically with time. Whatever the value of t, $z = 0$ whenever nx is an integral multiple of π, i.e. when $x = 0, \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}$, etc. and these points, called *nodes*, remain undisturbed during the wave motion. Thus, the two progressive waves moving in opposite senses merge into a stationary wave of varying amplitude whose frequency is $\frac{nc}{2\pi}$. This type of wave often occurs in electrical and mechanical vibrating systems. We shall see later how to solve (VIII.16) by separating the variables and thus to obtain the solution in a form similar to (VIII.20), which we shall find easier to apply to problems on vibration.

(6) The equation

$$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} = 0 \qquad \text{(VIII.21)}$$

where a, b, and c are constants, is often called a *homogeneous* equation of the second order, each term being of that order. The term "homogeneous" has already been used with two different meanings in Art. 132, Vol. I, and in Art. 65 of this book, and is also used to describe differential equations in which there are no terms

which contain only the independent variable or variables. In order to avoid confusion the reader must interpret the meaning of the word from the context. To solve (VIII.21) we assume that the method of operators developed in Chapter VI may be extended to cases where there are two or more independent variables.

If D_1 denotes $\frac{\partial}{\partial x}$, D_2 denotes $\frac{\partial}{\partial y}$, etc., (VIII.21) becomes

$$(aD_1^2 + bD_1D_2 + cD_2^2)z = 0$$

Factorizing the left-hand side as in algebra, we have

$$(n_1D_1 + m_1D_2)(n_2D_1 + m_2D_2)z = 0$$

which is satisfied by

$$(n_1D_1 + m_1D_2)z = 0, \text{ i.e. by } n_1\frac{\partial z}{\partial x} + m_1\frac{\partial z}{\partial y} = 0$$

and by $\quad (n_2D_1 + m_2D_2)z = 0, \text{ i.e. by } n_2\frac{\partial z}{\partial x} + m_2\frac{\partial z}{\partial y} = 0$

We see from Table VI that the solutions of these equations are

$$z = f_1(m_1x - n_1y) \text{ and } z = f_2(m_2x - n_2y) = 0$$

so that the general solution of (VIII.21) is

$$z = f_1(m_1x - n_1y) + f_2(m_2x - n_2y) \qquad . \text{(VIII.22)}$$

(7) Consider the equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0$$

By the use of operators as above

$$(D_1^2 - D_1D_2 - 2D_2^2)z = 0$$

i.e. $\quad (D_1 - 2D_2)(D_1 + D_2)z = 0$

This is satisfied by $\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial y} = 0$ or by $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

i.e. by $\quad z = f_1(2x + y)$ or by $z = f_2(x - y)$

and the general solution is

$$z = f_1(2x + y) + f_2(x - y)$$

(8) Consider the equation

$$a^2\frac{\partial^2 z}{\partial x^2} + 2ab\frac{\partial^2 z}{\partial x \partial y} + b^2\frac{\partial^2 z}{\partial y^2} = 0 \qquad . \text{(VIII.23)}$$

By operators $(a^2D_1^2 + 2abD_1D_2 + b^2D_2^2)z = 0$

i.e. $$(aD_1 + bD_2)^2 z = 0$$

Since both factors are identical, the only solution this gives is that of

$$(aD_1 + bD_2)z = 0$$

i.e. $$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 0$$

The solution of this equation is $z = f_1(bx - ay)$, which is seen to satisfy (VIII.23) on substitution but is not the general solution as this must contain two arbitrary functions. Let us assume that the general solution is

$$z = f_1(bx - ay) + xf_2(bx - ay) \quad . \qquad . \text{(VIII.24)}$$

Since $z = f_1(bx - ay)$ satisfies (VIII.23), we have to examine if $z = xf_2(bx - ay)$ also satisfies (VIII.23), in which case the sum of these two arbitrary functions will satisfy the given differential equation.

Putting $z = xf_2(bx - ay)$, we have

$$\frac{\partial z}{\partial x} = f_2(bx - ay) + bxf_2'(bx - ay)$$

$$\frac{\partial z}{\partial y} = -\, axf_2'(bx - ay)$$

$$\frac{\partial^2 z}{\partial x^2} = bf_2'(bx - ay) + bf_2'(bx - ay) + b^2xf_2''(bx - ay)$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\, af_2'(bx - ay) - abxf_2''(bx - ay)$$

and $$\frac{\partial^2 z}{\partial y^2} = a^2xf_2''(bx - ay)$$

This may be simplified by the omission of the bracket after each f, thus—

$$\frac{\partial z}{\partial x} = f_2 + bxf_2', \qquad \frac{\partial z}{\partial y} = -\, axf_2'$$

$$\frac{\partial^2 z}{\partial x^2} = 2bf_2' + b^2xf_2'', \qquad \frac{\partial^2 z}{\partial x \partial y} = -\, af_2' - abxf_2''$$

and $$\frac{\partial^2 z}{\partial y^2} = a^2xf_2''$$

By substitution of these values in (VIII.23)

$$\begin{aligned}\text{left-hand side} &= 2a^2bf_2' + a^2b^2xf_2'' - 2a^2bf_2' - 2a^2b^2xf_2'' + a^2b^2xf_2'' \\ &= 0\end{aligned}$$

Hence, both terms on the right-hand side of (VIII.24) satisfy (VIII.23), and (VIII.24) is the general solution.

The general solution of

$$(aD_1 + bD_2)^r z = 0 \quad . \quad . \quad . \text{(VIII.25)}$$

is

$$z = f_1(bx - ay) + xf_2(bx - ay) + x^2f_3(bx - ay) + \ldots + x^{r-1}f_r(bx - ay) \quad . \quad . \text{(VIII.26)}$$

(9) By the use of operators solve the equation

$$\frac{\partial^4 z}{\partial x^4} - 4\frac{\partial^4 z}{\partial x^3 \partial y} + 13\frac{\partial^4 z}{\partial x^2 \partial y^2} - 36\frac{\partial^4 z}{\partial x \partial y^3} + 36\frac{\partial^4 z}{\partial y^4} = 0 \quad . \text{(VIII.27)}$$

We have

$$(D_1^4 - 4D_1^3D_2 + 13D_1^2D_2^2 - 36D_1D_2^3 + 36D_2^4)z = 0$$

i.e.

$$(D_1 - 2D_2)^2(D_1^2 + 9D_2^2)z = 0$$

or

$$(D_1 - 2D_2)^2(D_1 + 3iD_2)(D_1 - 3iD_2)z = 0$$

where $i = \sqrt{-1}$

The general solution of (VIII.27) is, therefore

$$z = f_1(2x + y) + xf_2(2x + y) + f_3(3ix - y) + f_4(3ix + y) = 0 \quad \text{(VIII.28)}$$

The partial differential equation (VIII.16) may also be solved by the use of operators. Thus, if in $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$ we put D_1 for $\frac{\partial}{\partial t}$ and D_2 for $\frac{\partial}{\partial x}$, we have

$$(D_1^2 - c^2D_2^2)z = 0$$

i.e.

$$(D_1 - cD_2)(D_1 + cD_2)z = 0$$

This is satisfied if $\frac{\partial z}{\partial t} - c\frac{\partial z}{\partial x} = 0$ or $\frac{\partial z}{\partial t} + c\frac{\partial z}{\partial x} = 0$.

From Table VI we see that the solutions of these equations are respectively $z = f_1(ct + x)$ and $z = f_2(x - ct)$, the general solution being their sum as in (VIII.19).

78. **Particular Integral.** All the equations considered in the last section are of the type

$$k_0 \frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + k_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \ldots + k_n \frac{\partial^n z}{\partial y^n} = 0 \qquad \text{(VIII.29)}$$

where k_0, k_1, etc. are constant coefficients.

If in (VIII.29) the right-hand side is not zero but contains a function of one or more independent variables, we proceed as in the case of ordinary differential equations by finding a particular integral of the equation and adding it to the general solution obtained by equating the left-hand side of (VIII.29) to zero.

EXAMPLE 1

Solve the equation

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = d \quad . \quad . \quad . \quad . \quad \text{(VIII.30)}$$

where a, b, d are constants.

If we put $z = A(x + y)$, where A is a constant to be determined, (VIII.30) reduces to $A(a + b) = d$, from which $A = \dfrac{d}{a+b}$, so that $z = \dfrac{d}{a+b}(x + y)$ is the particular integral.

We have seen that the solution of

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 0 \text{ is } z = f(bx - ay)$$

Hence, the complete solution of (VIII.30) is

$$z = f(bx - ay) + \frac{d}{a+b}(x + y) \quad . \quad . \quad . \quad \text{(VIII.31)}$$

The first and second terms on the right of (VIII.31) correspond respectively to the complementary function and the particular integral in the solution of an ordinary differential equation.

EXAMPLE 2

Solve the equation

$$3\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} = 3x^2 y \quad . \quad . \quad . \quad \text{(VIII.32)}$$

Consider first

$$3\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

i.e.

$$\left(3\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) z = 0$$

the solution of which is the sum of the solutions of $3\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} = 0$ and $\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} = 0$, and is, therefore,

$$z = f_1(x + 3y) + f_2(x + y)$$

In finding the particular integral we note that the term on the right-hand side of (VIII.32) is of the third degree in x and y, and as the degree is reduced by 2 in second order differentiation, we assume that the particular integral is of the fifth degree, i.e. of the form

$$z = k_0x^5 + k_1x^4y + k_2x^3y^2 + k_3x^2y^3 + k_4xy^4 + k_5y^5$$

Substituting in (VIII.32), we have

$$3(20k_0x^3 + 12k_1x^2y + 6k_2xy^2 + 2k_3y^3) - 4(4k_1x^3 + 6k_2x^2y + 6k_3xy^2 + 4k_4y^3)$$
$$+ (2k_2x^3 + 6k_3x^2y + 12k_4xy^2 + 20k_5y^3) \equiv 3x^2y$$

i.e. $$x^3(60k_0 - 16k_1 + 2k_2) + x^2y(36k_1 - 24k_2 + 6k_3)$$
$$+ xy^2(18k_2 - 24k_3 + 12k_4) + y^3(6k_3 - 16k_4 + 20k_5) \equiv 3x^2y$$

Equating coefficients of corresponding terms on the two sides of this identity, we have

$$\left.\begin{aligned} 60k_0 - 16k_1 + 2k_2 &= 0 \\ 36k_1 - 24k_2 + 6k_3 &= 3 \\ 18k_2 - 24k_3 + 12k_4 &= 0 \\ \text{and} \quad 6k_3 - 16k_4 + 20k_5 &= 0 \end{aligned}\right\}$$

These are four equations involving six unknowns, and they can always be satisfied. Two of the constants can be given any convenient values.

If, for simplicity, we assume $k_4 = 0$ and $k_5 = 0$, we find successively $k_3 = 0$, $k_2 = 0$, $k_1 = \frac{1}{12}$, and $k_0 = \frac{1}{45}$. Thus, as a particular integral, we have $z = \frac{1}{45}x^5 + \frac{1}{12}x^4y$, which on substitution satisfies (VIII.32).

The complete solution is, therefore,

$$z = f_1(x + 3y) + f_2(x + y) + \tfrac{1}{45}x^5 + \tfrac{1}{12}x^4y \quad . \quad . \quad \text{(VIII.33)}$$

NOTE. Had we taken other values for k_4 and k_5, we should have obtained a different particular integral. This is to be expected, as any functions of $x + 3y$ and $x + y$ may be taken away from the particular integral and assumed to be included in the corresponding arbitrary functions.

79. **Operators $e^{x\alpha D}$ and $e^{x(\alpha D_1 + \beta D_2)}$.** The operator $e^{x\alpha D}$ is a short way of writing $1 + x\alpha D + \frac{1}{\lfloor 2}x^2\alpha^2D^2 + \frac{1}{\lfloor 3}x^3\alpha^3D^3 + \frac{1}{\lfloor 4}x^4\alpha^4D^4 + \ldots$ If $D = \frac{\partial}{\partial y}$, and $f(y)$ is a function of y, the effect of $e^{x\alpha D}$ operating on $f(y)$ is

$$e^{x\alpha D}f(y) = (1 + x\alpha D + \frac{1}{\lfloor 2}x^2\alpha^2D^2 + \frac{1}{\lfloor 3}x^3\alpha^3D^3$$
$$+ \frac{1}{\lfloor 4}x^4\alpha^4D^4 + \ldots)f(y)$$
$$= f(y) + x\alpha f'(y) + \frac{1}{\lfloor 2}x^2\alpha^2f''(y) + \frac{1}{\lfloor 3}x^3\alpha^3f'''(y)$$
$$+ \frac{1}{\lfloor 4}x^4\alpha^4f^{(4)}(y) + \ldots$$

where $f'(y)$, $f''(y)$, $f'''(y)$, etc. are the first, second, third, etc. derivatives of $f(y)$ with respect to y.

Thus $$e^{x\alpha D}f(y) = f(y + \alpha x) \qquad \text{(VIII.34)}$$

by Taylor's Theorem, Art. 58, Vol. I.

Similarly, if D_1 represents $\dfrac{\partial}{\partial y}$ and D_2 represents $\dfrac{\partial}{\partial z}$, then $e^{x(\alpha D_1 + \beta D_2)}$ operating on $f(y, z)$ gives

$$\begin{aligned} e^{x(\alpha D_1 + \beta D_2)}f(y, z) &= e^{x\alpha D_1}e^{x\beta D_2}f(y, z) \\ &= e^{x\alpha D_1}f(y, z + \beta x) \end{aligned}$$

i.e. $$e^{x(\alpha D_1 + \beta D_2)}f(y, z) = f(y + \alpha x, z + \beta x) \qquad \text{(VIII.35)}$$

EXAMPLE 1

Solve $6\dfrac{\partial u}{\partial x} + 11\dfrac{\partial u}{\partial y} = 0$

This equation is solved in Art. 77 (i). Here we shall use the method of operators. Writing D for $\dfrac{\partial}{\partial y}$, we have

$$\frac{\partial u}{\partial x} = -\tfrac{11}{6}Du$$

and treating D as a constant while integrating with respect to x,

$$u = Ae^{-\frac{11}{6}xD}$$

A is a function of y, so that writing $f(y)$ for A and placing it after the operator,

$$u = e^{-\frac{11}{6}xD}f(y)$$

From (VIII.34) we have $$u = f(y - \tfrac{11}{6}x)$$

which is equivalent to the solution $u = f(11x - 6y)$ already obtained.

EXAMPLE 2

Solve $\dfrac{\partial u}{\partial x} = a\dfrac{\partial u}{\partial y} + b\dfrac{\partial u}{\partial z}$

Here we write D_1 for $\dfrac{\partial}{\partial y}$ and D_2 for $\dfrac{\partial}{\partial z}$

and obtain $$\frac{\partial u}{\partial x} = (aD_1 + bD_2)u,$$

the solution of which is $u = Ae^{x(aD_1 + bD_2)}$, where $A = f(y, z)$

Thus, $$u = e^{x(aD_1 + bD_2)}f(y, z)$$

i.e. $$u = f(y + ax, z + bx)$$

The reader should verify by substitution that this result is correct.

EXAMPLE 3

Solve $ax\dfrac{\partial z}{\partial x} + by\dfrac{\partial z}{\partial y} = 0$

This equation is solved in Art. 77 (3). Here we put $x = e^{\theta}$ and $y = e^{\phi}$ as in Art. 65.

Then
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial \theta}$$

so that
$$x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \theta}$$

and similarly
$$y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \phi}$$

Substituting in the given equation, we have
$$a\frac{\partial z}{\partial \theta} = -b\frac{\partial z}{\partial \phi}$$

Dividing through by a and writing D for $\dfrac{\partial}{\partial \phi}$,
$$\frac{\partial z}{\partial \theta} = -\frac{b}{a}Dz$$

from which
$$\begin{aligned} z &= e^{-\frac{b}{a}\theta D}f(\phi) \\ &= f\left(\phi - \frac{b}{a}\theta\right) \\ &= f\left(\log_e y - \frac{b}{a}\log_e x\right) \\ &= f\left(\log_e \frac{y}{x^{\frac{b}{a}}}\right) \end{aligned}$$

i.e.
$$z = f_1\left(\frac{y^a}{x^b}\right) \text{ or } f_2\left(\frac{x^b}{y^a}\right), \text{ as before.}$$

EXAMPLE 4

Solve $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$

We have already solved this by a direct method [Art. 76 (5)] and by the use of two operators [Art. 77 (5)]. We shall now solve it by means of the operator $e^{x\alpha D}$.

Writing D for $\dfrac{\partial}{\partial x}$, we have
$$\frac{\partial^2 y}{\partial t^2} = c^2D^2y$$

and treating x and D as constants, we have on integration

$$y = Ae^{ctD} + Be^{-ctD}, \text{ where } A \text{ and } B \text{ are functions of } x.$$

Thus,
$$y = e^{ctD}f_1(x) + e^{-ctD}f_2(x)$$

i.e.
$$= yf_1(x + ct) + f_2(x - ct), \text{ as in (VIII.19).}$$

EXAMPLE 5

Solve $\dfrac{\partial^2 y}{\partial x^2} = c^2 \dfrac{\partial y}{\partial t}$, having given that $y = 0$ where x is infinite and $y = a \sin pt$ where $x = 0$.

Writing D for $\dfrac{\partial}{\partial t}$, we have $\dfrac{\partial^2 y}{\partial x^2} = c^2 Dy$, the solution of which is

$$y = e^{-cx\sqrt{D}} . A + e^{+cx\sqrt{D}} . B$$

where A and B are functions of t.

[Now $y = 0$ when $x = \infty$ at all times, so that $B = 0$ and

$$y = e^{-cx\sqrt{D}} . A$$

Substituting $x = 0$, $y = a \sin pt$, we have $A = a \sin pt$, and hence

$$y = ae^{-cx\sqrt{D}} \sin pt$$

In Chapter VI we saw that the operator D when operating on $\sin pt$ could be replaced by pi, where $i = \sqrt{-1}$, if the result of the operation were afterwards properly interpreted by the given rules.

We have then $\sqrt{D} = \sqrt{pi} = \sqrt{p} . \sqrt{i} = \sqrt{\dfrac{p}{2}}(1 + i)$, since $(1 + i)^2 = 2i$.

Substituting in $y = ae^{-cx\sqrt{D}} \sin pt$, we obtain

$$\begin{aligned} y &= ae^{-cx\sqrt{\frac{p}{2}}(1+i)} \sin pt \\ &= ae^{-cx\sqrt{\frac{p}{2}}} . e^{-cx\sqrt{\frac{p}{2}}\,i} \sin pt \\ &= ae^{-c\sqrt{\frac{p}{2}}\,x} . e^{-cx\frac{D}{\sqrt{2p}}} \sin pt \\ &= ae^{-c\sqrt{\frac{p}{2}}\,x} \sin\left\{p\left(t - \frac{cx}{\sqrt{2p}}\right)\right\} \text{ by (VIII.34)} \end{aligned}$$

$$\therefore \qquad y = ae^{-c\sqrt{\frac{p}{2}}\,x} \sin\left(pt - c\sqrt{\frac{p}{2}}\,x\right)$$

80. **Transverse Vibrations of a Uniform String.** Consider a string stretched taut between two points A and B (Fig. 53) and making small transverse vibrations in which every point of the string moves along a line perpendicular to AB. Let $\overline{AB} = l$ ft. Suppose the string to be perfectly flexible, i.e. to offer no resistance to bending, and to weigh w lb per foot run. Let T lb, the tension in the string, be so large that it is not appreciably changed by the slight alterations of length which occur during the motion. We shall neglect the effects of gravity.

Fig. 53 (a) shows the string in the position APB at time t sec, the vertical scale being greatly exaggerated for clearness. Let

(x, y) be the co-ordinates of P, as shown, and let Q be a neighbouring point of the string whose co-ordinates are $(x + \Delta x,\ y + \Delta y)$. Consider the motion of the element PQ of the string. This element is moving vertically upwards with acceleration $\frac{\partial^2 y}{\partial t^2}$. The forces acting on it are the two equal but not opposite forces T exerted by

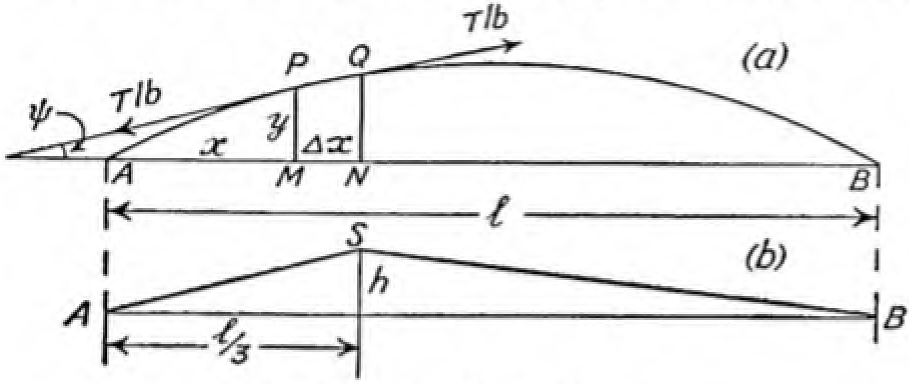


FIG. 53. VIBRATING STRING

the other parts of the string on PQ. Let ψ be the angle between the tangent at P and the straight line AB. The vertical component of the tension T at P is $T \sin \psi$ downwards, and the vertical component of the tension T at Q is $T \sin \psi + \frac{\partial}{\partial x}(T \sin \psi)\Delta x$ upwards. The net vertical component is thus $\frac{\partial}{\partial x}(T \sin \psi)\Delta x$ upwards. Hence, by the Second Law of Motion, Force = Mass $\times$ Acceleration, we have

$$\frac{w}{g}\Delta x \times \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}(T \sin \psi)\Delta x \qquad . \qquad . \text{(VIII.36)}$$

$$\therefore \qquad \frac{\partial^2 y}{\partial t^2} = \frac{gT}{w}\frac{\partial}{\partial x}(\sin \psi)$$

$$= \frac{gT}{w}\cos \psi \cdot \frac{\partial \psi}{\partial x}$$

Since ψ is small, we may neglect ψ^2, etc. in comparison with unity, and write $\cos \psi = 1$ and $\frac{\partial y}{\partial x} = \tan \psi = \psi$, whence $\frac{\partial \psi}{\partial x} = \frac{\partial^2 y}{\partial x^2}$

Thus
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}. \qquad . \qquad . \qquad . \text{(VIII.37)}$$

where
$$c^2 = \frac{gT}{w}$$

In Art. 77 we obtained the general solution of (VIII.37) in terms of two arbitrary functions representing waves travelling in opposite directions along AB with wave velocity c. In order to determine the motion of the string at any time we must know its position and motion at the beginning. Thus, suppose that we are told that the string is held displaced in the shape $y = f(x)$ up to time $t = 0$ and is then released. The general solution of (VIII.37) is

$$y = f_1(x + ct) + f_2(x - ct)$$

and we have the conditions—

(1) $y = 0$ when $x = 0$ for all values of t,

(2) $y = 0$ when $x = l$ for all values of t,

(3) $\dfrac{\partial y}{\partial t} = 0$ when $t = 0$ for all values of x,

and (4) $y = f(x)$ when $t = 0$.

The fitting of the general solution to these conditions is difficult, and we shall use an alternative method of solving (VIII.37) by separating the variables.

Let us write $y = XT$, where X is a function of x alone and T is a function of t alone. Then $\dfrac{\partial^2 y}{\partial t^2} = X\dfrac{\partial^2 T}{\partial t^2}$ and $\dfrac{\partial^2 y}{\partial x^2} = T\dfrac{\partial^2 X}{\partial x^2}$

Substituting these in (VIII.37), we have

$$X\frac{\partial^2 T}{\partial t^2} = c^2 T\frac{\partial^2 X}{\partial x^2}$$

and on dividing through by c^2XT

$$\frac{1}{c^2T}\frac{\partial^2 T}{\partial t^2} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2}$$

The left-hand side of this equation is a function of t alone and the right-hand side is a function of x alone, and, since the two sides are identically equal, i.e. equal for all values of x and t, each side must have the same constant value, say $-p^2$. Thus, instead of (VIII.37) we have the two ordinary differential equations

$$\frac{d^2T}{dt^2} + p^2c^2T = 0 \quad . \quad . \quad . \text{(VIII.38)}$$

and

$$\frac{d^2X}{dx^2} + p^2X = 0 \quad . \quad . \quad . \text{(VIII.39)}$$

The solution of (VIII.38) is $T = A \sin pct + B \cos pct$

and that of (VIII.39) is $X = C \sin px + D \cos px$

where A, B, C, and D are arbitrary constants, and p is also arbitrary. Since $y = XT$, a solution of (VIII.37) is

$$y = (A \sin pct + B \cos pct)(C \sin px + D \cos px)$$

which can be written in the form

$$y = R \sin (pct + \alpha) \sin (px + \beta) \quad . \quad . \quad \text{(VIII.40)}$$

where R, α and β are arbitrary constants. By varying p we can obtain an infinite number of solutions of (VIII.37), and the complete solution is the sum of all these. In the case of the vibrating string the conditions (1), (2), (3) and (4) must be satisfied by each term in the solution, i.e. by (VIII.40) for all values of p. Condition (1), $y = 0$ when $x = 0$ for all values of t, is satisfied if $\beta = 0$, and condition (2), $y = 0$ when $x = l$ for all values of t, is satisfied if $pl + \beta = n\pi$, i.e. $pl = n\pi$ (since $\beta = 0$), where n is any integer. Thus (VIII.40) becomes

$$y = R \sin \left(\frac{n\pi c}{l} t + \alpha\right) \sin \frac{n\pi}{l} x \quad . \quad . \quad \text{(VIII.41)}$$

in which n is any integer.

Now from (VIII.41) $\dfrac{\partial y}{\partial t} = R \dfrac{n\pi c}{l} \cos \left(\dfrac{n\pi c}{l} t + \alpha\right) \sin \dfrac{n\pi}{l} x$

By condition (3), $\dfrac{\partial y}{\partial t} = 0$ when $t = 0$ for all values of x, so that $\cos \alpha = 0$, i.e. $\alpha = \dfrac{\pi}{2}$, and (VIII.41) becomes

$$y = R \cos \frac{n\pi c}{l} t \sin \frac{n\pi}{l} x \quad . \quad . \quad \text{(VIII.42)}$$

This relation satisfies (VIII.37) and the conditions (1), (2) and (3) for all integral values of n. Adding these solutions, we have for the most general solution which satisfies the conditions

$$y = \sum_{n=1}^{n=\infty} R_n \cos \frac{n\pi c}{l} t \sin \frac{n\pi}{l} x \quad . \quad . \quad \text{(VIII.43)}$$

We have still to satisfy condition (4) that $y = f(x)$ when $t = 0$. Substituting $y = f(x)$ and $t = 0$ in (VIII.43), we have

$$f(x) = \sum_{n=1}^{n=\infty} R_n \sin \frac{n\pi}{l} x \quad . \quad . \quad . \quad \text{(VIII.44)}$$

Now, this is the expansion of $f(x)$ in a Fourier sine series, and by (III.14),

$$R_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx$$

from which, when $f(x)$ is known, the values of R_n can be found for all integral values of n.

Suppose that the string is set vibrating by the vertical displacement of the point S at $x = \frac{l}{3}$ through a small distance h ft [Fig. 53 (b)] and it is released from that stationary position at time $t = 0$. The shape of the string at the instant of release will be that shown in the figure for which $f(x) = \frac{3h}{l}x$ from $x = 0$ to $x = \frac{l}{3}$ and $f(x) = \frac{3h}{2l}(l - x)$ from $x = \frac{l}{3}$ to $x = l$.

Then $$R_n = \frac{2}{l}\left[\int_0^{\frac{l}{3}} \frac{3h}{l}x\sin\frac{n\pi}{l}x\,dx + \int_{\frac{l}{3}}^{l}\frac{3h}{2l}(l-x)\sin\frac{n\pi}{l}x\,dx\right]$$

which reduces to $$R_n = \frac{9h}{\pi^2 n^2}\sin\frac{n\pi}{3}$$

Substituting in (VIII.43) and giving n the values 1, 2, 3, etc. in order to obtain the series for y, we have

$$\begin{aligned} y = \frac{9\sqrt{3}h}{2\pi^2}\bigg[&\frac{1}{1^2}\cos\frac{\pi c}{l}t\sin\frac{\pi}{l}x + \frac{1}{2^2}\cos\frac{2\pi c}{l}t\sin\frac{2\pi}{l}x \\ &- \frac{1}{4^2}\cos\frac{4\pi c}{l}t\sin\frac{4\pi}{l}x - \frac{1}{5^2}\cos\frac{5\pi c}{l}t\sin\frac{5\pi}{l}x \\ &+ \frac{1}{7^2}\cos\frac{7\pi c}{l}t\sin\frac{7\pi}{l}x + \ldots\bigg]. \end{aligned} \qquad \text{(VIII.45)}$$

which gives the position of any point in the string at any time. Actually the vibrations would die away fairly quickly owing to the damping effects of internal friction in the string and the resistance of the air neither of which have been taken into account in the above investigation. If we assume the total damping force on the small element PQ of the string to be $\frac{2w}{g}k\frac{\partial y}{\partial t}\Delta x$, which is proportional to the product of its length and the first power of its speed, k being a

constant, this will be deducted from the force on the right-hand side of (VIII.36), and instead of (VIII.37) the equation of motion will be

$$\frac{\partial^2 y}{\partial t^2} + 2k\frac{\partial y}{\partial t} = c^2\frac{\partial^2 y}{\partial x^2} \quad . \quad . \quad . \text{(VIII.46)}$$

We shall try to separate the variables by putting $y = T \sin px$ where T is a function of t alone and p is a constant.

Then $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 T}{\partial t^2}\sin px$, $\dfrac{\partial y}{\partial t} = \dfrac{\partial T}{\partial t}\sin px$, and $\dfrac{\partial^2 y}{\partial x^2} = -Tp^2 \sin px$

Substituting these in (VIII.46) and dividing through by $\sin px$, we have

$$\frac{\partial^2 T}{\partial t^2} + 2k\frac{\partial T}{\partial t} + p^2c^2T = 0$$

and since T is a function of the single variable t, this is an ordinary differential equation

$$\frac{d^2T}{dt^2} + 2k\frac{dT}{dt} + p^2c^2T = 0$$

whose solution is from (VI.84) and (VI.88)

$$T = Re^{-kt}\cos(\sqrt{p^2c^2 - k^2}\,t + \beta)$$

where R and β are arbitrary constants.

We have separated the variables. Substituting this value of T in $y = T\sin px$, we have

$$y = Re^{-kt}\cos(\sqrt{p^2c^2 - k^2}\,t + \beta)\sin px$$

and putting $p = \dfrac{n\pi}{l}$, as before

$$y = Re^{-kt}\cos\left(\sqrt{\frac{n^2\pi^2c^2}{l^2} - k^2}\,t + \beta\right)\sin\frac{n\pi}{l}x$$

and, if $\beta = 0$

$$y = Re^{-kt}\cos\sqrt{\frac{n^2\pi^2c^2}{l^2} - k^2}\,t\sin\frac{n\pi}{l}x \quad . \text{(VIII.47)}$$

This in place of (VIII.42) satisfies conditions (1), (2) and (3), and the corresponding general solution is

$$y = \sum_{n=1}^{n=\infty} R_n e^{-kt}\cos\sqrt{\frac{n^2\pi^2c^2}{l^2} - k^2}\,t\sin\frac{n\pi}{l}x \quad . \text{(VIII.48)}$$

When $t = 0$, $e^{-kt} = 1$, so that the determination of R_n to suit condition (4) is carried out as before. The expanded form of (VIII.48) is, therefore

$$y = \frac{9\sqrt{3}he^{-kt}}{2\pi^2}\left[\frac{1}{1^2}\cos\sqrt{\frac{\pi^2c^2}{l^2} - k^2}\,t\sin\frac{\pi}{l}x\right.$$

$$+\frac{1}{2^2}\cos\sqrt{\frac{4\pi^2c^2}{l^2} - k^2}\,t\sin\frac{2\pi}{l}x$$

$$-\frac{1}{4^2}\cos\sqrt{\frac{16\pi^2c^2}{l^2} - k^2}\,t\sin\frac{4\pi}{l}x$$

$$-\frac{1}{5^2}\cos\sqrt{\frac{25\pi^2c^2}{l^2} - k^2}\,t\sin\frac{5\pi}{l}x$$

$$\left.+\frac{1}{7^2}\cos\sqrt{\frac{49\pi^2c^2}{l^2} - k^2}\,t\sin\frac{7\pi}{l}x + \ldots\right] \quad . \text{(VIII.49)}$$

Thus the effect of friction is to introduce the decay function e^{-kt} as a factor of each amplitude and to slightly reduce the frequencies of the components.

81. **Small Transverse Vibrations of a Uniform Rod.** Let APB be a thin uniform rod of length l ft and weight w lb per foot run which is vibrating about its mean position AMB, so that each point on it moves on a line perpendicular to AB (Fig. 54A). Let APB be the position of the rod at time t sec, and let the co-ordinates of the point

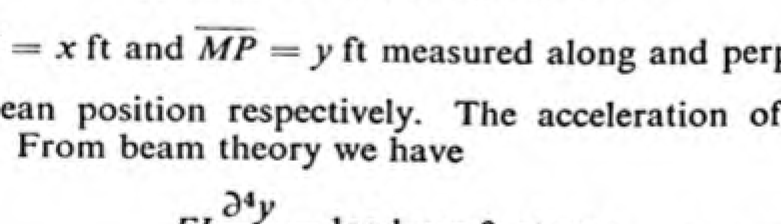


FIG. 54A. VIBRATING ROD OR BEAM

P be $\overline{AM} = x$ ft and $\overline{MP} = y$ ft measured along and perpendicular to the mean position respectively. The acceleration of P is $\dfrac{\partial^2 y}{\partial t^2}$ upwards. From beam theory we have

$$EI\frac{\partial^4 y}{\partial x^4} = \text{load per foot run} \quad . \quad . \quad . \text{(VIII.50)}$$

where E and I have their usual meanings and the units of length and force are the foot and the pound respectively. In this case gravity forces can be neglected, as we measure deflections, i.e. values of y, from the equilibrium position of the beam, and the forces acting are inertia forces. These are equal and opposite to the product of the mass of each small portion and its acceleration, and are called *reversed mass-accelerations*. The mass-acceleration of unit length of the rod at P is $\frac{w}{g}\frac{\partial^2 y}{\partial t^2}$ upwards, and the equivalent load per foot run is equal and opposite to this, i.e. is equal to $-\frac{w}{g}\frac{\partial^2 y}{\partial t^2}$, so that from (VIII.50) we have

$$EI\frac{\partial^4 y}{\partial x^4} = -\frac{w}{g}\frac{\partial^2 y}{\partial t^2} \qquad . \quad . \quad . \text{(VIII.51)}$$

or

$$\frac{\partial^4 y}{\partial x^4} = -\frac{w}{gEI}\frac{\partial^2 y}{\partial t^2} \qquad . \quad . \quad . \text{(VIII.52)}$$

We can solve (VIII.52) by substituting $y = XT$, where X and T are functions of x and t alone respectively, and separating the variables, but we shall shorten the analysis by assuming $T = \sin(pt + \alpha)$ and trying the solution $y = X\sin(pt + \alpha)$, in which p and α are arbitrary constants.

By differentiation $\quad \dfrac{\partial^4 y}{\partial x^4} = \dfrac{\partial^4 X}{\partial x^4}\sin(pt + \alpha)$

and $\quad \dfrac{\partial^2 y}{\partial t^2} = -p^2 X\sin(pt + \alpha)$

Substituting these values in (VIII.52) and dividing through by $\sin(pt + \alpha)$, we obtain $\frac{\partial^4 X}{\partial x^4} = \frac{wp^2}{gEI}X$, or writing m^4 for $\frac{wp^2}{gEI}$ and replacing the symbol ∂ by d

$$\frac{d^4 X}{dx^4} = m^4 X \qquad . \quad . \quad . \text{(VIII.53)}$$

the complete solution of which is

$$X = A\sinh mx + B\cosh mx + C\sin mx + D\cos mx$$

Thus, the complete solution of (VIII.52) is

$$y = \sin(pt + \alpha)(A\sinh mx + B\cosh mx + C\sin mx + D\cos mx) \qquad \text{(VIII.54)}$$

in which p, α, A, B, C, D are arbitrary constants whose values are fixed by the constraints at the ends of the rod and the initial disturbance which sets up the vibration. We shall consider three cases.

CASE (*a*)—ROD FREELY SUPPORTED AT THE ENDS. In this case the end conditions are

(1) $y = 0$ when $x = 0$ for all values of t

and (2) $y = 0$ when $x = l$ for all values of t

Since the bending moment $M = 0$ at the ends, and $M = EI\dfrac{d^2y}{dx^2}$ then

(3) $\dfrac{d^2y}{dx^2} = 0$ when $x = 0$ for all values of t

and (4) $\dfrac{d^2y}{dx^2} = 0$ when $x = l$ for all values of t

Conditions (1) and (3) when applied to (VIII.54) give $B + D = 0$ and $B - D = 0$, whence $B = D = 0$. Substituting these values and applying conditions (2) and (4) to (VIII.54), we have

$$A \sinh ml + C \sin ml = 0$$

and

$$A \sinh ml - C \sin ml = 0$$

from which $A \sinh ml = 0$ and $C \sin ml = 0$

Since $\sinh ml \neq 0$, then $A = 0$, and (VIII.54) reduces to

$$y = C \sin mx \sin (pt + \alpha) \qquad . \qquad . \text{(VIII.55)}$$

Since $C = 0$ would mean that the rod is at rest, we see from $C \sin ml = 0$ that ml must be an integral multiple of π, i.e. $ml = n\pi$, where n is a positive integer.

Since

$$m = \frac{n\pi}{l} \text{ and } m^4 = \frac{wp^2}{gEI}, \text{ then } p^2 = \frac{n^4\pi^4 gEI}{wl^4} \text{ and } p = \frac{n^2\pi^2}{l^2}\sqrt{\frac{gEI}{w}},$$

and (VIII.55) becomes

$$y = C \sin \frac{n\pi}{l} x \sin \left(\frac{n^2\pi^2}{l^2}\sqrt{\frac{gEI}{w}}\, t + \alpha\right) \qquad . \text{(VIII.56)}$$

For any positive integral value of n this represents a possible state of motion of the rod under the given end conditions. The constants C and α depend on how the motion is started. At any

given instant of time the shape of the rod will be that of a sine curve of $\frac{n}{2}$ wave-lengths. The motion of any particular point $x = x_1$ in the rod is given by

$$y = C' \sin\left(\frac{n^2\pi^2}{l^2}\sqrt{\frac{gEI}{w}}\,t + \alpha\right), \qquad \text{where } C' = C \sin\frac{n\pi}{l}x_1$$

First normal mode

Second normal mode

Third normal mode

FIG. 54B. NORMAL MODES OF VIBRATION

Thus, all points in the rod move with simple harmonic motion in the same period and with the same or opposite phases but with different amplitudes. This is known as a *normal mode* of motion. Fig. 54B shows the first, second and third normal modes for which $n = 1$, 2 and 3 respectively.

Since (VIII.56) satisfies the equation (VIII.52) for all integral values of n, the sum of these solutions

$$y = \sum_{n=1}^{n=\infty} C_n \sin\frac{n\pi}{l}x \sin\left(\frac{n^2\pi^2}{l^2}\sqrt{\frac{gEI}{w}}\,t + \alpha_n\right) \quad \text{(VIII.57)}$$

will also satisfy it. This is the complete solution of (VIII.52), and by suitable choice of the arbitrary constants C_n and α_n it can be made to represent any possible state of vibration of the rod under the given end conditions.

$p = \frac{n^2\pi^2}{l^2}\sqrt{\frac{gEI}{w}}$ is the *angular frequency* in the nth mode, and the time of a complete vibration in that mode is $\frac{2\pi}{p}$.

CASE (*b*)—ROD BUILT-IN AT ONE END AND FREE AT THE OTHER. In this case the end A is fixed horizontally and there is no support or constraint at the end B. The end conditions are, for all values of t

(1) $y = 0$ when $x = 0$,

(2) $\dfrac{\partial y}{\partial x} = 0$ when $x = 0$,

(3) $\dfrac{\partial^2 y}{\partial x^2} = 0$ when $x = l$, because bending moment $M = 0$ when $x = l$,

and (4) $\dfrac{\partial^3 y}{\partial x^3} = 0$ when $x = l$, because shearing force

$$S = -EI\frac{\partial^3 y}{\partial x^3} = 0 \text{ when } x = l.$$

Applying these conditions to (VIII.54) in turn, we have

$$B + D = 0 \qquad \text{(i)}$$

$$A + C = 0 \qquad \text{(ii)}$$

$$A \sinh ml + B \cosh ml - C \sin ml - D \cos ml = 0 \qquad \text{(iii)}$$

and

$$A \cosh ml + B \sinh ml - C \cos ml + D \sin ml = 0 \qquad \text{(iv)}$$

From the two equations (i) and (ii) we have $D = -B$ and $C = -A$ and by substitution in (iii) and (iv)

$$A(\sinh ml + \sin ml) + B(\cosh ml + \cos ml) = 0$$

and

$$A(\cosh ml + \cos ml) + B(\sinh ml - \sin ml) = 0$$

Eliminating A and B, we obtain

$$(\sinh ml + \sin ml)(\sinh ml - \sin ml) = (\cosh ml + \cos ml)^2$$

i.e. $\sinh^2 ml - \sin^2 ml = \cosh^2 ml + 2 \cosh ml \cos ml + \cos^2 ml$

whence

$$2 \cosh ml \cos ml = -2$$

or

$$\cos ml = -\operatorname{sech} ml \qquad \text{(VIII.58)}$$

Fig. 55 shows the graphs of $y = \cos x$ and $y = -\operatorname{sech} x$. Since $\operatorname{sech} x$ is numerically very small for values of $x > \pi$ (e.g. $\operatorname{sech} 6 < 0{\cdot}005$), then, except for the first root which is approximately

$x = 1{\cdot}88$, the successive roots are very approximately, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, $\frac{7\pi}{2}$, etc. These are values of ml for which the equation of motion is satisfied by (VIII.54), and substituting $C = -A$, $D = -B$, and $p = m^2\sqrt{\frac{gEI}{w}}$ in (VIII.54), we have

$$y = \sin\left(m^2\sqrt{\frac{gEI}{w}}\,t + \alpha\right)\ \{A(\sinh mx - \sin mx) + B(\cosh mx - \cos mx)\} \tag{VIII.59}$$

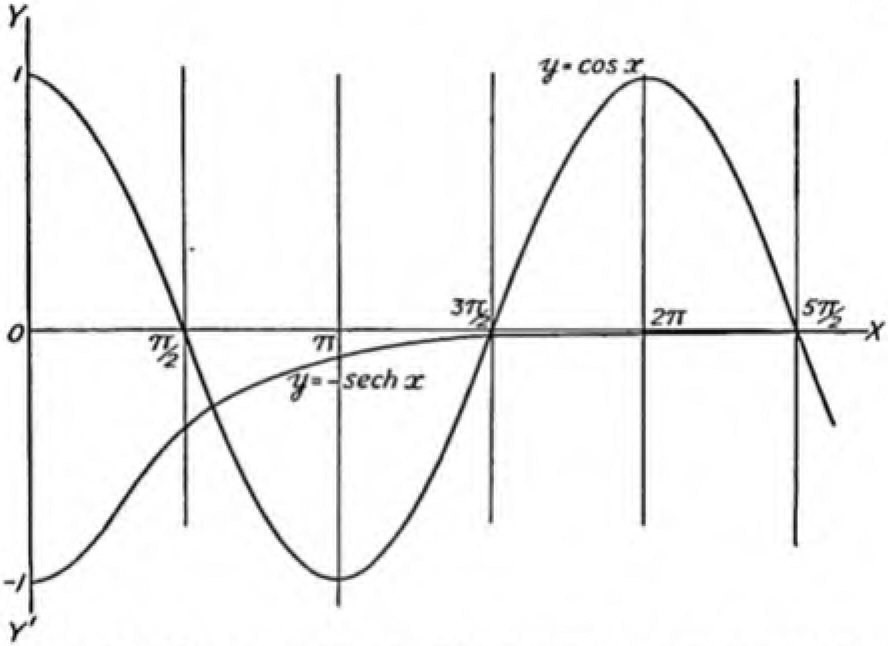


FIG. 55. SOLUTION OF THE EQUATION $\cos x = -\operatorname{sech} x$

Now from the equation (iii)

$$A(\sinh ml + \sin ml) = -B(\cosh ml + \cos ml) = R_n, \text{ say}$$

and substituting for A and B in (VIII.59), we obtain

$$y = R_n \sin\left(m^2\sqrt{\frac{gEI}{w}}\,t + \alpha\right)\ \left[\frac{\sinh mx - \sin mx}{\sinh ml + \sin ml} - \frac{\cosh mx - \cos mx}{\cosh ml + \cos ml}\right] \tag{VIII.60}$$

as a solution of (VIII.52) which also satisfies the end conditions.

For the first normal mode $m = \frac{1{\cdot}88}{l}$, and for the second, third, fourth, etc. normal modes the values of ml are very approximately $\frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$, etc., respectively.

CASE (*c*). A ROD AS IN CASE (*a*) BUT SUBJECTED TO A PULSATING LOAD OF MAGNITUDE $w_0 \sin \frac{\pi x}{l} \sin pt$ PER FOOT RUN. This will add a term of the given amount to the right-hand side of (VIII.51), and the equation of motion is

$$\frac{\partial^4 y}{\partial x^4} + \frac{w}{gEI}\frac{\partial^2 y}{\partial t^2} = \frac{w_0}{EI}\sin\frac{\pi x}{l}\sin pt. \qquad \text{(VIII.61)}$$

To find a particular integral of (VIII.61) we use operators, putting D_1^4 for $\frac{\partial^4}{\partial x^4}$ and D_2^2 for $\frac{\partial^2}{\partial t^2}$

Then,
$$y = \frac{w_0}{EI}\cdot\frac{1}{D_1^4 + \frac{w}{gEI}D_2^2}\sin\frac{\pi x}{l}\sin pt$$

$$= \frac{w_0}{EI}\cdot\sin\frac{\pi x}{l}\cdot\frac{1}{\frac{\pi^4}{l^4} + \frac{w}{gEI}D_2^2}\sin pt$$

$$= \frac{w_0}{EI\left(\frac{\pi^4}{l^4} - \frac{wp^2}{gEI}\right)}\sin\frac{\pi x}{l}\sin pt$$

i.e.
$$y = \frac{w_0 g l^4}{\pi^4 EIg - wp^2 l^4}\sin\frac{\pi x}{l}\sin pt \qquad \text{(VIII.62)}$$

is the particular integral.

This represents the forced vibrations, the amplitude of which becomes very large when $p = \frac{\pi^2}{l^2}\sqrt{\frac{EIg}{w}}$, that is, when the angular frequency of the pulsating load is equal to that of the first normal mode of vibration. The complementary function, i.e. the solution of (VIII.61) when the right-hand side is zero, has been found above.

The complete solution is, therefore

$$y = \sum_{n=1}^{n=\infty} C_n \sin\frac{n\pi x}{l} \sin\left(\frac{n^2\pi^2}{l^2}\sqrt{\frac{gEI}{w}}\,t + \alpha_n\right) + \frac{w_0 g l^4}{\pi^4 gEI - wp^2 l^4} \sin\frac{\pi x}{l} \sin pt \tag{VIII.63}$$

82. **Vibrations of a Shaft. Torsional Vibrations.** Let AB (Fig. 56 (*a*)) be a uniform shaft of length l ft which is in a state of torsional

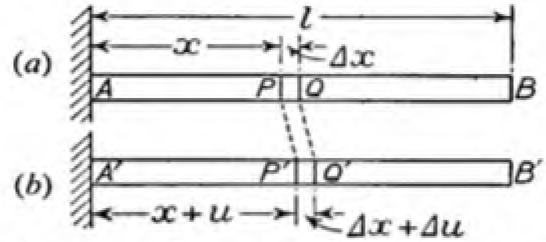


FIG. 56. TORSIONAL VIBRATIONS AND AXIAL VIBRATIONS

vibration in which all its particles move in planes perpendicular to the axis of the shaft at fixed distances from the axis. Let the dimensions and the elastic properties of the shaft be such that to twist one end of the shaft through an angle θ_0 radians relative to the other end requires the application at the ends of equal and opposite couples of magnitude $C_0\theta_0$ lb-ft, C_0 being the couple to produce unit twist. Let I_s engineers' units be the moment of inertia of the mass of the shaft about its axis. Consider the slice PQ between sections distant x and $x + \Delta x$ ft from A. Let θ be the angle of twist at P at time t sec. At the same instant the angle of twist at Q is $\theta + \Delta\theta$, i.e. $\theta + \dfrac{\partial\theta}{\partial x}\Delta x$, and the section at Q has twisted through an angle $\dfrac{\partial\theta}{\partial x}\Delta x$ relative to that at P. To produce this twist the couple C_P at P is

$$C_P = \frac{l}{\Delta x} C_0 \frac{\partial\theta}{\partial x}\Delta x = lC_0\frac{\partial\theta}{\partial x}$$

and the couple C_Q at Q is

$$C_Q = lC_0\frac{\partial\theta}{\partial x} + \frac{\partial}{\partial x}\left(lC_0\frac{\partial\theta}{\partial x}\right)\Delta x = lC_0\left(\frac{\partial\theta}{\partial x} + \frac{\partial^2\theta}{\partial x^2}\Delta x\right)$$

Now consider the motion of the element PQ of the shaft. The moment of inertia of its mass is $\frac{\Delta x}{l} I_s$ engineers' units. The resultant couple acting on it in the sense of θ increasing is

$$C_Q - C_P = lC_0 \frac{\partial^2 \theta}{\partial x^2} \Delta x$$

From dynamics we have

Couple = moment of inertia × angular acceleration

i.e.
$$lC_0 \frac{\partial^2 \theta}{\partial x^2} \Delta x = \frac{\Delta x}{l} I_s \frac{\partial^2 \theta}{\partial t^2}$$

Hence
$$\frac{\partial^2 \theta}{\partial t^2} = c^2 \frac{\partial^2 \theta}{\partial x^2} \quad . \quad . \quad . \text{(VIII.64)}$$

where
$$c^2 = \frac{C_0 l^2}{I_s} \qquad \text{or} \qquad c = l\sqrt{\frac{C_0}{I_s}}$$

This equation is identical with (VIII.37), and its solution is given in (VIII.40)

i.e.
$$\theta = R \sin(pct + \alpha) \sin(px + \beta) \quad . \quad . \text{(VIII.65)}$$

If we assume the shaft to be fixed at A as in the figure and set in motion by holding B displaced through an angle ϕ and releasing it at time $t = 0$, the conditions are

(1) $\theta = 0$ when $x = 0$ for all values of t,

(2) $\frac{\partial \theta}{\partial x} = 0$ when $x = l$ for all values of t,

(3) $\theta = \frac{x}{l}\phi$ when $t = 0$,

and (4) $\frac{\partial \theta}{\partial t} = 0$ when $t = 0$ for all values of x.

Substituting condition (1) in (VIII.65), we find $\beta = 0$. From condition (2) we have $\cos pl = 0$, whence $pl = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{(2n-1)\pi}{2}, \ldots$

Condition (4) gives $\cos \alpha = 0$, and, therefore, $\sin(pct + \alpha) = \cos pct$. Making these substitutions in (VIII.65), we have for the nth normal mode

$$\theta_n = R_n \cos \frac{(2n-1)\pi ct}{2l} \sin \frac{(2n-1)\pi x}{2l} \quad . \text{(VIII.66)}$$

and the actual motion being the sum of the normal modes is given by

$$\theta = \sum_{n=1}^{n=\infty} R_n \cos \frac{(2n-1)\pi ct}{2l} \sin \frac{(2n-1)\pi x}{2l} \quad . \text{(VIII.67)}$$

The values of R_1, R_2, R_3, . . . , R_n, . . . are determined from condition (3). Substituting condition (3)

$$\frac{\phi x}{l} = \sum_{n=1}^{n=\infty} R_n \sin \frac{(2n-1)\pi x}{2l} \quad . \quad . \text{(VIII.68)}$$

which is the expansion of $\frac{\phi x}{l}$ in a Fourier sine series. Multiplying both sides of (VIII.68) by $\sin \frac{(2n-1)\pi x}{2l}$ and integrating from $x = 0$ to $x = l$, we have

$$\frac{\phi}{l}\int_0^l x \sin \frac{(2n-1)\pi x}{2l}\, dx = \frac{l}{2} R_n$$

$$\therefore\ R_n = \frac{2\phi}{l^2} \times - \frac{2l}{(2n-1)\pi}\int_0^l x d\left\{\cos \frac{(2n-1)\pi x}{2l}\right\}$$

$$= -\frac{4\phi}{(2n-1)l\pi}\left[x \cos \frac{(2n-1)\pi x}{2l} - \int \cos \frac{(2n-1)\pi x}{2l}\, dx\right]_0^l$$

$$= -\frac{4\phi}{(2n-1)l\pi}\left[-\frac{2l}{(2n-1)\pi} \sin \frac{(2n-1)\pi x}{2l}\right]_0^l$$

$$= \frac{8\phi}{(2n-1)^2\pi^2}\left[\sin \frac{(2n-1)\pi}{2}\right]$$

$$= (-1)^{n+1} \frac{8\phi}{(2n-1)^2\pi^2}$$

Substituting this value of R_n in (VIII.67) we obtain the value of θ for the general motion, namely

$$\theta = \frac{8\phi}{\pi^2}\sum_{n=1}^{n=\infty}\frac{(-1)^{n+1}}{(2n-1)^2} \cos \frac{(2n-1)\pi ct}{2l} \sin \frac{(2n-1)\pi x}{2l} \quad . \text{(VIII.69)}$$

If the shaft carries a wheel of moment of inertia I at its free end conditions (1), (3) and (4) remain unchanged. Instead of condition (2) we have the condition that the couple in the shaft where $x = l$ must be that necessary to accelerate the wheel. We shall consider

only the normal modes of motion and shall therefore ignore condition (3). Condition (1) makes $\beta = 0$ in (VIII.65) as before, and

$$\theta = R \sin (pct + \alpha) \sin px$$

from which
$$\frac{\partial^2\theta}{\partial t^2} = - p^2c^2R \sin (pct + \alpha) \sin px$$

and
$$\frac{\partial\theta}{\partial x} = pR \sin (pct + \alpha) \cos px$$

The couple exerted on the wheel by the shaft is

$$IC_0 \left(\frac{\partial\theta}{\partial x}\right)_{x=l} = - IC_0pR \sin (pct + \alpha) \cos pl$$

negative because it is in the direction of θ decreasing.

But Couple = moment of inertia $\times$ angular acceleration

$$\therefore \quad IC_0pR \sin (pct + \alpha) \cos pl = Ip^2c^2R \sin (pct + \alpha) \sin pl$$

$$IC_0 \cot pl = Ipc^2$$

$$= \frac{I}{I_s} pC_0l^2$$

or
$$\cot pl = \frac{I}{I_s} pl \qquad . \quad . \quad . \text{(VIII.70)}$$

The values of pl which satisfy this are the roots of $\cot x = \frac{I}{I_s} x$. If the reader will sketch a graph he will see that the least root of this is less than $\frac{\pi}{2}$. If, as is usually the case, I_s is small compared with I, the roots beyond the first are very nearly π, 2π, 3π, etc. The angular frequency of the rth normal mode is, therefore, very nearly $cp_r = \frac{\pi c}{l}(r - 1)$. A graph covering the range $x = 0$ to $x = \frac{\pi}{2}$ would give the frequency of the first normal mode for any given value of $\frac{I}{I_s}$. Writing (VIII.70) in the form $pl \tan pl = \frac{I_s}{I}$ and expanding $\tan pl$ in powers of pl

$$pl \left(pl + \frac{p^3l^3}{3} + \ldots\right) = \frac{I_s}{I}$$

Neglecting higher powers of pl, which is small compared with unity since I_s is small compared with I, we have

$$p^2l^2 \left(1 + \frac{p^2l^2}{3}\right) = \frac{I_s}{I}$$

As a first approximation we neglect the second term in brackets and have

$$p^2l^2 = \frac{I_s}{I}$$

Substituting this in the bracket

$$p^2l^2 = \frac{I_s}{I}\,\frac{1}{1 + \dfrac{I_s}{3I}} = \frac{I_s}{I + \frac{1}{3}I_s}$$

is a second, and closer, approximation to the value of p^2l^2. The angular frequency of the first normal mode of vibration is therefore

$$pc = \sqrt{\frac{I_s}{I + \frac{1}{3}I_s}}\sqrt{\frac{C_0}{I_s}} \text{ approximately}$$

or

$$\text{Frequency} = \frac{1}{2\pi}\sqrt{\frac{C_0}{I + \frac{1}{3}I_s}} \text{ vibrations per second}$$

from which it appears that in calculating the frequency of the first normal mode the shaft may be treated as massless if $\frac{1}{3}$rd of its moment of inertia is added to that of the wheel.

Longitudinal Vibrations. Now consider the axial or longitudinal vibrations of the shaft. Let w lb be the weight of the shaft per foot run, and E lb be the force per foot stretch of the whole shaft. Let (a), Fig. 56, represent the undisturbed position of the shaft and (b) the position when vibrating at time t sec. $P'Q'$ is the position of the slice originally at PQ. Let $A'P' = x + u$; then $P'Q' = \Delta(x + u)$, and the stretch, or increase in length, of PQ is $\Delta(x + u) - \Delta x = \Delta u$. The tension in PQ is, therefore

$$E\frac{l}{\Delta x}\,\Delta u = El\frac{\Delta u}{\Delta x}$$

If T_P lb is the axial tension at P, then

$$T_P = \underset{\Delta x \to 0}{\text{limit}}\, El\frac{\Delta u}{\Delta x} = El\frac{\partial u}{\partial x}$$

Let T_Q lb be the tension at Q; then

$$T_Q = T_P + \frac{\partial T_P}{\partial x}\,\Delta x$$

so that

$$T_Q - T_P = El\frac{\partial^2 u}{\partial x^2}\,\Delta x$$

But $T_Q - T_P$ is the resultant force on the slice PQ, and $\dfrac{\partial^2(u+x)}{\partial t^2}$ is its acceleration to the right. Hence from

$$\text{Force} = \text{mass} \times \text{acceleration},$$

we have
$$EI\frac{\partial^2 u}{\partial x^2}\,\Delta x = \frac{w\Delta x}{g} \times \frac{\partial^2(u+x)}{\partial t^2}$$

or, since x is independent of t

$$\frac{\partial^2 u}{\partial t^2} = \frac{gEI}{w}\frac{\partial^2 u}{\partial x^2} \qquad . \quad . \quad . \text{(VIII.71)}$$

or
$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

where $c = \sqrt{\dfrac{gEI}{w}}$. This is the same equation as for torsional vibrations. If, with the end A fixed, the vibrations are set up by holding the end B displaced and releasing it at time $t = 0$, then with appropriate changes in the symbols the two solutions are identical.

83. **Flow of Heat.** In Fig. 57, OX, OY, and OZ are rectangular axes. P is the point (x, y, z) and Q is the point $(x + \Delta x, y + \Delta y,$

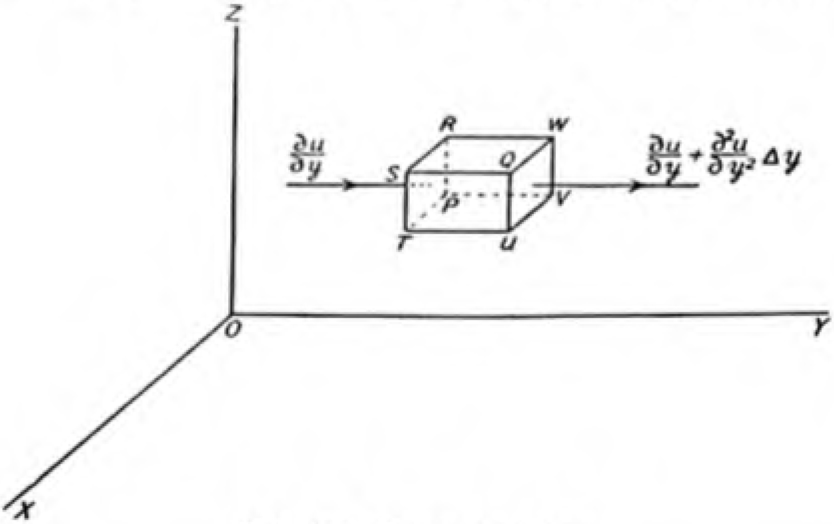


FIG. 57. FLOW OF HEAT

$z + \Delta z)$ in a space supposed to be filled with homogeneous material, and the parallelepiped shown has its faces parallel to the co-ordinate planes. If u is the temperature at P, the rate of increase of the temperature with distance in the direction OY is $\dfrac{\partial u}{\partial y}$. This is the temperature gradient at the face $PRST$ in the direction OY. The

temperature gradient in the same direction at the face $QWVU$ is $\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\Delta y = \frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2}\Delta y$. Heat flows in a direction opposite to that of increase of temperature, so that heat will flow outwards, i.e. in the sense of y decreasing, through the face $PRST$ at a rate equal to the continued product of the area of the face, the temperature gradient, and k, the coefficient of conduction. Thus, we have—

Rate of outflow of heat through the face

$$PRST = k\frac{\partial u}{\partial y}\Delta x \,.\, \Delta z \text{ per sec}$$

Similarly, rate of inflow of heat through the face

$$QWVU = k\left(\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2}\Delta y\right)\Delta x \,.\, \Delta z$$

$\therefore$ Net rate of inflow through the faces

$$PRST \text{ and } QWVU = k\frac{\partial^2 u}{\partial y^2}\Delta x \,.\, \Delta y \,.\, \Delta z$$

Also, net rate of inflow through the faces

$$PRWV \text{ and } QUTS = k\frac{\partial^2 u}{\partial x^2}\Delta x \,.\, \Delta y \,.\, \Delta z$$

and net rate of inflow through the faces

$$PTUV \text{ and } QSRW = k\frac{\partial^2 u}{\partial z^2}\Delta x \,.\, \Delta y \,.\, \Delta z$$

The sum of these three net rates is the rate at which the heat in the parallelepiped is increasing.

But the rate of increase of heat in the parallelepiped is the continued product of $\frac{\partial u}{\partial t}$ (the rate of increase of the temperature with respect to the time), the volume $\Delta x \,.\, \Delta y \,.\, \Delta z$, the density w, and the specific heat s.

Thus $\frac{\partial u}{\partial t} ws\Delta x \,.\, \Delta y \,.\, \Delta z = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\Delta x \,.\, \Delta y \,.\, \Delta z$

Hence
$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right). \qquad \text{(VIII.72)}$$

where $c = \sqrt{\frac{k}{ws}}$

We shall apply this to the case of a slab of homogeneous material, one boundary of which is the whole upper half of the plane $y = 0$, with a parallel boundary which is the whole upper half of the plane $y = d$, and a third boundary which is the plane $z = 0$. Thus the slab is d units thick, rests on the horizontal plane $z = 0$, and extends to infinity in the directions of the positive and negative axes of x and also in the direction of the positive axis of z. Fig. 58 shows the elevation of the slab on the plane ZOY. Both vertical faces are kept at zero temperature and the base is kept at temperature θ. We have to find an expression for the temperature u at any point in the plate. Since the slab extends to infinity on both sides of the plane ZOY, there will be no temperature gradient in the direction of OX, and

$$\frac{\partial^2 u}{\partial x^2} = 0$$

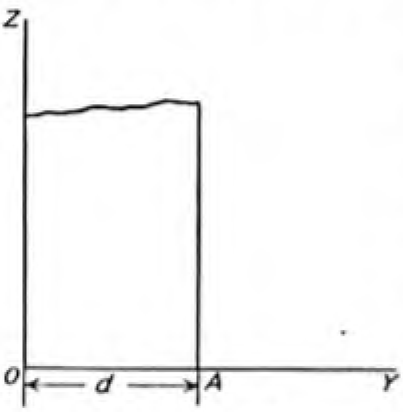


FIG. 58. HEAT FLOW IN INFINITE SLAB

Equation (VIII.72) becomes

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad . \text{(VIII.73)}$$

We shall consider the case in which $\theta = \theta_0 =$ constant, and shall assume that a state of steady flow has been attained. In this case $\frac{\partial u}{\partial t} = 0$, and (VIII.73) becomes

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad . \quad . \quad . \text{(VIII.74)}$$

Substitute $u = YZ$, where Y is a function of y only and Z is a function of z only. Then on separating the variables we have

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -p^2, \text{ where } p \text{ is a constant}$$

Thus $\frac{\partial^2 Y}{\partial y^2} + p^2 Y = 0$ . . . . (VIII.75)

and $$\frac{\partial^2 Z}{\partial z^2} = p^2 Z \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \text{(VIII.76)}$$

The solution of (VIII.75) is $Y = A \sin (py + \alpha)$, and that of (VIII.76) is $Z = Be^{pz} + Ce^{-pz}$, where A, B, C, α are arbitrary constants. The general solution of (VIII.74) is, therefore,

$$u = A \sin (py + \alpha)(Be^{pz} + Ce^{-pz})$$

i.e.

$$u = (Re^{pz} + Se^{-pz}) \sin (py + \alpha) \quad . \quad . \text{(VIII.77)}$$

where $R = A \times B$ and $S = A \times C$.

R, S and α are determined from the given conditions, which are as follows—

(1) $u = 0$ when $y = 0$ for all values of t,

(2) $u = 0$ when $y = d$ for all values of t,

and (3) $u = \theta_0$ when $z = 0$ for all values of t.

First we conclude that at $z = \infty$ the effect of the temperature θ_0 at $z = 0$ will not be apparent, and as e^{pz} becomes infinite there, we see from (VIII.77) that $R = 0$. Substituting in (VIII.77) we have from condition (1), $\sin \alpha = 0$, whence $\alpha = 0$, and then from condition (2), $\sin pd = 0$, whence $pd = n\pi$, i.e. $p = \frac{n\pi}{d}$, where n is any positive integer. Thus (VIII.77) becomes

$$u_n = S_n e^{-\frac{n\pi z}{d}} \sin \frac{n\pi y}{d} \quad . \quad . \quad . \text{(VIII.78)}$$

which satisfies (VIII.74) and the conditions (1) and (2) for all values of n. The sum of the solutions u_n, $n = 1, 2, 3$, etc.

i.e.

$$u = \sum_{n=1}^{n=\infty} S_n e^{-\frac{n\pi z}{d}} \sin \frac{n\pi y}{d} \quad . \quad . \text{(VIII.79)}$$

also satisfies (VIII.74) and the conditions (1) and (2), and it will satisfy (3) also, if

$$\theta_0 = \sum_{n=1}^{n=\infty} S_n \sin \frac{n\pi y}{d} \quad . \quad . \quad . \text{(VIII.80)}$$

This is the Fourier series for the constant θ_0, and by (III.14)

$$S_n = \frac{2}{d}\int_0^d \theta_0 \sin \frac{n\pi y}{d}\, dy = -\frac{2\theta_0}{n\pi}\left[\cos \frac{n\pi y}{d}\right]_0^d \quad . \text{(VIII.81)}$$

$$\therefore \qquad S_n = \frac{2\theta_0}{n\pi}[1 - \cos n\pi]$$

If n is even, $\cos n\pi = 1$ and $S_n = 0$; if n is odd, $\cos n\pi = -1$, and $S_n = \frac{4\theta_0}{n\pi}$

The solution of (VIII.74) which satisfies all the conditions is therefore, from (VIII.79),

$$u = \frac{4\theta_0}{\pi}\left[e^{-\frac{\pi z}{d}} \sin \frac{\pi y}{d} + \tfrac{1}{3}e^{-\frac{3\pi z}{d}} \sin \frac{3\pi y}{d} + \tfrac{1}{5}e^{-\frac{5\pi z}{d}} \sin \frac{5\pi y}{d} + \ldots\right] \qquad \text{(VIII.82)}$$

If the temperature when $z = 0$ is not constant but is given by $u = f(y)$, then we substitute $f(y)$ for θ_0, and obtain for the value of S_n

$$S_n = \frac{2}{d}\int_0^d f(y) \sin \frac{n\pi y}{d}\, dy \quad . \qquad . \qquad . \text{(VIII.83)}$$

If, for example, $f(y) = y$, i.e. the temperature varies uniformly from zero at O to d degrees at A, we have on integrating by parts

$$S_n = \frac{2}{d}\left[-\frac{d}{n\pi} y \cos \frac{n\pi y}{d} + \frac{d^2}{n^2\pi^2} \sin \frac{n\pi y}{d}\right]_0^d$$

i.e. $$S_n = -\frac{2d}{n\pi} \cos n\pi$$

If n is even, $S_n = -\frac{2d}{n\pi}$, and if n is odd, $S_n = \frac{2d}{n\pi}$, so that from (VIII.82)

$$u = \frac{2d}{\pi}\left[e^{-\frac{\pi z}{d}} \sin \frac{\pi y}{d} - \tfrac{1}{2}e^{-\frac{2\pi z}{d}} \sin \frac{2\pi y}{d} + \tfrac{1}{3}e^{-\frac{3\pi z}{d}} \sin \frac{3\pi y}{d} \right.$$
$$\left. - \tfrac{1}{4}e^{-\frac{4\pi z}{d}} \sin \frac{4\pi y}{d} + \ldots\right] \quad . \qquad . \text{(VIII.84)}$$

84. **Flow of Heat in One Dimension.** If a slab of material such as that in Fig. 58 is assumed to extend to infinity in the directions of y positive and negative and of z positive and negative, and the temperatures on the opposite faces $x = 0$ and $x = d$ are constant but unequal, heat flows only in the direction of the axis of x, and the relation (VIII.72) reduces to

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad . \qquad . \qquad . \text{(VIII.85)}$$

This relation applies to the flow of heat along a long uniform rod whose lateral surface is covered with heat insulation material, or to the flow through a small area of a uniform finite slab of material whose thickness is small compared with its other dimensions, provided that the area is well away from the edges of the slab.

EXAMPLE 1

Two opposite sides of an infinite slab of thickness l are kept at temperatures θ_0 at $x = 0$ and θ_l at $x = l$. Find the temperature distribution inside the slab.

From elementary considerations it is easily seen that the distribution is $u = \dfrac{l - x}{l}\theta_0 + \dfrac{x}{l}\theta_l$. We shall obtain the result from (VIII.85). Since the conditions are steady, $\dfrac{\partial u}{\partial t} = 0$, and (VIII.85) becomes $\dfrac{d^2u}{dx^2} = 0$, from which $u = ax + b$. Since $u = \theta_0$ when $x = 0$ and $u = \theta_l$ when $x = l$, then $b = \theta_0$ and $a = \dfrac{\theta_l - \theta_0}{l}$, and for the temperature distribution we have

$$u = \frac{\theta_l - \theta_0}{l}x + \theta_0 \quad . \quad . \quad . \quad . \quad (\text{VIII.86})$$

which agrees with the above result.

When $\dfrac{\partial u}{\partial t} \neq 0$, the flow is not steady, and it is necessary to solve (VIII.85) subject to certain conditions. In some cases the solution is best obtained by separating the variables. As in Art. 80 we assume that $u = XT$, where X is a function of x alone and T of t alone.

Thus
$$\frac{\partial u}{\partial t} = X\frac{\partial T}{\partial t}$$

and
$$\frac{\partial^2 u}{\partial x^2} = T\frac{\partial^2 X}{\partial x^2}$$

and substituting in (VIII.85)

$$X\frac{\partial T}{\partial t} = c^2T\frac{\partial^2 X}{\partial x^2}$$

i.e.
$$\frac{1}{c^2T}\frac{\partial T}{\partial t} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2}$$

which express the equality of a function of t alone and a function of x alone. This equality can hold only if each side is constant. Let this constant be $-p^2$, so that

$$\frac{1}{c^2T}\frac{\partial T}{\partial t} = -p^2 \quad . \quad . \quad . \quad (\text{VIII.87})$$

and $$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -p^2 \qquad . \quad . \quad . \text{(VIII.88)}$$

The solutions of these equations are respectively

$$T = Ae^{-p^2c^2t} \qquad . \quad . \quad . \text{(VIII.89)}$$

and $$X = B\sin(px + \alpha) \qquad . \quad . \quad . \text{(VIII.90)}$$

Since $u = XT$, a solution of (VIII.85) is

$$u = Re^{-p^2c^2t}\sin(px + \alpha) \qquad . \quad . \text{(VIII.91)}$$

where R, p, and α are arbitrary constants.

EXAMPLE 2

A large iron plate of thickness d is at a uniform temperature θ_0, and its opposite faces are suddenly cooled to 0°C, and kept at that temperature. Find the temperature distribution t sec later.

Assuming that x is measured perpendicular to the parallel faces of the plate, we have as on page 323

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad . \quad . \quad . \quad . \text{(VIII.92)}$$

and the conditions are

(1) $u = 0$ when $x = 0$ for all values of t,

(2) $u = 0$ when $x = d$ for all values of t,

and (3) $u = \theta_0$ when $t = 0$ for all values of x.

Substituting from (1) in (VIII.91), we have $0 = Re^{-p^2c^2t}\sin\alpha$ for all values of t, whence $\alpha = 0$. Substituting from (2), we have $0 = Re^{-p^2c^2t}\sin pd$ for all values of t, whence $\sin pd = 0$, i.e. $p = \frac{n\pi}{d}$, where n is any positive integer. With these values of α and p substituted, (VIII.91) becomes

$$u_n = R_n e^{-\frac{n^2c^2\pi^2}{d^2}t}\sin\frac{n\pi x}{d} \qquad . \quad . \quad . \text{(VIII.93)}$$

where R_n is written for R. This satisfies (VIII.92) and conditions (1) and (2) for all integral values of n, and so also does the sum of any number of terms on the right of (VIII.93) obtained from $n = 1, 2, 3$, etc. The most general solution which satisfies the two conditions is, therefore,

$$u = \sum_{n=1}^{n=\infty} R_n e^{-\frac{n^2c^2\pi^2}{d^2}t}\sin\frac{n\pi x}{d} \qquad . \quad . \quad . \text{(VIII.94)}$$

Substituting condition (3), we have

$$\theta_0 = \sum_{n=1}^{n=\infty} R_n\sin\frac{n\pi x}{d} \qquad . \quad . \quad . \quad . \text{(VIII.95)}$$

which is a sine series for θ_0, and by (III.14),

$$\begin{aligned} R_n &= \frac{2}{d}\int_0^d \theta_0 \sin\frac{n\pi x}{d}\,dx \\ &= \frac{2\theta_0}{n\pi}\left[-\cos\frac{n\pi x}{d}\right]_0^d \\ &= \frac{2\theta_0}{n\pi}[1-\cos n\pi] \end{aligned}$$

Thus, $R_n = 0$ when n is even, and $R_n = \frac{4\theta_0}{n\pi}$ when n is odd.

Substituting these values in (VIII.93), we have

$$u = \frac{4\theta_0}{\pi}\left[e^{-\frac{c^2\pi^2}{d^2}t}\sin\frac{\pi x}{d} + \frac{1}{3}e^{-\frac{9c^2\pi^2}{d^2}t}\sin\frac{3\pi x}{d} + \frac{1}{5}e^{-\frac{25c^2\pi^2}{d^2}t}\sin\frac{5\pi x}{d} + \ldots\right]. \quad \text{(VIII.96)}$$

EXAMPLE 3

Assume the thickness d of the slab in Ex. 2 to be infinite. If the face at $x = 0$ is given a fluctuating temperature $u = u_0 \sin pt$, determine the temperature distribution throughout the slab. Assume that $u = 0$ when x is infinite.

We have $$\frac{\partial u}{\partial t} - c^2\frac{\partial^2 u}{\partial x^2} = 0,$$

with the conditions (1) $u = 0$ when x is infinite for all values of t,
and (2) $u = u_0 \sin pt$ when $x = 0$.

We shall assume that $u = Re^{-\alpha x}\sin(pt - qx)$, the factor $e^{-\alpha x}$ being suggested by condition (1) and the factor $\sin(pt - qx)$ having the form of condition (2) when $x = 0$.

We have $\frac{\partial u}{\partial t} = Rpe^{-\alpha x}\cos(pt - qx)$

and $$\frac{\partial^2 u}{\partial x^2} = Re^{-\alpha x}[(\alpha^2 - q^2)\sin(pt - qx) + 2\alpha q\cos(pt - qx)]$$

If our assumption is correct, these values of $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ should satisfy the equation $\frac{\partial u}{\partial t} - c^2\frac{\partial^2 u}{\partial x^2} = 0$ identically, so that

$$Re^{-\alpha x}[(p - 2c^2\alpha q)\cos(pt - qx) + c^2(q^2 - \alpha^2)\sin(pt - qx)] = 0$$

It follows that $p - 2c^2\alpha q = 0$ and $q^2 - \alpha^2 = 0$, which give $q = \alpha = \frac{1}{c}\sqrt{\frac{p}{2}}$.

The solution is, therefore,

$$u = Re^{-\frac{1}{c}\sqrt{\frac{p}{2}}x}\sin\left(pt - \frac{1}{c}\sqrt{\frac{p}{2}}x\right)$$

or, since $u = u_0 \sin pt$ when $x = 0$, so that $R = u_0$,

$$u = u_0 e^{-\frac{1}{c}\sqrt{\frac{p}{2}}x}\sin\left(pt - \frac{1}{c}\sqrt{\frac{p}{2}}x\right) \quad . \quad . \quad \text{(VIII.97)}$$

The solution (VIII.97) satisfies the differential equation and the given conditions. It represents the temperature distribution in the slab. It also represents the flow of heat along an infinitely long uniform insulated rod under the same end conditions. (An alternative method of solution is shown in Ex. 5, Art. 79.)

85. **Flow of Electric Current.** Suppose that an electric current is flowing through a long wire such as a telephone wire with an earth

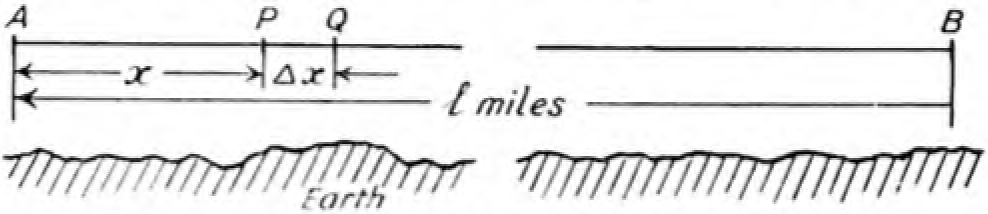


FIG. 59. FLOW OF ELECTRICITY IN A TELEPHONE WIRE

return or twin wires one of which is the return wire. Suppose also that for each mile of wire there is resistance R ohms, inductance L henrys, capacity C farads, and leakance G mhos. We show such a wire in Fig. 59. Consider a small length of the wire $\overline{PQ} = \Delta x$ miles, P being distant x miles from the sending end A.

The fall in voltage difference from P to Q is $-\dfrac{\partial V}{\partial x}\Delta x$, where V is the voltage difference at P and, therefore, $V + \dfrac{\partial V}{\partial x}\Delta x$ is that at Q.

The fall of voltage between P and Q is $iR\Delta x$ due to the resistance, where i is the current at P, and $L\dfrac{\partial i}{\partial t}\Delta x$ due to the inductance, where t sec is the time. Thus, we have the relation

$$iR\Delta x + L\frac{\partial i}{\partial t}\Delta x = -\frac{\partial V}{\partial x}\Delta x, \text{ and dividing through by } \Delta x$$

$$Ri + L\frac{\partial i}{\partial t} = -\frac{\partial V}{\partial x} \quad . \quad . \quad . \quad . \quad . \text{(VIII.98)}$$

The current loss between P and Q is $-\dfrac{\partial i}{\partial x}\Delta x$. This is made up of $C\dfrac{\partial V}{\partial t}\Delta x$ due to capacity and $GV\Delta x$ due to leakance.

Thus
$$C\frac{\partial V}{\partial t}\Delta x + GV\Delta x = -\frac{\partial i}{\partial x}\Delta x$$

i.e.
$$C\frac{\partial V}{\partial t} + GV = -\frac{\partial i}{\partial x} \quad . \quad . \text{(VIII.99)}$$

(VIII.98) and (VIII.99) are simultaneous differential equations with x and t as independent variables. Writing D_1 for $\frac{\partial}{\partial t}$ and D_2 for $\frac{\partial}{\partial x}$, and re-arranging

$$(R + LD_1)i + D_2V = 0 \quad . \quad \text{(VIII.100)}$$

and

$$D_2 i + (CD_1 + G)V = 0 \quad . \quad \text{(VIII.101)}$$

Multiplying through (VIII.100) by D_2 and (VIII.101) by $(R + LD_1)$, and subtracting

$$[D_2^2 - (R + LD_1)(CD_1 + G)]V = 0$$

i.e.

$$LC\frac{\partial^2 V}{\partial t^2} + (CR + LG)\frac{\partial V}{\partial t} + RGV = \frac{\partial^2 V}{\partial x^2} \quad . \quad \text{(VIII.102)}$$

Also, multiplying through (VIII.100) by $CD_1 + G$ and (VIII.101) by D_2, and subtracting

$$[(CD_1 + G)(R + LD_1) - D_2^2]i = 0$$

i.e.

$$LC\frac{\partial^2 i}{\partial t^2} + (CR + LG)\frac{\partial i}{\partial t} + RGi = \frac{\partial^2 i}{\partial x^2} \quad . \quad \text{(VIII.103)}$$

which is (VIII.102) with i substituted for V.

(A) Assume that there is no leakance or inductance and that $\overline{AB} = l$ miles is so large that it may be considered to be of infinite length. If there is a pulsating voltage $V = V_0 \sin pt$, find the distribution of V.

Putting $G = L = 0$ in (VIII.102), we have

$$CR\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} \quad . \quad . \quad \text{(VIII.104)}$$

the same equation as in Ex. 3, Art. 84 for the flow of heat along an insulated rod or through an infinite plate. If we replace u by V, u_0 by V_0, and c by $\frac{1}{\sqrt{CR}}$, (VIII.97) gives the voltage distribution

i.e.

$$V = V_0 e^{-\sqrt{\frac{1}{2}pCR}\,x} \sin\left(pt - \sqrt{\tfrac{1}{2}pCR}\,x\right) \quad . \quad \text{(VIII.105)}$$

Now putting $L = 0$ in (VIII.98)

$$i = -\frac{1}{R}\frac{\partial V}{\partial x} \quad . \quad . \quad . \quad \text{(VIII.106)}$$

and writing θ for $pt - \sqrt{\tfrac{1}{2}pCR}\, x$, we have

$$i = \frac{V_0}{R}\sqrt{\tfrac{1}{2}pCR}\, e^{-\sqrt{\frac{1}{2}pCR}\,x}(\sin\theta + \cos\theta)$$

$$= \frac{V_0}{R}\sqrt{pCR}\, e^{-\sqrt{\frac{1}{2}pCR}\,x}\sin\left(\theta + \frac{\pi}{4}\right)$$

i.e.

$$i = \frac{V_0}{R}\sqrt{pCR}\, e^{-\sqrt{\frac{1}{2}pCR}\,x}\sin\left(pt - \sqrt{\tfrac{1}{2}pCR}\, x + \frac{\pi}{4}\right) \qquad \text{(VIII.107)}$$

which shows that the current leads the voltage at all times and places by 45°.

(B) If the inductance and capacity are negligible compared with the leakance and resistance, relations (VIII.98) and (VIII.99) reduce to $Ri = -\dfrac{\partial V}{\partial x}$ and $\dfrac{\partial i}{\partial x} = -GV$. It follows that $\dfrac{\partial^2 V}{\partial x^2} = RGV$, the solution of which is

$$V = A\cosh\sqrt{RG}\,x + B\sinh\sqrt{RG}\,x \qquad \text{(VIII.108)}$$

where A and B are arbitrary constants.

For an infinity long line $V = 0$ when $x = \infty$, and since $\sinh\sqrt{RG}\,x$ and $\cosh\sqrt{RG}\,x$ tend to equality as x approaches ∞, $B = -A$, and (VIII.109) becomes

$$V = A(\cosh\sqrt{RG}\,x - \sinh\sqrt{RG}\,x) = Ae^{-\sqrt{RG}\,x}$$

and if $V = V_0$ when $x = 0$

$$V = V_0 e^{-\sqrt{RG}\,x} \quad . \quad . \qquad \text{(VIII.110)}$$

Since

$$i = -\frac{1}{R}\frac{\partial V}{\partial x}$$

$$i = V_0\sqrt{\frac{G}{R}}\, e^{-\sqrt{RG}\,x} \quad . \qquad \text{(VIII.111)}$$

in which V_0 may be a constant or a function of time.

If $l \neq \infty$ and a resistance r is connected between the wires at B, or between the single wire and the earth if the earth is the return lead, the voltage drop across r is ir, and we have then the relation (VIII.108) with the end conditions

(1) $V = V_0$ when $x = 0$ and (2) $V = ir$ when $x = l$.

From (VIII.108) and the relation $Ri = -\dfrac{\partial V}{\partial x}$

$$i = -\sqrt{\frac{G}{R}}(A \sinh \sqrt{RG}\, x + B \cosh \sqrt{RG}\, x) \quad \text{(VIII.112)}$$

From condition (2)

$$A \cosh \sqrt{RG}\, l + B \sinh \sqrt{RG}\, l$$
$$= -r\sqrt{\frac{G}{R}}(A \sinh \sqrt{RG}\, l + B \cosh \sqrt{AG}\, l) \quad . \text{(VIII.113)}$$

and from condition (1) $\qquad A = V_0$

Substituting for A in (VIII.113)

$$B = -V_0 \frac{\cosh \sqrt{RG}\, l + r\sqrt{\frac{G}{R}} \sinh \sqrt{RG}\, l}{\sinh \sqrt{RG}\, l + r\sqrt{\frac{G}{R}} \cosh \sqrt{RG}\, l}$$

On substitution in (VIII.108)

$$V = V_0 \left[\cosh \sqrt{RG}\, x - \frac{\cosh \sqrt{RG}\, l + r\sqrt{\frac{G}{R}} \sinh \sqrt{RG}\, l}{\sinh \sqrt{RG}\, l + r\sqrt{\frac{G}{R}} \cosh \sqrt{RG}\, l} \sinh \sqrt{RG}\, x\right] \quad \text{(VIII.114)}$$

which gives the voltage everywhere in terms of V_0, the voltage at the sending end. The corresponding expression for the current i is

$$i = V_0 \sqrt{\frac{G}{R}} \left[\frac{\cosh \sqrt{RG}\, l + r\sqrt{\frac{G}{R}} \sinh \sqrt{RG}\, l}{\sinh \sqrt{RG}\, l + r\sqrt{\frac{G}{R}} \cosh \sqrt{RG}\, l} \cosh \sqrt{RG}\, x - \sinh \sqrt{RG}\, x\right] \quad \text{(VIII.115)}$$

(C) If there is no leakance (VIII.102) takes the form

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + CR\frac{\partial V}{\partial t} \quad . \quad \text{(VIII.116)}$$

Suppose that the voltage at the sending end A is $V_0 \sin pt$. Owing to the impedance the voltage will decrease as the distance x increases and will be very small when x is very large. We have then the two conditions—

(1) $V = 0$ when $x = \infty$, which suggests that the expression for V will contain a decay factor $e^{-\alpha x}$, the other factor being a function of x and t.

(2) $V = V_0 \sin pt$ when $x = 0$. From this it appears that the other factor may be $\sin(pt - qx)$ where q is a constant. We shall see if by suitable choice of q and α, (VIII.116) is satisfied by

$$V = V_0 e^{-\alpha x} \sin(pt - qx) \quad . \quad \text{(VIII.117)}$$

From (VIII.117)

$$\frac{\partial V}{\partial t} = pV_0 e^{-\alpha x} \cos(pt - qx)$$

$$\frac{\partial^2 V}{\partial t^2} = -p^2 V_0 e^{-\alpha x} \sin(pt - qx)$$

and $\dfrac{\partial^2 V}{\partial x^2} = V_0 e^{-\alpha x}[(\alpha^2 - q^2)\sin(pt - qx) + 2\alpha q \cos(pt - qx)]$

Substituting these in (VIII.116)

$$V_0 e^{-\alpha x}[(\alpha^2 - q^2)\sin(pt - qx) + 2\alpha q\cos(pt - qx)]$$
$$= V_0 e^{-\alpha x}[-LCp^2 \sin(pt - qx) + CRp\cos(pt - qx)]$$

i.e. $\quad (\alpha^2 - q^2 + LCp^2)\sin(pt - qx) = (CRp - 2\alpha q)\cos(pt - qx)$

and since these are to be equal for all values of t and x, they must each be zero.

Hence
and

$$\left.\begin{aligned} \alpha^2 - q^2 + LCp^2 &= 0 \\ CRp - 2\alpha q &= 0 \end{aligned}\right\} \quad . \quad \text{(VIII.118)}$$

Eliminating α by substituting from the second equation in the first,

$$\frac{C^2R^2p^2}{4q^2} - q^2 + LCp^2 = 0$$

or

$$4q^4 - 4LCp^2q^2 - C^2R^2p^2 = 0$$

Solving for q^2

$$q^2 = \frac{4LCp^2 \pm \sqrt{16L^2C^2p^4 + 16C^2R^2p^2}}{8}$$

$$\therefore \qquad q^2 = \tfrac{1}{2}LCp^2\left(1 + \sqrt{1 + \frac{R^2}{p^2L^2}}\right)$$

the negative sign being inadmissible.

Writing N^2 for the quantity in the bracket, we have $q^2 = \frac{1}{2}LCp^2N^2$, from which $q = pN\sqrt{\frac{LC}{2}}$, so that

$$\alpha = \frac{CRp}{2q} = \frac{CR}{2} \times \frac{\sqrt{2}}{N\sqrt{LC}} = \frac{R}{N}\sqrt{\frac{C}{2L}}$$

With these values of q and α, (VIII.117) satisfies (VIII.116) and conditions (1) and (2). The solution is, therefore

$$V = V_0 e^{-\frac{R}{N}\sqrt{\frac{C}{2L}}x} \sin p\left(t - N\sqrt{\frac{LC}{2}}\,x\right). \qquad \text{(VIII.119)}$$

where $\qquad N^2 = 1 + \sqrt{1 + \frac{R^2}{p^2L^2}}$

If $\frac{R}{pL}$ is so small that its square may be neglected in comparison with unity, $N = \sqrt{2}$, and (VIII.119) becomes

$$V = V_0 e^{-\frac{R}{2}\sqrt{\frac{C}{L}}x} \sin p(t - \sqrt{LC}x) \quad . \qquad \text{(VIII.120)}$$

EXAMPLE 1

When the leakance and inductance effects in a telegraph cable can be neglected $\frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t}$. Under steady conditions the voltage at the sending end is V_A and that at the receiving end is V_B. The receiving end is suddenly grounded. Find the voltage and current at time t sec after the grounding.

Before the grounding, conditions are steady and $\frac{\partial V}{\partial t} = 0$. The equation becomes $\frac{\partial^2 V}{\partial x^2} = 0$, whose solution is $V = ax + b$, where a and b are constants. But if l miles is the length of the cable, $V = V_A$ when $x = 0$ and $V = V_B$ when $x = l$. From these, $b = V_A$ and $a = \frac{V_B - V_A}{l}$. The steady voltage before grounding is then

$$V = \frac{V_B - V_A}{l}x + V_A \qquad . \qquad . \qquad . \text{(VIII.121)}$$

Let V_T be the transient voltage and V_{T_0} its value when $t = 0$.

After the disturbance due to grounding the transient state will die away and the final steady-state voltage V' is found by putting $V_B = 0$ in (VIII.121), giving

$$V' = V_A\left(1 - \frac{x}{l}\right) \quad . \quad . \quad . \quad . \text{(VIII.122)}$$

Thus at time $t = 0$, the transient voltage is V_{T_0} where $V_{T_0} = V - V'$, whence

$$V_{T_0} = \frac{V_B - V_A}{l}x + V_A - V_A\left(1 - \frac{x}{l}\right)$$

i.e.
$$V_{T_0} = \frac{V_B}{l}x \quad . \quad . \quad . \quad . \text{(VIII.123)}$$

The equation
$$\frac{\partial^2 V_T}{\partial x^2} = RC\frac{\partial V_T}{\partial t} \quad . \quad . \quad . \quad . \text{(VIII.124)}$$

is to be satisfied together with the conditions

(1) $V_T = 0$ when $x = 0$ for all values of t,

(2) $V_T = \frac{V_B}{l}x$ for all values of x when $t = 0$,

and (3) $V_T = 0$ when $x = l$ for all values of t.

We obtained a solution of (VIII.124) in Art. 84, where we found, with the necessary change in notation,

$$V_T = Re^{-\frac{p^2}{RC}t}\sin(px + \alpha) \quad . \quad . \quad . \text{(VIII.125)}$$

Substituting condition (1), we have $\sin\alpha = 0$, whence $\alpha = 0$.

Substituting condition (3), we have $0 = Re^{-\frac{p^2}{RC}t}\sin pl$, so that $\sin pl = 0$ and $p = \frac{n\pi}{l}$, where n is any positive integer.

Thus,
$$V_T = R_n e^{-\frac{n^2\pi^2}{RCl^2}t}\sin\frac{n\pi x}{l} \quad . \quad . \quad . \text{(VIII.126)}$$

satisfies equation (VIII.124) and conditions (1) and (3) for all integral values of n. So also does

$$V_T = \sum_{n=1}^{n=\infty} R_n e^{-\frac{n^2\pi^2}{RCl^2}t}\sin\frac{n\pi x}{l} \quad . \quad . \quad . \text{(VIII.127)}$$

(VIII.126) will satisfy condition (2) if

$$\frac{V_B}{l}x = \sum_{n=1}^{n=\infty} R_n \sin\frac{n\pi x}{l}$$

This is a Fourier sine series, and by (III.14)

$$R_n = \frac{2V_B}{l^2}\int_0^l x\sin\frac{n\pi x}{l}\,dx$$
$$= \frac{2V_B}{n\pi l}\left[-x\cos\frac{n\pi x}{l} + \int\cos\frac{n\pi x}{l}\,dx\right]_0^l$$

Hence, $$R_n = \frac{2V_B}{n\pi}[-\cos n\pi] = \frac{2V_B}{n\pi}(-1)^{n+1}$$

Substituting in (VIII.126), we have

$$V_T = \frac{2V_B}{\pi}\sum_{n=1}^{n=\infty}\frac{(-1)^{n+1}}{n}e^{-\frac{n^2\pi^2}{RCl^2}t}\sin\frac{n\pi x}{l} \quad . \text{(VIII.128)}$$

The actual voltage V in the wire at time t is $V_T + V_A\left(1-\frac{x}{l}\right)$

The current is given by $I = -\frac{1}{R}\frac{\partial V}{\partial x}$

i.e. $$I = -\frac{1}{R}\frac{\partial V_T}{\partial x} + \frac{1}{Rl}V_A$$

i.e. $$I = \frac{V_A}{Rl} + \frac{2V_B}{l}\sum_{n=1}^{n=\infty}(-1)^n e^{-\frac{n^2\pi^2}{RCl^2}t}\cos\frac{n\pi x}{l} \quad . \text{(VIII.129)}$$

When none of the four characteristics is negligible, the voltage is found by solving (VIII.103) subject to the given end conditions. If a voltage $V_0 \sin pt$ is impressed on the sending end, forced vibrations of the same frequency will occur in the circuit which will decrease in amplitude with distance, the amplitude becoming zero at infinity. The phase of the vibrations will change as x increases. Thus we may reasonably assume that the voltage at a point x miles from the sending end is

$$V = V_0 e^{-\alpha x}\sin p\left(t-\frac{x}{c}\right) \quad . \quad . \text{(VIII.130)}$$

where α is constant and c is the wave velocity along the wire, or wires. From this

$$\frac{\partial V}{\partial t} = pV_0 e^{-\alpha x}\cos p\left(t-\frac{x}{c}\right)$$

$$\frac{\partial^2 V}{\partial t^2} = -p^2 V_0 e^{-\alpha x}\sin p\left(t-\frac{x}{c}\right)$$

and

$$\frac{\partial^2 V}{\partial x^2} = V_0 e^{-\alpha x}\left[\alpha^2 \sin p\left(t-\frac{x}{c}\right) + \frac{2\alpha p}{c}\cos p\left(t-\frac{x}{c}\right) - \frac{p^2}{c^2}\sin p\left(t-\frac{x}{c}\right)\right]$$

Substituting these in (VIII.102), dividing through by $V_0e^{-\alpha x}$ and equating the coefficients of $\sin p\left(t-\dfrac{x}{c}\right)$ and $\cos p\left(t-\dfrac{x}{c}\right)$ on the two sides of the equation

$$RG - p^2LC = \alpha^2 - \frac{p^2}{c^2} \quad . \quad . \quad (\text{VIII.131})$$

and
$$p(CR + LG) = \frac{2\alpha p}{c}$$

or
$$CR + LG = \frac{2\alpha}{c} \quad . \quad . \quad . \quad (\text{VIII.132})$$

From (VIII.131)
$$\frac{1}{c^2} = LC + \frac{\alpha^2 - RG}{p^2} \quad . \quad (\text{VIII.133})$$

which shows that the wave velocity c is a function of the frequency p. If a disturbance containing several frequencies is transmitted the higher frequencies will travel fastest and the received wave-form will be a distorted copy of the transmitted wave. This distortion does not take place if $\alpha^2 = RG$ for then from (VIII.133), $c = \dfrac{1}{\sqrt{LC}}$ which is constant. Putting these values of α and c in (VIII.132) we have

$$CR + LG = 2\sqrt{LCRG}$$
$$CR - 2\sqrt{CR \times LG} + LG = 0$$
$$(\sqrt{CR} - \sqrt{LG})^2 = 0$$

i.e.
$$CR = LG \quad . \quad (\text{VIII.134})$$

This result is due to Heaviside, and a circuit whose characteristics satisfy (VIII.134) is known as a *distortionless circuit*.

EXAMPLES VIII

In each of the following cases eliminate the arbitrary function or functions and thus form a partial differential equation.

(1) (i) $z = f(ax - by)$, where a and b are constants

(ii) $z = 2x - 3y + f(4x - 5y)$

(iii) $z = f\left(\dfrac{x^d}{y^c}\right)$

(2) (i) $z = f_1(x + 4y) + f_2(x - 4y)$

(ii) $u = f_1(s)f_2(t)$

(iii) $u = f_1(s + at) + sf_2(s + at)$, where a is a constant

(3) (i) $u = e^{-x}f(x - 2t)$

(ii) $z = f(xy)$

(iii) $z = f_1(2x + y) + f_2(x - 2y)$

(4) Show that $y = A \sin(kct + \alpha) \sin(kx + \beta)$ satisfies the equation

$$\frac{\partial y^2}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

and that $y = Ae^{-kt} \cos(\sqrt{a^2c^2 - k^2} \,.\, t + \alpha) \sin ax$ satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} + 2k \frac{\partial y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2}$$

[c, A, k, a, α, β are constants.]

(5) Solve the following equations—

(i) $10 \dfrac{\partial z}{\partial x} + 7 \dfrac{\partial z}{\partial y} = 0$

(ii) $10y \dfrac{\partial z}{\partial x} + 7x \dfrac{\partial z}{\partial y} = 0$

(iii) $x \dfrac{\partial z}{\partial x} - y \dfrac{\partial z}{\partial y} = 0$

(6) (i) $10x \dfrac{\partial z}{\partial x} + 7y \dfrac{\partial z}{\partial y} = 0$

(ii) $\dfrac{\partial^2 z}{\partial x \partial y} = 0$

(iii) $\dfrac{\partial^2 u}{\partial y^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}$, where a is a constant

(iv) $2 \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial x \partial y} - 3 \dfrac{\partial^2 u}{\partial y^2} = 0$

(7) (i) $4 \dfrac{\partial^2 u}{\partial x^2} + 20 \dfrac{\partial^2 u}{\partial x \partial y} + 25 \dfrac{\partial^2 u}{\partial y^2} = 0$

(ii) $\dfrac{\partial^2 z}{\partial x^2} = -\dfrac{\partial^2 z}{\partial t^2}$

(iii) $\dfrac{\partial^2 z}{\partial x^2} - 4 \dfrac{\partial^2 z}{\partial x \partial y} + 5 \dfrac{\partial^2 z}{\partial y^2} = 0$

(8) (i) $\dfrac{\partial^3 u}{\partial x^3} + 2 \dfrac{\partial^3 u}{\partial x^2 \partial y} - \dfrac{\partial^3 u}{\partial x \partial y^2} - 2 \dfrac{\partial^3 u}{\partial y^3} = 0$

(ii) $\dfrac{\partial^4 z}{\partial x^4} - 2 \dfrac{\partial^4 z}{\partial x^3 \partial y} + 5 \dfrac{\partial^4 z}{\partial x^2 \partial y^2} - 8 \dfrac{\partial^4 z}{\partial x \partial y^3} + 4 \dfrac{\partial^4 z}{\partial y^4} = 0$

(iii) $\dfrac{\partial^4 u}{\partial x^4} + 4 \dfrac{\partial^4 u}{\partial x^2 \partial y^2} + 4 \dfrac{\partial^4 u}{\partial y^4} = 0$

(9) Solve $\dfrac{\partial^2 z}{\partial x^2} = c^2 \dfrac{\partial^2 z}{\partial y^2}$ making use of the substitutions $r = y + cx$, $s = y - cx$.

(10) With x as abscissa and z as ordinate, sketch a graph of $z = f(x)$ taking this, for simplicity, as the upper half of the circumference of a circle with centre at the origin. Show on your graph the positions of the travelling waves $z = f(x + at)$ and $z = f(x - at)$ where a is constant at times $t = 1$, $t = 2$ and $t = 3$. In a second diagram show one undulation of the stationary wave produced by the merging of the two travelling waves $z = A \sin(x + at)$, $z = A \sin(x - at)$ when the stationary wave is in its position of maximum displacement.

(11) A perfectly flexible uniform string of length l ft weighing w lb per foot run is stretched to tension T lb between two points A and B distant l ft apart. The mid-point of the string is held displaced through a small distance h ft perpendicular to AB and is then released from rest in that position. Show that the motion of the string at time t sec after the instant of release is given by the relation

$$y = \frac{8h}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l} - \frac{1}{3^2}\cos\frac{3\pi ct}{l}\sin\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi ct}{l}\sin\frac{5\pi x}{l} - \ldots\right]$$

where $c = \sqrt{\dfrac{gT}{w}}$ and x is the distance from A.

(12) Find a particular integral of $\dfrac{\partial^2 z}{\partial x \partial y} = x^2 + y^2$ and find the complete solution of the equation. Prove by substitution that your solution is correct.

(13) If $x = \alpha + \beta$, $y = \alpha - \beta$, where x and y, and α and β are pairs of independent variables, and if V is a function of x and y (and therefore also of α and β), show that

$$\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial \alpha \partial \beta}$$

Hence, or otherwise, find a particular solution of the equation

$$\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2} = x^2 + y^2 \qquad \text{(U.L.)}$$

(14) Find the complete solutions of—

(i) $3\dfrac{\partial z}{\partial x} + 5\dfrac{\partial z}{\partial y} = (x + y) + 48$

(ii) $6\dfrac{\partial z}{\partial x} + 7\dfrac{\partial z}{\partial y} = x^2 + y + 26$

(iii) $\dfrac{\partial^4 y}{\partial x^4} + c^4\dfrac{\partial^2 y}{\partial t^2} = a \sin qx \sin pt$

To solve (iii) proceed as in Art. 81, case (*c*). In (i) and (ii) it is easier to find the particular integral for each separate term on the right by the method of Ex. 2, Art. 78.

(15) Solve $8\dfrac{\partial^4 z}{\partial x^4} - 20\dfrac{\partial^4 z}{\partial x^3 \partial y} - 18\dfrac{\partial^4 z}{\partial x^2 \partial y^2} + 81\dfrac{\partial^4 z}{\partial x \partial y^3} - 54\dfrac{\partial^4 z}{\partial y^4} = 0$

and find a particular integral of $\dfrac{\partial^4 y}{\partial x^4} + 9\dfrac{\partial^2 y}{\partial t^2} = 4$.

(16) Solve $\dfrac{\partial^2 z}{\partial x^2} = a^2\dfrac{\partial z}{\partial t}$ having given that $z = z_0 \sin pt$ when $x = 0$ for all values of t, and $z = 0$ when x is very large.

(17) Find a solution of the equation in (16) having given that $z = 0$ when t is very large and $z = 0$ where $x = 0$ or $x = l$ for all values of t.

Assuming that $z = z_0$ for all values of x when $t = 0$, find the value of z at any subsequent time and place. State a problem of which this is the solution.

(18) Find the frequencies of the normal modes of motion for a uniform horizontal rod freely supported at the ends when making small vertical vibrations.

(19) Repeat the last Example for the case of a uniform cantilever.

(20) A uniform horizontal rod as in Ex. (18) is subjected to a pulsating load $w_0 \sin \frac{\pi x}{l} \sin pt$ lb per foot run. For what value of p does the amplitude of vibration become excessive? Show that this is the angular frequency of the first normal mode of vibration of the rod.

(21) If the end A of the string in (11) is given a simple harmonic motion $x = d \sin pt$ and the end B is fixed show that $y = \dfrac{\sin \frac{px}{c}}{\sin \frac{pl}{c}} d \sin pt$.

[NOTE. Fit the solution $y = A \sin (pt + \alpha) \sin \frac{p}{c}(x + \beta)$ to the end conditions $x = 0, y = 0$ and $x = l, y = d \sin pt$ at all times.]

(22) A uniform shaft of length l ft whose section has a polar moment of inertia I_s in engineers' units is in a state of torsional vibration. C_0 lb-ft is the couple for unit twist and θ radians is the angle of twist at a section distant x ft from a fixed section. Show that $\frac{\partial^2\theta}{\partial t^2} = c^2 \frac{\partial^2\theta}{\partial x^2}$ where $c^2 = \frac{C_0 l^2}{I_s}$. Assuming that one end of the shaft is fixed and the shaft is set vibrating by turning the other end through ϕ radians and releasing it after a short pause, find θ in terms of x and t.

(23) Obtain an expression to represent the longitudinal vibrations of the shaft in (22) where w lb is the weight per foot run and E lb is the force per foot stretch. Assume the vibrations to be started by holding the end B displaced through d ft and releasing it at time $t = 0$.

(24) A uniform straight rod of length l ft and total weight wl lb is supported freely at its ends as a horizontal beam. If the beam is set oscillating by being released from a displaced position in the form of a parabola of maximum displacement h at mid-span, find the deflection of any point of the rod at any later time t sec.

(25) Heat flows through a thick slab of material with parallel faces. The slab may be supposed to extend to infinity in directions perpendicular to its thickness. If the axis of x is taken in the direction of the thickness and the temperature is uniform over each face show that the temperature u at the distance x from one face at time t sec is given by $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $c^2 = \frac{k}{ws}$ where k is the coefficient of conductivity, w is the weight of unit volume of the material and s its specific heat.

(26) If the slab in the previous Example is d units thick and the whole slab is initially at temperature θ_0, find the temperature distribution t sec after both faces are brought to, and kept at, zero temperature.

(27) A steady voltage distribution of 20 volts at the sending end and 12 volts at the receiving end is maintained in a telephone wire l miles long. At time $t = 0$ the receiving end is grounded. Find the voltage and the current t sec later. Neglect leakance and inductance.

(28) Show that when finding the frequency of vibration of a mass of W lb weight vibrating at the lower end of a spiral spring of w lb weight where w is small compared with W the spring mass may be neglected if the vibrating mass is supposed to be one of $W + w/3$ lb weight.

(29) Find by an accurate method an expression for the frequency in the last Example and compare it with that given by the approximate method and with that found by neglecting w.

(30) If the equation $x^2 \dfrac{\partial^2 u}{\partial x^2} + x \dfrac{\partial u}{\partial x} + \dfrac{\partial^2 u}{\partial y^2} = 0$ has a solution of the form $u = XY$, where X and Y are, respectively, functions of x and y only, find the differential equations satisfied by X and Y and solve them when Y involves real trigonometrical functions only.

If $\dfrac{\partial u}{\partial x} = -\cos 2y$ when $x = a$ and u tends to zero as x tends to infinity, find u. [U.L.]

(31) A very long telephone cable is of very high resistance but of negligible inductance and leakance. If a pulsating voltage $E = E_0 \sin pt$ is applied at the sending end, show that the current leads the voltage by nearly 45° everywhere and at all times.

CHAPTER IX

APPLIED MATHEMATICS—LINE, SURFACE AND VOLUME INTEGRALS

86. **Plane Motion of a Rigid Body.** A *rigid body* is one in which the particles are, and remain, at rest relative to one another. If all its particles move parallel to a fixed plane, the body is said to be in *plane motion*. It is convenient to look upon a body in plane motion

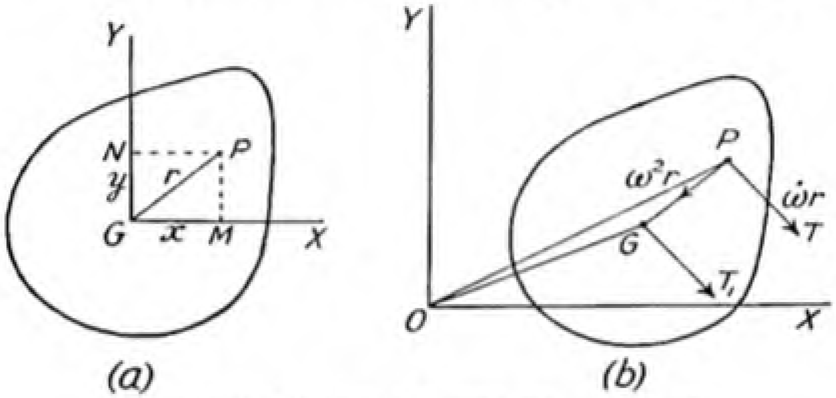


FIG. 60. PLANE MOTION OF A LAMINA

as a lamina, or flat plate, the mass of any particle in the lamina being assumed to be the sum of the masses of the particles of the body whose orthogonal projections on the lamina coincide in positions with that particle. We saw in Vol. I that in the motion of the lamina the velocities, but not the accelerations, of all its particles at any given instant of time are the same as they would be if the lamina were rotating at an appropriate speed about a fixed point called the instantaneous centre of rotation. As the lamina moves, the position of this point changes continuously relative to the lamina as well as to the fixed plane in which the lamina moves. The locus of the instantaneous centre on the lamina is known as the body-centrode and that on the fixed plane the space-centrode. The motion of the lamina may be regarded as due to the rolling, without slipping, of the body-centrode on the space-centrode.

In Fig. 60 (*a*) we represent a plane lamina of which G is the centre of mass and P is the position of any particle, mass m engineers' units. Referred to GX, GY as rectangular axes of reference the co-ordinates of P are $\overline{GM} = x$ ft and $\overline{GN} = y$ ft. Let $\overline{GP} = r$ ft.

If now we multiply each position vector $\overrightarrow{GP}$ by the mass at P, then the sum of the resulting products over the whole of the lamina will be $\Sigma m\overrightarrow{GP}$, and, since $\overrightarrow{GP} = \overrightarrow{GM} + \overrightarrow{GN}$, $\Sigma m\overrightarrow{GP} = \Sigma m\overrightarrow{GM} + \Sigma m\overrightarrow{GN}$.

The magnitude of the vector $\Sigma m\overrightarrow{GM}$ is Σmx and that of $\Sigma m\overrightarrow{GN}$ is Σmy, and $\Sigma mx = \Sigma my = 0$, since G is the centre of mass.

Thus
$$\Sigma m\overrightarrow{GP} = 0 \qquad . \quad . \quad . \quad \text{(IX.1)}$$

Assume now that the body is in plane motion (Fig. 60 (b)) in which at a given instant of time t sec the acceleration of G is $\mathbf{A}_G$, that of P is $\mathbf{A}_P$, and that of P relative to G is ${}_G\mathbf{A}_P$.

Then we have vectorially

Acceleration of P = acceleration of G

+ acceleration of P relative to G

or
$$\mathbf{A}_P = \mathbf{A}_G + {}_G\mathbf{A}_P \qquad . \quad . \quad . \quad \text{(IX.2)}$$

If the angular speed of the lamina is ω radn/sec in the clockwise sense about G, then ${}_G\mathbf{A}_P$ has two components, $\omega^2 r$ ft/sec² from P towards G and $r\dfrac{d\omega}{dt} = \dot{\omega}r$ through P along the perpendicular PT to PG. The first component requires for its production a force $m\omega^2 r = m\omega^2\overrightarrow{PG}$ lb, and the second component requires a force $m\dot{\omega}r = m\dot{\omega}\overrightarrow{PT}$, where $\overline{PT} = \overline{PG}$. This second component is equivalent to a force $m\dot{\omega}\overrightarrow{GT_1}$ at G, GT_1 being equal and parallel to PT, and a couple of magnitude $m\dot{\omega}r \times r = m\dot{\omega}r^2$ lb-ft about an axis perpendicular to plane XOY. Thus, to accelerate the particle m at P there must be (1) a force $m\mathbf{A}_G$, (2) a force $m\omega^2\overrightarrow{PG}$, (3) a force $m\dot{\omega}\overrightarrow{GT_1}$, and (4) a couple $m\dot{\omega}r^2$. The forces (1), (2) and (3) all act through G. Summing these quantities for all the particles of the moving lamina, we have the following three forces all of which pass through G.

(1) $\Sigma m\mathbf{A}_G = \mathbf{A}_G\Sigma m = M\mathbf{A}_G$ lb, where $M = \Sigma m$ is the total mass;

(2) $\Sigma m\omega^2\overrightarrow{PG} = \omega^2\Sigma m\overrightarrow{PG} = 0$, by (IX.1);

(3) $\Sigma m\dot{\omega}\overrightarrow{GT_1} = \dot{\omega}\Sigma m\overrightarrow{GT_1} = 0$, since the system of vectors $\Sigma m\overrightarrow{GT_1}$ is the system $\Sigma m\overrightarrow{PG}$ turned through 90°, and this latter system is equal to zero; together with

(4) a couple of moment $\Sigma m\dot{\omega}r^2 = \dot{\omega}\Sigma mr^2 = I\dot{\omega}$ lb-ft about an axis perpendicular to plane of lamina, I being the polar moment of inertia of the lamina about G.

It is seen that these four quantities reduce to (a) the force needed to produce the acceleration of a particle at G of mass $M = \Sigma m$ whose motion is the same as that of G, and (b) a couple $I\dot{\omega}$ about an axis perpendicular to plane of lamina. *If* $\mathbf{F}$ is the resultant force acting on the lamina and C is the total moment of F about G, then

$$\mathbf{F} = \text{mass} \times \text{acceleration}$$

i.e.
$$\mathbf{F} = M\mathbf{A}_G \qquad . \quad . \quad . \quad . \quad . \qquad \text{(IX.3)}$$

and
$$\text{Couple} = \text{moment of inertia} \times \text{angular acceleration}$$

i.e.
$$C = I\dot{\omega} = I\ddot{\theta} \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(IX.4)}$$

where θ is the angle between any line fixed in the lamina and a line fixed in the plane in which the lamina moves. We treat C as a scalar since the direction of its axis is fixed.

If impulsive forces act on the lamina, i.e. if F is large and the time of action very short, then, since $\mathbf{A}_G = \frac{d}{dt}\mathbf{V}_G = \dot{\mathbf{V}}_G$, where $\mathbf{V}_G$ is the velocity of G, we have

$$\int_0^t \mathbf{F}dt = \int_0^t M\mathbf{A}_G dt = \int_0^t M\frac{d}{dt}\mathbf{V}_G dt$$

i.e.
$$\int_0^t \mathbf{F}dt = M(\mathbf{V}_2 - \mathbf{V}_1)$$

The left-hand side of this relation is the product of the time average of the force and the time during which it acts, and is called the impulse of the force. On the right-hand side $\mathbf{V}_1$ is the velocity of G before impact and $\mathbf{V}_2$ is that after impact. Since the time of impact is very short, the direction of $\mathbf{F}$ does not change during impact, and the sense and direction of the impulse are the same as those of $\mathbf{F}$. Writing $\mathbf{P}$ for the impulse $\int_0^t \mathbf{F}dt$, we have

$$\mathbf{P} = M(\mathbf{V}_2 - \mathbf{V}_1) \quad . \quad . \quad . \quad . \qquad \text{(IX.5)}$$

or
$$\text{Impulse} = \text{change of linear momentum}$$

Similarly, if K is the impulsive couple, i.e. if $K = \int_0^t Cdt$, we have

$$K = I(\omega_2 - \omega_1) \quad . \quad . \quad . \qquad \text{(IX.6)}$$

or
$$\text{Impulsive couple} = \text{change of angular momentum}$$

where $I\omega$ represents the angular momentum, or moment of momentum, of the lamina when rotating at an angular speed ω about G.

We see from the above that, when the lamina is moving under the action of non-impulsive forces, we may consider the system of forces and couples acting on it as equivalent to the inertia force needed to accelerate the whole mass assumed concentrated at its centre of mass, together with the couple needed to produce the angular acceleration of the lamina about its centre of mass. If the inertia force and couple are reversed in sense, they will balance the external forces and couples, and the problem of determining the relation between the inertia effects and the applied forces becomes one of statics. This is also true in cases where the acting forces and couples are impulsive, except that the statement must be altered to allow for the fact that finite changes in the motion take place practically instantaneously. In such a case we say that the system of applied impulsive forces acting on the lamina is equivalent to the single force through its centre of mass needed to produce the change of velocity of a particle at that point whose mass is that of lamina, together with a couple needed to produce the change of the angular momentum of the lamina about its centre of mass. We shall solve examples on impulsive forces by reversing the inertia forces and determining the unknowns by statical methods.

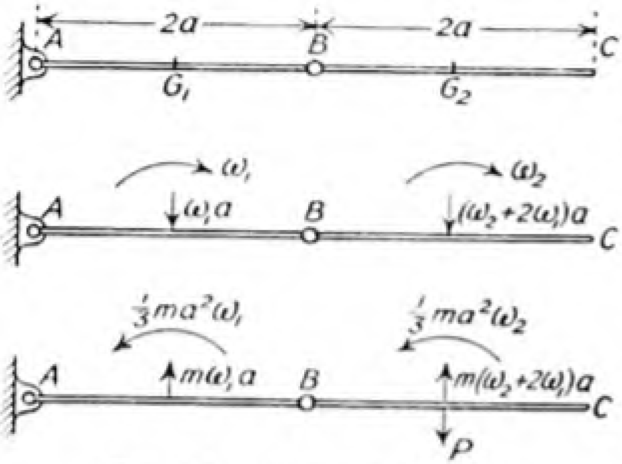


FIG. 61. IMPULSIVE FORCES

EXAMPLE 1

Two uniform thin rigid rods AB, BC, of equal length and mass are freely hinged together at B, and the extremity A is freely hinged to a fixed point. The rods are at rest in a straight line when an impulse is applied at the mid-point of BC perpendicular to the rods. Show that the initial angular velocity of the rod BC is double that of the rod AB. Find also the initial kinetic energies of the rods. (U.L.)

Let the mass of each rod be m and the length $2a$, and let G_1 and G_2 be the respective centres of mass. The top diagram in Fig. 61 shows the rods just before impact, and the middle diagram shows the motion just after impact. ω_1 and ω_2 are the angular speeds of AB and BC respectively, both assumed to be clockwise. The velocity of G_1 is $\omega_1 a$ and that of G_2 is $(\omega_2 + 2\omega_1)a$. The moment

of inertia of each rod about its mid-point is $\frac{1}{3}ma^2$. The bottom diagram shows the changes of linear and angular momentum, with their directions reversed, and the impulse P. The equations of equilibrium are now formed just as if ABC were a beam acted on by the given forces and couples. First considering the rod BC and taking moments about B, we have

$$Pa = \tfrac{1}{3}ma^2\omega_2 + m(\omega_2 + 2\omega_1)a^2 \quad . \quad . \quad . \quad . \quad (1)$$

Considering next both rods and taking moments about A, we have

$$P \times 3a = \tfrac{1}{3}ma^2(\omega_1 + \omega_2) + m\omega_1 a^2 + m(\omega_2 + 2\omega_1)a \times 3a$$

i.e.
$$3Pa = \tfrac{22}{3}\, ma^2\omega_1 + \tfrac{10}{3}\, ma^2\omega_2 \quad . \quad . \quad . \quad . \quad (2)$$

Eliminating P from (1) and (2), we obtain

$$ma^2\omega_2 + 3m(\omega_2 + 2\omega_1)a^2 = \tfrac{22}{3}\, ma^2\omega_1 + \tfrac{10}{3}\, ma^2\omega_2, \text{ which leads to}$$

$$\omega_2 = 2\omega_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Thus, the initial angular speed of BC is twice that of AB, both in the same sense.

Let E_1 and E_2 be the respective initial kinetic energies of AB and BC.

Then $\qquad E_1 = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}mv_1^2 \quad \text{and} \quad E_2 = \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}mv_2^2$

where $I_1 = I_2 = \frac{1}{3}ma^2$, $v_1 = \omega_1 a$, and $v_2 = (\omega_2 + 2\omega_1)a = 4\omega_1 a$, by (3).

Hence,
$$\frac{E_1}{E_2} = \frac{\frac{1}{6}ma^2\omega_1^2 + \frac{1}{2}ma^2\omega_1^2}{\frac{1}{6}ma^2 \times 4\omega_1^2 + \frac{1}{2}m \times 16a^2\omega_1^2} = \frac{\frac{1}{6} + \frac{1}{2}}{\frac{2}{3} + 8} = \frac{1}{13}$$

which is the ratio of the initial kinetic energies.

From (3) and (1), ω_1 and ω_2 can be expressed in terms of P. If the upward impulsive reaction of AB on BC at B is P_B, then by considering the vertical forces acting on BC we have $P = P_B + m(\omega_2 + 2\omega_1)a$, from which P_B can be found. The impulsive reaction at A can be found similarly.

EXAMPLE 2

A lamina is moving in its own plane. Determine the changes in the motion of the lamina if it is acted upon by an impulse whose line of action is in the plane of the lamina.

A uniform thin rod AB, of mass M, has a particle of mass m fixed to it at B. The rod is spinning on a smooth horizontal table with angular speed ω about a smooth fixed pivot at A. Suddenly A is released and the mid-point C of AB is fixed. Determine the new angular speed of the rod and the impulsive reaction at C. (U.L.)

The changes in the motion of the lamina are explained above. Let $2a$ be the length of the rod and ω' its new angular speed. The top diagram in Fig. 62 shows the motion before impact. The centre of mass C has a velocity ωa perpendicular to the rod, and the rod rotates about C at angular speed ω. The particle at B has a velocity $2\omega a$ perpendicular to the rod. The middle diagram shows the speeds after impact, ω' being the new angular speed. The point C is at rest, and B is moving perpendicular to the rod with velocity $\omega' a$. The inertia effects due to the fixing of C are changes of linear momentum $m\omega' a - 2m\omega a$ at B and $M(0 - \omega a) = -M\omega a$ at C, and a couple $\frac{1}{3}Ma^2(\omega' - \omega)$ clockwise. These

inertia effects are shown reversed in the bottom diagram, where they balance P, the impulse on the bar due to fixing C. Again by statical methods we have, resolving perpendicular to the rod,

$$P + ma(\omega' - 2\omega) = M\omega a$$

i.e.
$$P = M\omega a - ma(\omega' - 2\omega) \quad . \quad . \quad . \quad (1)$$

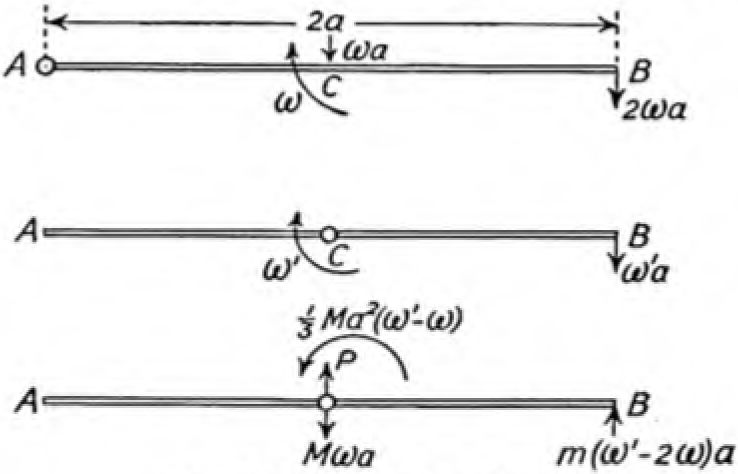


FIG. 62. SUDDEN STOPPAGE

Taking moments about C, we have

$$\tfrac{1}{3}Ma^2(\omega' - \omega) + ma^2(\omega' - 2\omega) = 0$$

i.e.
$$\omega'(\tfrac{1}{3}M + m) = \omega(\tfrac{1}{3}M + 2m)$$

whence
$$\omega' = \frac{M + 6m}{M + 3m}\omega \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (2) and (1),
$$P = \omega(Ma + 2ma) - ma\omega'$$

$$= \omega a(M + 2m) - ma\omega\frac{M + 6m}{M + 3m}, \text{ which gives}$$

$$P = \frac{M(M + 4m)}{M + 3m}\omega a \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is the impulsive reaction at C.

EXAMPLE 3

Equal uniform bars PQ, QR, each of mass m, freely jointed at Q, lie at rest on a smooth horizontal table, inclined to each other at an obtuse angle $\pi - \alpha$. A horizontal blow of impulse I is applied at P in the direction perpendicular to PQ. Show that the velocity given to P has a component along PQ of magnitude

$$\frac{6I \sin \alpha \cos \alpha}{m(16 + 9 \sin^2 \alpha)} \qquad \text{(U.L.)}$$

Let $2a$ be the length of each bar, and G_1, G_2 their centres of mass. The top diagram in Fig. 63 shows the bars at rest before impact. The middle diagram shows the motion just after impact. Let v_1 and v_2 be the component velocities of

Q in directions perpendicular to PQ and along QP respectively, and ω_1 counter-clockwise and ω_2 clockwise the angular speeds of PQ and QR respectively. The components of the linear velocity of G_1 are $v_1 + \omega_1 a$ perpendicular to PQ and v_2 towards P. Those of G_2 are v_1 in direction perpendicular to PQ, v_2 parallel to QP due to the motion of Q, and $\omega_2 a$ perpendicular to QR due to the motion of

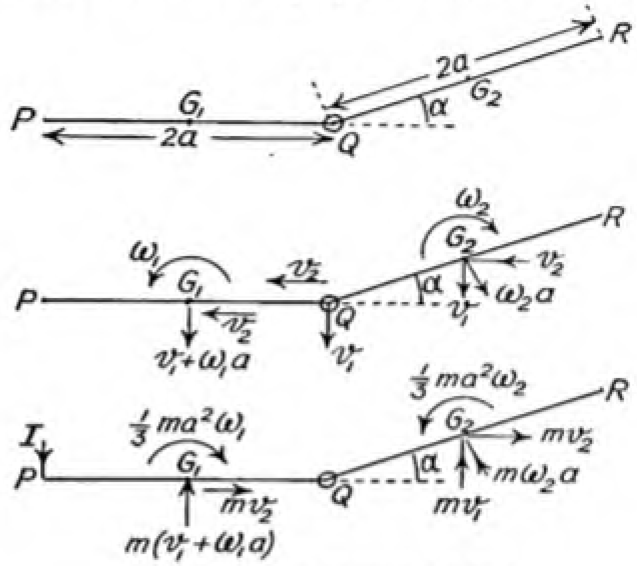


FIG. 63. IMPULSIVE BLOW

QR relative to Q. In addition we have the angular speeds ω_1 and ω_2 about G_1 and G_2 respectively. The bottom diagram shows the inertia effects reversed, these being in equilibrium with I.

Resolving along PQ, we have

$$m\omega_2 a \sin\alpha = 2mv_2$$

i.e.
$$a\omega_2 \sin\alpha = 2v_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Considering the bar QR, and taking moments about Q, we have

$$mv_2 a \sin\alpha = mv_1 a \cos\alpha + m\omega_2 a^2 + \tfrac{1}{3}m\omega_2 a^2$$

i.e.
$$3v_2 \sin\alpha = 3v_1 \cos\alpha + 4\omega_2 a \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Resolving perpendicular to PQ for whole system, we have

$$I = m(v_1 + \omega_1 a) + mv_1 + m\omega_2 a \cos\alpha$$

i.e.
$$I = m(2v_1 + \omega_1 a + \omega_2 a \cos\alpha) \quad . \quad . \quad . \quad . \quad (3)$$

Taking moments about Q for the bar PQ, we have

$$2aI = \tfrac{1}{3}ma^2\omega_1 + m(v_1 + \omega_1 a)a$$

whence
$$6I = m(4a\omega_1 + 3v_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From these four equations the unknown quantities v_1, v_2, ω_1, ω_2, can be found. Eliminating v_2 from (1) and (2), and simplifying, we have

$$3\omega_2 a \sin^2\alpha = 6v_1 \cos\alpha + 8\omega_2 a$$

whence
$$v_1 = \frac{\omega_2 a}{6\cos\alpha}(3\sin^2\alpha - 8) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Eliminating ω_1 by multiplying (3) by 4 and subtracting from (4), we have

$$2I = -5mv_1 - 4m\omega_2 a \cos\alpha$$

and on substituting for v_1 from (5), this becomes

$$2I = -5m\frac{\omega_2 a}{6\cos\alpha}(3\sin^2\alpha - 8) - 4m\omega_2 a\cos\alpha$$

$$= m\omega_2 a\left(\frac{40 - 15\sin^2\alpha}{6\cos\alpha} - 4\cos\alpha\right)$$

$$= \frac{m\omega_2 a}{6\cos\alpha}(40 - 15\sin^2\alpha - 24\cos^2\alpha)$$

$$= \frac{m\omega_2 a}{6\cos\alpha}(16 + 9\sin^2\alpha)$$

Hence, $$\omega_2 = \frac{12\,I\cos\alpha}{ma(16 + 9\sin^2\alpha)}$$

From (1), $$v_2 = \frac{a\sin\alpha}{2}\,\omega_2 = \frac{6I\sin\alpha\cos\alpha}{m(16 + 9\sin^2\alpha)}$$

which is the required component.

EXAMPLE 4

A uniform thin bar whose centre of gravity is G is of length $2a$ and rests horizontally on two props at points distant b and c from G. If either prop is suddenly removed, show that the load on the other is instantaneously increased or decreased according as $a^2 > 3bc$ or $a^2 < 3bc$. (U.L.)

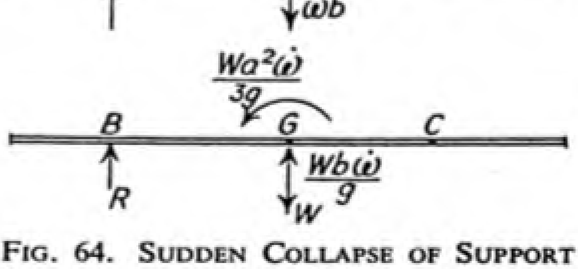


FIG. 64. SUDDEN COLLAPSE OF SUPPORT

The top diagram in Fig. 64 shows the bar on its supports at B and C. $\overline{BG} = b$ and $\overline{GC} = c$. Length of bar $= 2a$. Weight of bar $= W$ lb. I_G of rod $= \frac{Wa^2}{3g}$ engineers' units, the length unit being a foot. The initial pressures on the props

at B and C are respectively $\frac{cW}{b+c}$ lb and $\frac{bW}{b+c}$ lb. The middle diagram shows the initial motion when the prop is removed from C. Let ω be the initial angular velocity of BC in radn/sec. Then the initial linear acceleration of B is $\frac{d}{dt}(\omega b)$ $= \dot{\omega}b$ ft/sec² downwards. The couple required to produce the angular acceleration is $I_G\dot{\omega} = \frac{Wa^2\dot{\omega}}{3g}$ lb-ft, and the force required to produce the linear acceleration is $\frac{W\dot{\omega}b}{g}$ lb. These reversed are shown in the bottom diagram along with the weight W lb through G and the upward reaction R lb of the prop at B, the whole forming a system in equilibrium.

Resolving vertically, we have

$$R = W - \frac{W\dot{\omega}b}{g} \qquad \qquad (1)$$

Taking moments about B,

$$Wb = \frac{Wa^2\dot{\omega}}{3g} + \frac{Wb^2\dot{\omega}}{g}$$

whence

$$\dot{\omega} = \frac{3gb}{a^2 + 3b^2} \qquad \qquad (2)$$

Substituting from (2) in (1), we obtain

$$R = W\left(1 - \frac{3b^2}{a^2 + 3b^2}\right) = \frac{Wa^2}{a^2 + 3b^2}$$

Hence, increase of pressure on prop at $B = R - \frac{cW}{b+c}$

$$= W\left(\frac{a^2}{a^2 + 3b^2} - \frac{c}{b+c}\right)$$

$$= W\frac{b(a^2 - 3bc)}{(a^2 + 3b^2)(b+c)}$$

This expression is positive if $a^2 > 3bc$ and negative if $a^2 < 3bc$, so that the pressure is increased or decreased according as $a^2 > 3bc$ or $a^2 < 3bc$. This also applies to the case in which the prop at B is removed and the pressure is that on the prop at C, for an interchange of the values b and c leaves the condition unchanged.

87. **Equivalent Dynamical Systems.** We have seen that the motion of a plane lamina in its own plane is such that the resultant force on it is equal to the product of the mass and the acceleration of its centre of mass and that the couple acting on it is equal to the product of its moment of inertia about the mass-centre and its angular acceleration. Suppose now we have a second lamina, which we will consider to be rigid but without mass, and which has attached to it

two heavy particles the sum of whose masses is equal to the mass of the first lamina, whose centre of mass coincides with that of the first lamina, and whose moment of inertia about the common mass-centre is the same as that of the first lamina. When we apply relations (IX.3) and (IX.4) to these two systems in turn we obtain identical solutions, which indicates that a system of forces acting on either the lamina or the system of two particles will produce the same rate of change of motion. It is often convenient to consider the motion of the two particles rather than that of a rigid body, and the former is known as an *equivalent dynamical system* to the rigid body. In Fig 65 (*a*), L represents the lamina whose mass-centre is at G, its mass is M engineers' units and its radius of gyration about G is k feet. Let m_1 at P_1 and m_2 at P_2 be the equivalent system of particles. P_1P_2 must pass through G but its direction is arbitrary. Let $\overline{GP_1} = p_1$ and $\overline{P_2G} = p_2$, both in feet. Then we have, since

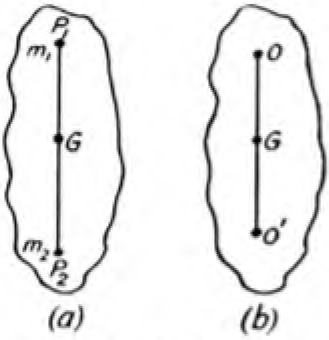


FIG. 65. EQUIVALENT DYNAMICAL SYSTEMS

G is the mass centre $\qquad m_1p_1 = m_2p_2 \quad . \quad . \quad . \quad$ (IX.7)

the total mass is $M \qquad m_1 + m_2 = M \quad . \quad . \quad . \quad$ (IX.8)

the moment of inertia about G is Mk^2

$$\therefore \qquad m_1p_1^2 + m_2p_2^2 = Mk^2 \quad . \quad . \quad . \quad \text{(IX.9)}$$

From (IX.7) and (IX.8)

$$m_1 = \frac{p_2}{p_1 + p_2} M \text{ and } m_2 = \frac{p_1}{p_1 + p_2} M \quad . \quad \text{(IX.10)}$$

Substituting these in (IX.9) and dividing through by M

$$\frac{p_1^2p_2 + p_1p_2^2}{p_1 + p_2} = k^2$$

i.e.

$$p_1p_2 = k^2 \quad . \quad . \quad . \quad \text{(IX.11)}$$

We see then that the particles must be placed on a line through G but on opposite sides of it at distances inversely proportional to their masses and so that the product of the distances is the square of the radius of gyration. As there are four unknowns m_1, m_2, p_1 and p_2 to be determined and only three conditions to be satisfied, we may give any suitable value to one of the unknowns. Thus if the lamina is oscillating as a compound pendulum about a horizontal axis

through O, Fig. 65 (b), we take one particle m_1 at O and the other at a point O' on OG produced. From (IX.11) $p_2 = \dfrac{k^2}{p_1}$ and $\overline{OO'} = p_1 + p_2 = \dfrac{k^2 + p_1^2}{p_1}$. One particle is thus fixed in position at O and the other moves as would the bob of a simple pendulum of length $OO' = \dfrac{k^2 + p_1^2}{p_1}$. Compare this with Ex. 5, Art 138, Vol. I.

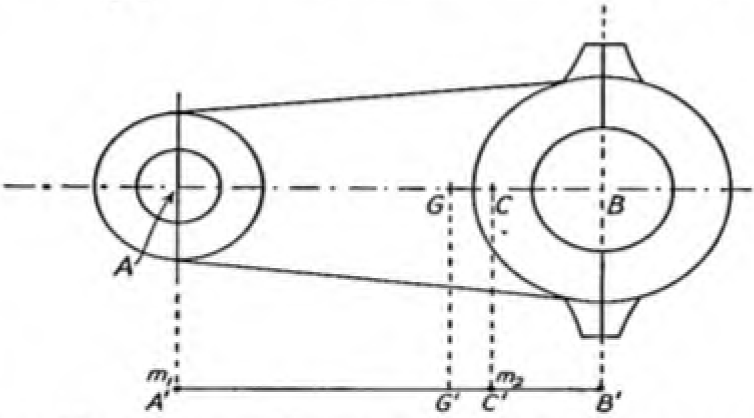


FIG. 66. PARTICLES EQUIVALENT TO CONNECTING-ROD

O' is called the *centre of oscillation* of the compound pendulum and also the *centre of percussion*.

In the case of a connecting-rod, or a link in a mechanism, it is convenient to assume one particle to be at one end. In the connecting-rod AB, Fig. 66, we place one particle, say m_1, at the small end A. Let the length AB be 3 ft and the centre of mass be G on AB, 2 ft from A. Suppose the mass to be 5 engineers' units, its weight being $5g = 161$ lb. Let C on GB be the position of the particle m_2. The particles are shown at A', C' in the lower figure; B' and G' are the projections of B and G respectively. Let the radius of gyration of the rod be $k = 0{\cdot}75$ ft. Then with the above notation $p_1 = 2$ and $p_2 = \dfrac{k^2}{p_1} = \frac{9}{32}$ ft.

Also $$2m_1 = \tfrac{9}{32}\, m_2$$

$\therefore$ $$m_1 = \tfrac{9}{64}\, m_2$$

Hence from (IX.10), $m_1 = \frac{9}{73} M = \frac{45}{73} = 0{\cdot}6164$ engineers' units

and $\quad m_2 = \frac{64}{73} M = \frac{320}{73} = 4{\cdot}384$ engineers' units.

Thus two particles, one of mass 0·616 units at A and one of mass 4·384 units at C when $\overline{GC} = \frac{9}{32}$ ft beyond G are an equivalent

dynamical system to the rod. The accelerations of these points are found by means of an acceleration image, see *Theory of Machines*, and the mass-accelerations of the points are the forces acting on the particles. The resultant of these is the resultant force acting on the rod.

EXAMPLE

Find a two-particle system corresponding to a bar weighing 500 lb with a radius of gyration about its mass-centre of 2 ft, pivoted at a point 6 in. from the mass-centre. Assume one mass to be at the pivot. Find the force acting through the mass-centre which till give to the bar an angular acceleration of 12 radn/sec².

We have $p_1 = 0{\cdot}5$, $k^2 = 4$. Hence from (IX.11), $p_2 = 4/0{\cdot}5 = 8$. Then from (IX.10) $m_2 = 1/17 \times 500 = 29{\cdot}4$ lb mass and $m_1 = 470{\cdot}6$ lb mass. The equivalent system has one particle weighing 471 lb at the pivot and one weighing 29·4 lb distant 8 ft from the centre of mass and on the other side of it. The particle at the pivot has zero acceleration and the normal acceleration of the other is $12(p_1 + p_2) = 12 \times 8{\cdot}5 = 102$ ft/sec². The force on the particle to produce this acceleration is $\dfrac{29{\cdot}4}{g} \times 102 = 93{\cdot}2$ lb perpendicular to the bar. As the force is to be applied at the mass-centre the necessary force is $\dfrac{93{\cdot}2}{0{\cdot}5} \times 8{\cdot}5 = 1\,580$ lb very nearly.

Second Moment of Area. When finding the second moment of a plane area which can be divided up into triangles we may make use of a fictitious system of particles. The second moment of a plane triangle of area A and height h about its base is $\frac{1}{6}Ah$. If we suppose the area divided into three equal portions $\frac{1}{3}A$ and one of these concentrated as a "particle" at the mid-point of each side, the second moment of the particle system about the base is

$$\tfrac{1}{3}A\left(\frac{h}{2}\right)^2 + \tfrac{1}{3}A\left(\frac{h}{2}\right)^2 + \tfrac{1}{3}A \times 0^2 = \tfrac{1}{6}Ah$$

the same as that of the triangle. It is easy to see that the three "particles" have the same centroid as has the triangle so that the particles have the same second moment as the triangle about any axis in the plane parallel to the base. As we may take any one of the three sides as base, this means that the momental ellipse of the area and that of the particle system have the same centre and intersect in 6 points and must therefore coincide entirely (see Vol. I). Thus the system of particles and the area have the same second moment about any point in their plane and are therefore equivalent for finding second moments.

If the area is a lamina of uniform density, w lb being the weight per square foot, and the particles are each assumed to weigh $\dfrac{wA}{3}$ lb, the particles and the lamina are equivalent dynamical systems.

EXAMPLE

Find the second moment of the area of a square of side a about any axis in its plane through its centre. Fig. 67 shows the square and the particles each of magnitude $\frac{1}{6}a^2$.

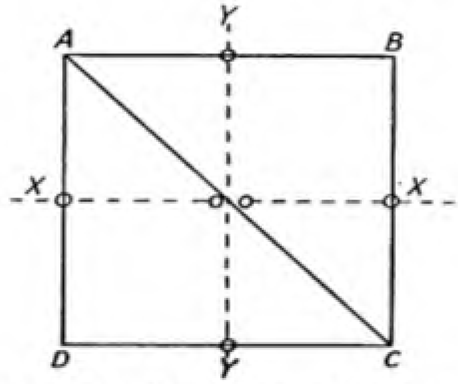


FIG. 67. EQUIVALENT SYSTEMS

We have—

$$\text{second moment about either diagonal} = 4 \times \tfrac{1}{6}a^2 \times \left(\frac{\sqrt{2}a}{4}\right)^2 = \tfrac{1}{12}a^4$$

$$\text{also} \quad \text{second moment about } XX \text{ or } YY = 2 \times \tfrac{1}{6}a^2 \times \left(\frac{a}{2}\right)^2 = \tfrac{1}{12}a^4$$

The second moment about any of the four axes is the same, i.e. $\frac{1}{12}a^4$, hence the momental ellipse is a circle and the second moment is the same about all axes in the plane which pass through the centroid.

88. **Beam Flexure. Clapeyron's Theorem of Three Moments.** When a beam is continuous over three or more supports, it is possible to find a relation between the bending moments at any three consecutive supports in terms of the loads and the relative levels of the supports. This relation, first stated by Clapeyron, is given the above name. In Vol. I we proved the relation

$$EI\frac{d^2y}{dx^2} = M \qquad . \quad . \quad . \quad \text{(IX.12)}$$

where M is the bending moment at a point (x, y) on the axis of the beam. The reader should refer to Vol. I for the meanings of the symbols and the conventions concerning algebraic signs. Consider the beam AB shown in the upper diagram of Fig. 68. It rests on supports at A and B, and carries a load of w lb per in. uniformly distributed and also a concentrated load W lb at C distant e in. from A and d in. from B. Let $\overline{AB} = l = e + d$, and let M, F, i, y represent the bending moment, shear force, slope, and "deflection"

below OX respectively at E whose co-ordinates are x and y referred to any horizontal line OX as axis of x and OY drawn vertically downwards and passing through A as axis of y. A letter A, C, or B used as a subscript with any of the above symbols will indicate the particular value of the variable at that point. We shall assume M to be positive when it tends to make the curvature convex upwards and F to be positive when it tends to move the right-hand portion

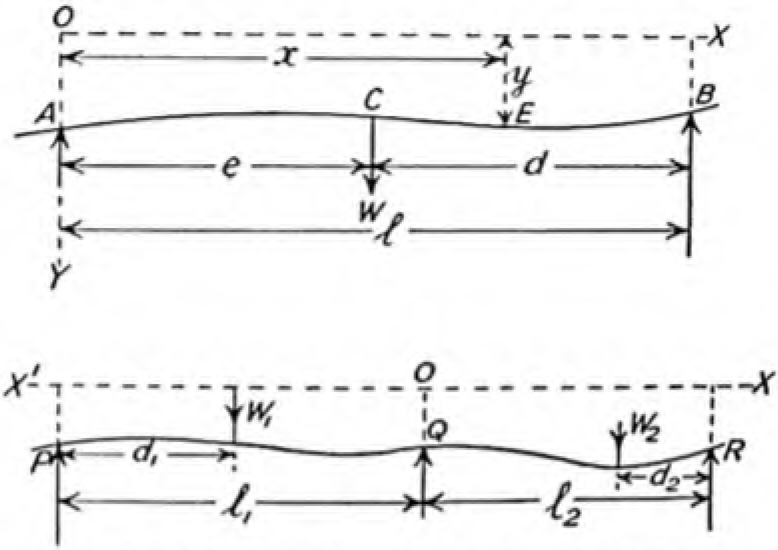


Fig. 68. Theorem of Three Moments

of the beam downwards relative to the other portion. From (IX.12)

$$EI\frac{d^2y}{dx^2} = \text{bending moment at } E$$

i.e.
$$EI\frac{d^2y}{dx^2} = M_A - F_Ax + \frac{wx^2}{2} + [W(x-e)] \quad . \quad \text{(IX.13)}$$

the square brackets indicating that the quantity inside must be ignored if $x < e$.

Integrating with respect to x

$$EI\frac{dy}{dx} = EIi_A + M_Ax - \tfrac{1}{2}F_Ax^2 + \frac{wx^3}{6} + \left[\frac{W}{2}(x-e)^2\right]. \quad \text{(IX.14)}$$

EIi_A is the constant of integration. In integrating the term in square brackets we have done so with respect to $x - e$ instead of x. This is permissible because $\int f(x)dx$ and $\int f(x)d(x-e)$ differ only by a constant. This is known as *Macaulay's Method*, and enables us to integrate in one step the discontinuous function in (IX.13).

Integrating again

$$EIy = EIy_A + EIi_A x + \tfrac{1}{2}M_A x^2 - \tfrac{1}{6}F_A x^3 + \frac{wx^4}{24} + \left[\frac{W}{6}(x-e)^3\right] \tag{IX.15}$$

(IX.14) and (IX.15) may be used to find the slope and deflection at any point of AB if y_A and i_A are known. Putting $x = l$, (IX.15) gives

$$EI(y_B - y_A) = EIi_A l + \tfrac{1}{2}M_A l^2 - \tfrac{1}{6}F_A l^3 + \frac{wl^4}{24} + \frac{W}{6}d^3 \quad . \tag{IX.16}$$

Now taking moments about B of the forces and couples acting on AB

$$F_A l = M_A - M_B + \frac{wl^2}{2} + Wd \quad . \quad . \tag{IX.17}$$

Substituting for $F_A l$ in (IX.16)

$$EI(y_B - y_A) = EIi_A l + \tfrac{1}{2}M_A l^2 - \tfrac{1}{6}l^2\left(M_A - M_B + \frac{wl^2}{2} + Wd\right) + \frac{wl^4}{24} + \frac{W}{6}d^3$$

i.e.

$$EI(y_B - y_A) = EIi_A l + \tfrac{1}{6}l^2(2M_A + M_B) - \frac{wl^4}{24} - \tfrac{1}{6}Wd(l^2 - d^2) \tag{IX.18}$$

We apply this to the continuous beam of which two spans are shown in the lower diagram in Fig. 68. PQ has a span l_1 in. and carries a load of w_1 lb per in. run together with a concentrated load W_1 lb at d_1 in. from P, while QR has a span l_2 in. and carries a load of w_2 lb per in. run together with a concentrated load W_2 lb at d_2 in. from R. First consider the portion PQ. Let OX' be the axis of x, in this case drawn to the left, so that owing to this reversal of sense i and F will change sign. Applying (IX.18) to the portion PQ we have,

$$EI(y_P - y_Q) = -EIi_Q l_1 + \tfrac{1}{6}l_1^2(2M_Q + M_P) - \frac{w_1 l_1^4}{24} - \tfrac{1}{6}W_1 d_1(l_1^2 - d_1^2) \quad . \tag{IX.19}$$

Similarly for the portion QR with OX as the axis of x, we have

$$EI(y_R - y_Q) = EIi_Q l_2 + \tfrac{1}{6}l_2^2(2M_Q + M_R) - \frac{w_2 l_2^4}{24} - \tfrac{1}{6}W_2 d_2(l_2^2 - d_2^2) \quad . \quad \text{(IX.20)}$$

Dividing (IX.19) and (IX.20) through by l_1 and l_2 respectively and adding, we obtain

$$EI\left(\frac{y_P - y_Q}{l_1} + \frac{y_R - y_Q}{l_2}\right)$$
$$= \tfrac{1}{6}(M_P l_1 + 2M_Q\overline{l_1 + l_2} + M_R l_2) - \tfrac{1}{24}(w_1 l_1^3 + w_2 l_2^3)$$
$$- \tfrac{1}{6}\left\{\frac{W_1 d_1}{l_1}(l_1^2 - d_1^2) + \frac{W_2 d_2}{l_2}(l_2^2 - d_2^2)\right\}$$

i.e. $M_P l_1 + 2M_Q(l_1 + l_2) + M_R l_2$

$$= \tfrac{1}{4}(w_1 l_1^3 + w_2 l_2^3) + \frac{W_1 d_1}{l_1}(l_1^2 - d_1^2) + \frac{W_2 d_2}{l_2}(l_2^2 - d_2^2)$$
$$+ 6EI\left(\frac{y_P - y_Q}{l_1} + \frac{y_R - y_Q}{l_2}\right) \quad . \quad . \quad . \quad \text{(IX.21)}$$

If all the supports are at the same level, then $y_P = y_Q = y_R$ and (IX.21) becomes

$$M_P l_1 + 2M_Q(l_1 + l_2) + M_R l_2 = \tfrac{1}{4}(w_1 l_1^3 + w_2 l_2^3) + \frac{W_1 d_1}{l_1}(l_1^2 - d_1^2) + \frac{W_2 d_2}{l_2}(l_2^2 - d_2^2) \quad . \quad \text{(IX.22)}$$

If there are several concentrated loads in one span, it is easily seen that in place of a term $\frac{Wd}{l}(l^2 - d^2)$ there will appear $\sum \frac{Wd}{l}(l^2 - d^2)$.

If the three supports are at the same level and there are no concentrated loads but simply a uniformly distributed load w lb per in. run, then from (IX.22)

$$M_P l_1 + 2M_Q(l_1 + l_2) + M_R l_2 = \frac{w}{4}(l_1^3 + l_2^3) \quad . \quad \text{(IX.23)}$$

If the length unit is the foot and w is the load per foot length the bending moments are in ft-lb units whilst the units of E and I are the pound per square foot and the (foot)^4.

EXAMPLE 1

Prove the theorem of three moments in the form

$$L_1a + 2L_2(a + b) + L_3b = \frac{w}{4}(a^3 + b^3)$$

A uniform beam of weight w per unit length and of length $a + b$ is clamped horizontally at each end and is propped level with the ends at a point distant a from one end. Show that the bending moment at the prop is

$$\frac{w}{12}(a^2 + b^2 - ab)$$

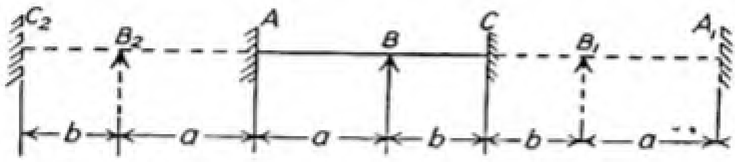


FIG. 69

Find also the thrust on the prop. (U.L.)

Here L is used to indicate bending moment instead of the usual M. With the necessary change of letters the required relation is (IX.23) above. It is left as an exercise for the reader to establish this theorem *ab initio* by the method of this section.

Let ABC be the beam (Fig. 69), $\overline{AB} = a$ and $\overline{BC} = b$. Further, let CB_1A_1 be the image of CBA about C, i.e. $\overline{CB_1} = \overline{CB}$ and $\overline{CA_1} = \overline{CA}$, and similarly let AB_2C_2 be the image of ABC about A. If we imagine AB_2C_2 and CB_1A_1 to be continuations of the beam ABC, each carrying load w per unit length and propped at B_2 and B_1 to the same level as ABC, it is clear from symmetry that

$$M_B = M_{B_1} = M_{B_2}$$

Applying the theorem of three moments to the portions B_2AB, ABC, and BCB_1 in turn, we have

$$2aM_B + 4aM_A = \frac{wa^3}{2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$aM_A + 2(a + b)M_B + bM_C = \frac{w}{4}(a^3 + b^3) \quad . \quad . \quad (2)$$

and

$$2bM_B + 4bM_C = \frac{wb^3}{2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (1),

$$M_A = \frac{wa^2}{8} - \tfrac{1}{2}M_B \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (3),

$$M_C = \frac{wb^2}{8} - \tfrac{1}{2}M_B \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting from (4) and (5) in (2), we obtain

$$\frac{wa^3}{8} - \tfrac{1}{2}aM_B + 2(a + b)M_B + \frac{wb^3}{8} - \tfrac{1}{2}bM_B = \frac{w}{4}(a^3 + b^3)$$

$$\therefore \qquad \frac{3}{2}(a+b)M_B = \frac{w}{8}(a^3+b^3)$$

Hence, $$M_B = \frac{w}{12}(a^2-ab+b^2) = \text{bending moment at the prop}$$

From (4), $$M_A = \frac{wa^2}{8} - \frac{w}{24}(a^2-ab+b^2)$$

i.e. $$M_A = \frac{w}{24}(2a^2+ab-b^2)$$

From (5), $$M_C = \frac{wb^2}{8} - \frac{w}{24}(a^2-ab+b^2)$$

i.e. $$M_C = \frac{w}{24}(2b^2+ab-a^2)$$

Let R_A, R_B, and R_C be the pressures on the supports at A, B, and C respectively. Then, taking moments about B of the actions on AB, we have

$$M_A + \frac{wa^2}{2} = aR_A + M_B$$

$$\therefore \qquad R_A = \frac{M_A - M_B}{a} + \frac{wa}{2} = \frac{wb}{8a}(a-b) + \frac{wa}{2} = \frac{w}{8a}(4a^2+ab-b^2)$$

Similarly, taking moments about B of the actions on BC, we have

$$M_C + \frac{wb^2}{2} = bR_C + M_B$$

$$\therefore \qquad R_C = \frac{M_C - M_B}{b} + \frac{wb}{2} = \frac{wa}{8b}(b-a) + \frac{wb}{2} = \frac{w}{8b}(4b^2+ab-a^2)$$

Resolving vertically, we have

$$R_A + R_B + R_C = w(a+b)$$

$$\therefore \qquad R_B = w(a+b) - \frac{w}{8a}(4a^2+ab-b^2) - \frac{w}{8b}(4b^2+ab-a^2)$$

$$= \frac{w}{8ab}(8a^2b + 8ab^2 - 4a^2b - ab^2 + b^3 - 4ab^2 - a^2b + a^3)$$

$$= \frac{w}{8ab}(a^3 + 3a^2b + 3ab^2 + b^3)$$

i.e. $$R_B = \frac{w(a+b)^3}{8ab} = \text{pressure on prop}$$

EXAMPLE 2

A uniform beam is supported at its ends and carries a uniformly distributed load along the middle half. Show that the additional deflection due to the load is $\frac{57}{64}$ times the additional deflection had the load been concentrated at the mid-point. (U.L.)

Let AD be the beam (Fig. 70), and y the deflection at P where $\overline{AP} = x$, $\overline{AB} = \overline{CD} = a$, and $\overline{BC} = 2a$. Let w be the distributed load per unit length. Then, considering the portion of the beam to the left of P and using Macaulay's method, we have

$$EI\frac{d^2y}{dx^2} = M_P = -\,wax + \left[\frac{w}{2}(x-a)^2\right] \qquad . \quad . \quad . \quad (1)$$

Integrating,
$$EI\frac{dy}{dx} = -\,\tfrac{1}{2}wax^2 + \left[\frac{w}{6}(x-a)^3\right] + k \qquad . \quad . \quad . \quad (2)$$

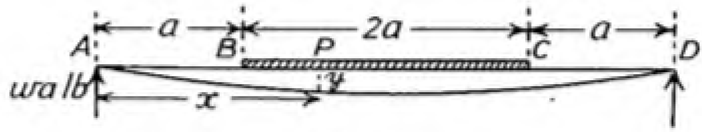


FIG. 70

Since $\dfrac{dy}{dx} = 0$ when $x = 2a$, then by substitution in (2),

$$a = -\,2wa^3 + \tfrac{1}{6}wa^3 + k, \text{ whence } k = \tfrac{11}{6}wa^3$$

Substituting the value of k in (2) and integrating again, we have

$$EIy = -\,\tfrac{1}{6}wax^3 + \left[\frac{w}{24}(x-a)^4\right] + \tfrac{11}{6}wa^3x \qquad . \quad . \quad . \quad (3)$$

the constant of integration being zero since $y = 0$ when $x = 0$.

If Δ is the central deflection, then from (3)

$$EI\Delta = -\,\tfrac{4}{3}wa^4 + \tfrac{1}{24}wa^4 + \tfrac{11}{3}wa^4 = \tfrac{57}{24}wa^4 \qquad . \quad . \quad . \quad (4)$$

The value of Δ_0, the central deflection on the assumption that the whole load is concentrated at mid-span, may be found by the above method, but is best obtained from the standard result

$$EI\Delta_0 = \tfrac{1}{48} \text{ total load} \times (\text{span})^3$$

i.e.
$$EI\Delta_0 = \tfrac{1}{48} \times 2wa \times (4a)^3 = \tfrac{8}{3}wa^4 \qquad . \quad . \quad . \quad . \quad (5)$$

From (4) and (5) by division,

$$\frac{\Delta}{\Delta_0} = \frac{57}{64}, \text{ which is the required ratio.}$$

89. **Field of Force. Conservation of Energy.** A *field of force* is a region in space of two or three dimensions in which a force is associated with each point in the region. Examples of such are the gravitational and magnetic fields of the earth. In Fig. 71, P represents a point in a two-dimensional field of force, and $\boldsymbol{F}$ the force of magnitude F which the field exerts on a certain particle placed at P. Let O be an origin fixed in the field and $\boldsymbol{r} = \overrightarrow{OP}$ the

position vector of P. Let Q be a point very near to P and $\boldsymbol{r}_Q$ its position vector. Then $\boldsymbol{r}_Q = \boldsymbol{r} + \Delta\boldsymbol{r}$, and $\overrightarrow{PQ} = \Delta\boldsymbol{r}$. The work done by $\boldsymbol{F}$ on the particle as it moves from P to Q is $\boldsymbol{F} \cdot \Delta\boldsymbol{r} = F\Delta s \cos \theta$, where $\Delta s = \overline{PQ}$ and θ is the angle between the direction of $\boldsymbol{F}$ and the tangent at P. The total work done on the particle by the force $\boldsymbol{F}$ as the particle moves from P_1 to P_2 along the curve is

$$\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{r=r_1}^{r=r_2} F_c ds \quad . \quad . \quad . \quad \text{(IX.24)}$$

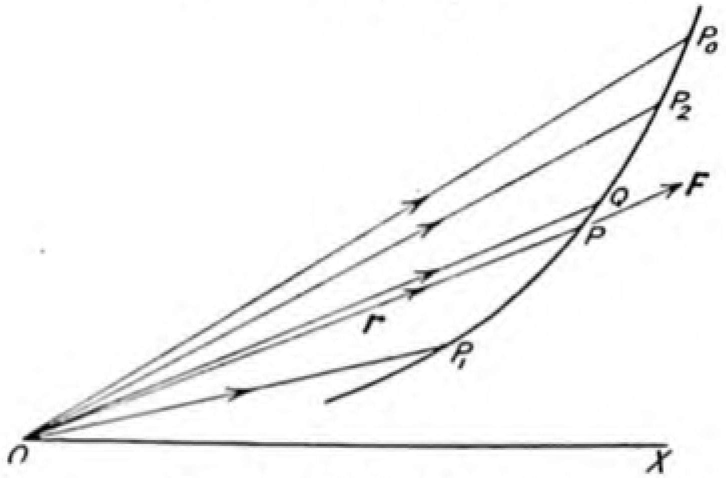


FIG. 71. FIELD OF FORCE

where $F_c = F \cos \theta$ is the component force along the curve, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the position vectors of P_1 and P_2, and r_1 and r_2 are their magnitudes.

If the work done in the displacement from P_1 to P_2 is independent of the actual path and depends only on the initial and final positions P_1 and P_2, the force is called a *conservative force* and the field a *conservative field of force*. The earth's gravitational field is a conservative field because the work done in moving a given mass to a higher level against gravity depends only on the height through which the mass is raised. A field in which there are frictional forces is a non-conservative field because the work done on a body against a frictional force depends upon the path of the body and the force of friction at any point of the field depends upon the direction of the sliding motion.

As a particle moves from P_1 to P_2 in a conservative field, the field forces will do work on the particle of amount given by (IX.24). This capacity for doing work is inherent in the field, and we call it the *potential energy* of the field, or, simply, the potential. Suppose

that we choose P_0, Fig. 71, as a standard position, and define the potential energy when the particle is at P as the work done by the field forces on the particle as the particle moves from P to P_0, then, if V denote the potential at P

$$V = \int_{r}^{r_0} \mathbf{F} \cdot d\mathbf{r} = \int_{P}^{P_0} F_c ds \qquad . \qquad . \quad \text{(IX.25)}$$

V is independent of the path of the particle from P to P_0, and is, therefore, a function of the position of P. The position of P_0 is arbitrary, and fixes the level from which the potential energy is measured. The excess of the potential at P_1 over that at P_2 is

$$V_1 - V_2 = \int_{P_1}^{P_0} F_c ds - \int_{P_2}^{P_0} F_c ds$$

and, assuming that the first integral is evaluated along a path which passes through P_2

$$V_1 - V_2 = \int_{P_1}^{P_2} F_c ds + \int_{P_2}^{P_0} F_c ds - \int_{P_2}^{P_0} F_c ds$$

i.e.

$$V_1 - V_2 = \int_{P_1}^{P_2} F_c ds \qquad . \qquad . \qquad . \qquad . \qquad . \quad \text{(IX.26)}$$

As the particle moves from P to the neighbouring point Q, the loss of potential is ΔV, and $\Delta V = - F_c \Delta s$. Proceeding to the limit, we have

$$\frac{dV}{ds} = - F_c \qquad . \qquad . \qquad . \qquad . \quad \text{(IX.27)}$$

The relation (IX.27) means that at any point of the field the field force in a given direction is equal and opposite to the rate of increase of the potential in that direction. Suppose that a particle of mass M engineers' units is moving in the field with velocity v ft/sec at time t sec. The field force is doing work on the particle at the rate $-\dfrac{dV}{dt}$ ft-lb/sec, the force in the direction of motion is $-\dfrac{1}{v}\dfrac{dV}{dt}$ lb, and the acceleration is $-\dfrac{1}{Mv}\dfrac{dV}{dt}$ ft/sec^2, i.e. $\dfrac{dv}{dt} = -\dfrac{1}{Mv}\dfrac{dV}{dt}$ or

$$Mv\frac{dv}{dt} + \frac{dV}{dt} = 0$$

Integrating, $\frac{1}{2}Mv^2 + V = C$, where C is a constant.

Now $\frac{1}{2}Mv^2$ is the kinetic energy $= T$, say

Then

$$T + V = C \qquad . \qquad . \qquad . \quad \text{(IX.28)}$$

Thus, the sum of the kinetic energy and the potential energy is constant. This is the law of the conservation of energy, and it is for this reason that we call the field conservative. If the particle is acted upon by external forces as well as by the field forces, (IX.23) will still apply provided that the external forces do no work, positive or negative, on the particle. If, for example, the external forces are those due to fixed frictionless constraints, they will do no work and the energy will be conserved. The above is only a partial statement of the law of the conservation of energy as it is concerned with mechanical energy only and takes no account of other forms of energy, such as heat energy and electrical energy. The proof given applies only to the case of a single particle. A rigid body may be considered as a collection of particles at fixed distances apart. In addition to the field and external forces there are internal forces between the particles, these internal forces occurring in pairs, one of each pair being an action and the other an equal and opposite reaction. Owing to the fixed distance between any two particles, the work done by one of these is equal to that done against the other, and the total work done by the internal forces is zero. The law of the conservation of energy, therefore, applies to the motion of a rigid body if all the acting forces are conservative.

EXAMPLE 1

A uniform plank of thickness $2h$ rests across the top of a fixed circular cylinder of radius a, whose axis is horizontal. The plank is allowed to turn slowly without slipping, through an angle θ, in a vertical plane parallel to its length, so as to remain in contact with the cylinder. Prove that the gain of potential energy is

$$W[a\theta \sin \theta - (a + h)(1 - \cos \theta)]$$

where W is the weight of the plank.

Deduce the condition for stable equilibrium in the original position. (U.L.)

The plank is shown in the displaced position in Fig. 72. R and Q coincide in the initial position, and since there is no slipping, $\overline{RP}$ = arc $QP = a\theta$. The height of G, the centre of gravity of the plank, was originally $a + h$, and in the displaced position it is the sum of the vertical projections of OP, PR, and RG, i.e. $a \cos \theta + a\theta \sin \theta + h \cos \theta$. Thus, the increase in height of G is

$$(a + h) \cos \theta + a\theta \sin \theta - (a + h)$$

so that the increase in potential energy is $W \times$ this increase in height

i.e. $$W[a\theta \sin \theta - (a + h)(1 - \cos \theta)]$$

[This result is true even if the plank is not of uniform thickness provided that its lower edge is straight and the plank balances symmetrically on the cylinder, h being now the height of the centre of gravity above the original point of contact.]

If G is on the left of P, the weight acting through G will tend to restore the

plank to its position of equilibrium. Hence, for stability, $\overline{GR}\sin\theta$ must be less than $\overline{RP}\cos\theta$, i.e. $h\sin\theta < a\theta\cos\theta$ or $h < a\dfrac{\theta}{\tan\theta}$. But $\underset{\theta\to 0}{Lt.}\dfrac{\theta}{\tan\theta} = 1$, so that for stability, h must be less than a.

Let us consider the case of a body in a conservative field of force If the body moves with one degree of freedom, as in the above example, the field forces will do work on the body when the potential is falling, and the body will do work against the field

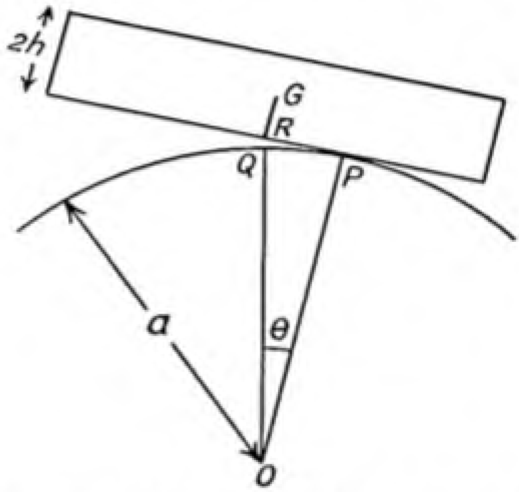


FIG. 72. STABILITY OF EQUILIBRIUM

forces when the potential is rising. Let θ be the variable which fixes the position of the body, V the potential energy, and T the kinetic energy of the body. Then we see that work is done on the body by the field forces when $\dfrac{dV}{d\theta}$ is negative, and work is done by the body against the field forces when $\dfrac{dV}{d\theta}$ is positive. When $\dfrac{dV}{d\theta} = 0$, no work is done either on or by the body, and the forces on it are in equilibrium. Thus, the condition for equilibrium is that the potential energy must have a stationary value. If the stationary value is one at which V is a minimum and the body is given a small velocity when at rest in that position, V will increase and, by (VI.23), T will decrease in the subsequent motion and will quickly become zero when the body will have come to rest. Thus, if V is a minimum, the position is one of stable equilibrium. If, on the other hand, the stationary value is one at which V is a maximum and the body is given a small velocity when at rest in that position, V will decrease and T will increase in the subsequent motion, and the body will

move more quickly the further it gets from the position of equilibrium. Thus, if V is a maximum, the position is one of unstable equilibrium. When a body is in stable equilibrium in a conservative field of force, the potential energy is a minimum. This is the well-known *principle of least energy*.

Applying this principle to the above Example, we have

$$V = W[a\theta \sin \theta - (a + h)(1 - \cos \theta)] + \text{constant}$$

$$\frac{dV}{d\theta} = W[a\theta \cos \theta - h \sin \theta]$$

The equation $\frac{dV}{d\theta} = 0$ has two roots, one of which is $\theta = 0$, and the other corresponds to the position in which GP in Fig. 72 is vertical.

$$\frac{d^2V}{d\theta^2} = W[(a - h) \cos \theta - a\theta \sin \theta]$$

When $\theta = 0$, $\frac{d^2V}{d\theta^2} = W(a - h)$, which is positive if $a > h$. Hence V is a minimum and $\theta = 0$ is a position of stable equilibrium if $a > h$.

In Vol. I we have examined the conditions for stability of a floating body and proved that the condition for stability of equilibrium is that the metacentre should be above the centre of gravity. If M is the metacentre, H the centre of buoyancy, i.e. the centre of gravity of the displaced fluid, and G the centre of gravity of the floating body, then, if the body is symmetrical about a vertical plane, H and G will lie on a vertical line, M being a point on HG or HG produced. The length of HM is given by

$$\overline{HM} = \frac{Ak^2}{V} \quad . \quad . \quad . \quad . \quad (\text{IX.29})$$

where A is the area of the water-line section of the body, V is the volume of fluid displaced by the body, and k is the radius of gyration of the area A about its axis of symmetry.

EXAMPLE 2

Determine the condition of stability of equilibrium for a solid of revolution floating with its axis vertical. A cylindrical hole, of radius r, is drilled centrically through a cube at right angles to a pair of faces. If the cube floats in a liquid of double its density with the axis of the hole vertical, prove that the equilibrium is

stable provided the radius of the hole lies between the positive roots of the equation

$$6\pi r^4 - 3\pi a^2 r^2 + a^4 = 0$$

where a is the length of the edge of the cube. (U.L.)

We leave the first part of the question as an exercise for the reader. In the case of the cube, Ak^2 is the moment of inertia of the liquid-line section about any axis through the centroid of the section, and $Ak^2 = \frac{a^4}{12} - \frac{\pi r^4}{4}$, whether that axis is parallel to the sides of the section or not; V = volume of displaced liquid $= \frac{1}{2}$ vol. of solid $= \frac{1}{2}(a^3 - \pi r^2 a)$

Hence $$HM = \frac{Ak^2}{V} = \frac{a^4 - 3\pi r^4}{6(a^3 - \pi r^2 a)}$$

The centre of buoyancy H and the centre of gravity G of the solid are at heights $\frac{a}{4}$ and $\frac{a}{2}$ respectively above the base of the solid, so that $HG = \frac{a}{2} - \frac{a}{4} = \frac{a}{4}$. Since for stability M must be above G, i.e. $HM > HG$, then

$$\frac{a^4 - 3\pi r^4}{6(a^3 - \pi r^2 a)} > \frac{a}{4},$$

which leads to

$$6\pi r^4 - 3\pi a^2 r^2 + a^4 < 0 \qquad . \quad . \quad . \quad . \quad (1)$$

If r_1^2 and r_2^2 are the roots of the equation $6\pi r^4 - 3\pi a^2 r^2 + a^4 = 0$ regarded as a quadratic in r^2, (1) can be written in the form

$$6\pi(r^2 - r_1^2)(r^2 - r_2^2) < 0$$

which is satisfied if, and only if, r^2 lies between r_1^2 and r_2^2. Here r is essentially positive, so that the condition for stability is that stated in the question.

EXAMPLE 3

A uniform right circular solid cylinder of radius r, length $2l$, and specific gravity $\frac{1}{2}$, floats in water with its axis horizontal. Find approximately the work done in rotating the cylinder slowly through a small angle θ about a horizontal axis through its centre perpendicular to its axis of symmetry. Deduce that the original position of the cylinder is stable for this type of displacement if $l > r$. (U.L.)

As the specific gravity is $\frac{1}{2}$, the axis of the cylinder lies on the surface of the water, and the displaced water being in the form of a semi-cylinder has its centre of gravity, i.e. the centre of buoyancy, at a distance $\frac{4r}{3\pi}$ from the axis. Thus, with the usual notation, $\overline{HG} = \frac{4r}{3\pi}$. Here Ak^2 = moment of inertia of water-line section about axis of rotation $= \frac{2r \times (2l)^3}{12} = \frac{4}{3}rl^3$, and V = volume of displaced water $= \frac{1}{2}\pi r^2 \times 2l = \pi r^2 l$.

Thus, if M is the metacentre, $\overline{HM} = \frac{Ak^2}{V} = \frac{\frac{4}{3}rl^3}{\pi r^2 l} = \frac{4l^2}{3\pi r}$

For stability, $\overline{HM} > \overline{HG}$, so that

$$\frac{4l^2}{3\pi r} > \frac{4r}{3\pi}, \text{ i.e. } l^2 > r^2, \text{ i.e. } l > r$$

The weight W of the cylinder acting through G and the buoyancy W acting upwards through M form a couple of moment $W \times \overline{GM} \sin \theta$, i.e. $W \times \overline{GM}\theta$, since θ is small. A couple C of this magnitude but of opposite sense is required to keep the cylinder in its displaced position, so that

$$\begin{aligned} C &= W \times \overline{GM}\theta \\ &= w \times \pi r^2 l(\overline{HM} - \overline{HG})\theta, \text{ where } w = \text{wt/unit vol. of water} \\ &= w \times \pi r^2 l \left(\frac{4l^2}{3\pi r} - \frac{4r}{3\pi}\right)\theta \end{aligned}$$

i.e.
$$C = \tfrac{4}{3} wlr(l^2 - r^2)\theta$$

The work done during the small rotation is $\int_0^\theta C d\theta = \frac{2}{3} wlr(l^2 - r^2)\theta^2$. From this we can deduce the condition of stability found above, for, if $l > r$, positive work must be done in displacing the cylinder from its original position, which is, therefore, one of stable equilibrium.

EXAMPLE 4

A homogeneous solid floating in equilibrium is removed from the liquid and placed in the inverted position in a second liquid of such density that the plane of flotation is the same as before. Show that the two positions are both stable or both unstable.

A uniform lamina in the form of a regular pentagon floats symmetrically, with its plane vertical, with three edges completely immersed. Prove that the equilibrium is stable for displacements in the vertical plane of the lamina if not more than one-quarter of each of the remaining edges is immersed. (U.L.)

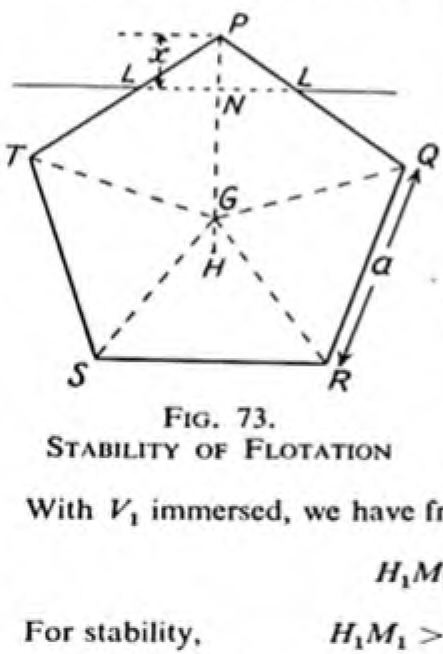


FIG. 73.
STABILITY OF FLOTATION

Let V_1 and V_2 be the volumes of the parts into which the solid is divided by the plane of flotation, and let H_1 and H_2 be the respective centres of gravity of these parts. The centre of gravity G of the whole solid is on the line H_1H_2, and, by moments about G,

$$V_1 \times \overline{H_1G} = V_2 \times \overline{H_2G}$$

With V_1 immersed, we have from (IX.29), if M_1 is the metacentre,

$$H_1M_1 = \frac{Ak^2}{V_1}, \ A \text{ and } k \text{ having their usual meanings.}$$

For stability, $\quad H_1M_1 > H_1G$, i.e. $Ak^2 > V_1 \times \overline{H_1G}$ . . . . (1)

The corresponding condition with V_2 immersed is

$$Ak^2 > V_2 \times \overline{H_2G} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As A and k are the same in (1) and (2), and the right-hand sides of (1) and (2) are equal, the two conditions are identical, and the two positions are either both stable or both unstable.

To solve the second part of the question we use (IX.29) and regard the lamina as a floating body of thickness t, where t is small.

Let $PQRST$ be the pentagon, centre G, side a (Fig. 73) and LL the liquid-line surface. Also let $PN = x$, where N is the point in which PG cuts LL.

$$\text{Area of pentagon} = \tfrac{5}{4}a^2 \tan 54^\circ = 1\cdot7205a^2$$

$$\text{Area } PLL = x^2 \tan 54^\circ = 1\cdot3764x^2$$

$$\text{Area under liquid} = 1\cdot7205a^2 - 1\cdot3764x^2$$

$$\overline{GP} = \tfrac{1}{2}a \sec 54^\circ = 0\cdot8507a$$

H, the centroid of the immersed area, lies on PG produced, and, by moments of areas about P,

$$\overline{HP}\,(1\cdot7205a^2 - 1\cdot3764x^2) + 1\cdot3764x^2 \times \frac{2x}{3} = \overline{GP} \times 1\cdot7205a^2$$

whence
$$\overline{HP} = \frac{1\cdot4636a^3 - 0\cdot9176x^3}{1\cdot7205a^2 - 1\cdot3764x^2}$$

From (IX.29),
$$\overline{HM} = \frac{Ak^2}{V} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where Ak^2 = moment of inertia of liquid-line surface about axis of rotation $= \dfrac{t \times LL^3}{12} = \dfrac{t \times (2x \tan 54^\circ)^3}{12} = 1\cdot737tx^3$, and V = vol. of displaced liquid $= t(1\cdot7205a^2 - 1\cdot3764x^2)$.

Substituting in (3) and cancelling t,
$$\overline{HM} = \frac{1\cdot737x^3}{1\cdot7205a^2 - 1\cdot3764x^2}$$

Now
$$\overline{HG} = \overline{HP} - \overline{GP} = \frac{1\cdot4636a^3 - 0\cdot9176x^3}{1\cdot7205a^2 - 1\cdot3764x^2} - 0\cdot8507a$$

$$= \frac{1\cdot1709ax^2 - 0\cdot9176x^3}{1\cdot7205a^2 - 1\cdot3764x^2}$$

The condition for stability is $\overline{HM} > \overline{HG}$

i.e.
$$1\cdot737x^3 > 1\cdot1709ax^2 - 0\cdot9176x^3$$

i.e.
$$2\cdot655x > 1\cdot171a$$

or
$$x > 0\cdot4412a$$

Since $x = \overline{PL} \cos 54^\circ$, the condition becomes

$$\overline{PL} > \frac{0\cdot4412}{0\cdot5878}\,a, \text{ i.e. } PL > 0\cdot75a \text{ (approx.)}$$

For stability, therefore, not more than one-quarter of each of the sides PQ and PT will be immersed.

90. **Virtual Work.** We can frequently obtain solutions to statical problems by imagining a body which is in equilibrium under the action of a system of forces to receive a small displacement, determining the corresponding small displacements of the points of application of the several forces, and equating to zero the total work done by the forces in such displacements, this total work being a small quantity of the second order. This is the principle of *virtual work*.

(See Vol. I, Art. 41.) Thus, if F_1, F_2, F_3, etc. are the forces and Δs_1, Δs_2, Δs_3, etc. are the respective small displacements of their points of application

$$\left.\begin{aligned} F_1\Delta s_1 + F_2\Delta s_2 + F_3\Delta s_3 + \ldots &= 0 \\ \text{or} \qquad \Sigma F\Delta s &= 0 \end{aligned}\right\} \quad . \quad \text{(IX.30)}$$

If the displacement of the body takes place in time Δt, $\Sigma F\dfrac{\Delta s}{\Delta t}$ is of the first order of small quantities, and as Δt approaches the limit zero, we have

$$\Sigma Fv = 0 \quad . \quad . \quad . \quad . \quad \text{(IX.31)}$$

where v is the velocity of the point of application of a force in the direction of the force. ΣFv is the rate at which the forces are doing work on the body, and (IX.31) expresses the fact that, when a body is passing through a position of equilibrium, the rate of doing work is zero.

EXAMPLE 1

A square $ABCD$ of equal smoothly-jointed light rods each of length a lies on a smooth horizontal table; OA, OB, OC, OD are four equal light rigid rods,

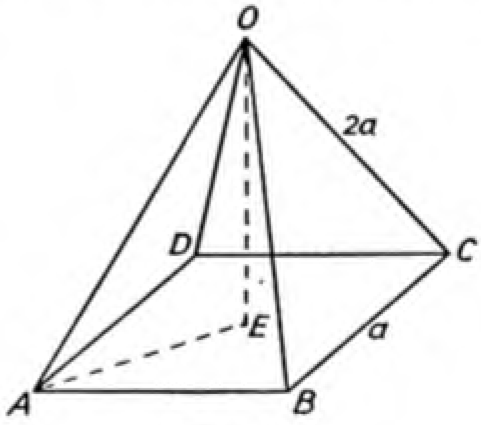


FIG. 74

each of length $2a$, smoothly jointed at O, A, B, C, D, so that all the rods form the edges of a pyramid. A weight W is suspended in equilibrium at O. By the method of virtual work, or otherwise, find the stress in a rod such as OA, and a rod such as AB. (U.L.)

If E is the centre of $ABCD$ (Fig. 74), then $\overline{AE} = \dfrac{a}{\sqrt{2}}$, and if $\overline{OE} = h$, then

$$h^2 + \frac{a^2}{2} = \overline{OA}^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We shall assume that the weight at O is displaced upwards through a small distance Δh, and that A, B, C, D remain fixed whilst each of the sloping rods

extends its length from l to $l + \Delta l$. It is necessary to introduce a new symbol l for $\overline{OA}$ because we wish to keep constant $\overline{AB} = a$ whilst varying $\overline{OA} = 2a$. We shall afterwards put $l = 2a$.

Thus, we write (1) as $$h^2 + \frac{a^2}{2} = l^2 \qquad (2)$$

Then $$2h\Delta h = 2l\Delta l$$

i.e. $$h\Delta h = l\Delta l \qquad (3)$$

The work done on W in raising it through a height Δh is $W\Delta h$, and if T is the stress in each sloping rod, the work done by these stresses is $4T\Delta l$.

From (IX.25) we have

$$\text{virtual work of forces} = \Sigma F\Delta s = 0$$

and in this case,

$$4T\Delta l - W\Delta h \doteq 0 \qquad (4)$$

From (3) and (4), $$4T\Delta l = W\frac{l}{h}\Delta l$$

i.e. $$T = \frac{Wl}{4h}$$

Now $l = 2a$, and from (2), $h = \sqrt{4a^2 - \frac{a^2}{2}} = \frac{\sqrt{14}a}{2}$

Hence, $$T = \frac{\sqrt{14}W}{14}$$

To find the stress S in each horizontal rod we assume its length to change whilst that of each sloping rod remains unchanged. From (2)

$$2h\Delta h + a\Delta a = 0 \qquad (5)$$

The work done by each of the eight forces S is $S \times \frac{1}{2}\Delta a$, and from (IX.30),

$$\text{virtual work of forces} = \Sigma F\Delta s = 0$$

which in this case gives

$$8S \times \tfrac{1}{2}\Delta a + W\Delta h = 0 \qquad (6)$$

From (5) and (6), $$\frac{W}{2h} = \frac{4S}{a}$$

i.e. $$S = \frac{Wa}{8h} = \frac{W}{4\sqrt{14}} = \frac{\sqrt{14}W}{56}$$

A rod such as OA is subjected to a stress of $\frac{\sqrt{14}W}{14}$ in compression, and one such as AB is subjected to a stress of $\frac{\sqrt{14}W}{56}$ in tension.

The stresses are much more easily found by resolution of forces. Thus, resolving vertically for the equilibrium of W at O, we have

$$4T\cos\widehat{EOA} = W$$

Since $\cos\widehat{EOA} = \dfrac{h}{2a} = \dfrac{\sqrt{14}}{4}$, then $T = \dfrac{W}{\sqrt{14}} = \dfrac{\sqrt{14}W}{14}$

Resolving horizontally for the equilibrium of the forces acting at A, we have

$$2S\cos 45^\circ = T\cos\widehat{EAO} = \frac{\sqrt{14}W}{14} \times \frac{\sqrt{2}}{4}, \text{ so that}$$

$$S = \frac{\sqrt{14}W}{56}$$

EXAMPLE 2

Four equal particles A, B, C, D, each of weight W, are connected by light strings AB, BC, CD, DA, each of length l, and then placed at rest symmetrically on a smooth sphere of radius R $[l < \frac{1}{2}\pi R]$.

Prove by virtual work, or otherwise, that the tension in each string is

$$\tfrac{1}{2}W\sin(l/R)\sqrt{\sec(l/R)}$$

[The string between two particles lies along a great circle on the sphere.]

(U.L.)

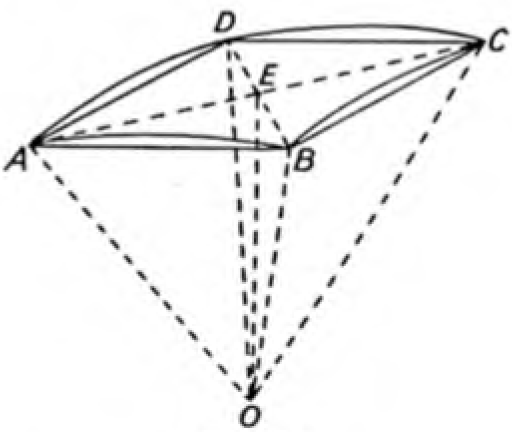


FIG. 75

In Fig. 75 the curved lines represent the strings, O is the centre of the sphere, and E is the centre and $\overline{AE}$ the radius of the section of the sphere made by the plane $ABCD$. $OABCD$ is an inverted pyramid with base at height $\overline{OE} = h$ above O. Let the angles AOB, BOC, COD, and DOA be each equal to θ. Then $l = R\theta$ and $\overline{AB} = 2R\sin\dfrac{\theta}{2}$. Also $\overline{AE} = \overline{AB}\cos 45^\circ = \sqrt{2}R\sin\dfrac{\theta}{2}$, and from the right-angled triangle AEO, $\overline{OE}^2 + \overline{AE}^2 = \overline{OA}^2$, i.e. $h^2 + 2R^2\sin^2\dfrac{\theta}{2} = R^2$,

whence $h = R\sqrt{\cos\theta}$. If Δh is the change in h due to a small change $\Delta\theta$ in θ we have

$$\Delta h = -\frac{R\sin\theta}{2\sqrt{\cos\theta}}\,\Delta\theta = -\tfrac{1}{2}R\sin\left(\frac{l}{R}\right)\sqrt{\sec\left(\frac{l}{R}\right)}\;\Delta\theta. \qquad (1)$$

Also, since $l = R\theta$,
$$\Delta l = R\Delta\theta \qquad (2)$$

and from (1) and (2),
$$\Delta h = -\tfrac{1}{2}\sin\left(\frac{l}{R}\right)\sqrt{\sec\left(\frac{l}{R}\right)}\;\Delta l \qquad (3)$$

During these displacements, Δh and Δl, the work done on the particles by gravity is $-4W\Delta h$, the sign being negative since Δh is an upward displacement, and that done by the tensions T in the strings is $-4T\Delta l$. The only other forces acting on the particles are the reactions of the spherical surface. As there is no friction, these do no work. From (IX.30), the equation of virtual work is

$$-4W\Delta h - 4T\Delta l = 0$$

whence
$$T = -W\frac{\Delta h}{\Delta l}$$

Using (3), we have
$$T = -W \times -\tfrac{1}{2}\sin\left(\frac{l}{R}\right)\sqrt{\sec\left(\frac{l}{R}\right)}$$

i.e.
$$T = \tfrac{1}{2}W\sin\left(\frac{l}{R}\right)\sqrt{\sec\left(\frac{l}{R}\right)}$$

91. **Conservation of Angular Momentum.** When a rigid body is rotating about a fixed axis with angular velocity ω radn/sec under the action of a couple C lb-ft, the equation of motion is $C = \dfrac{d}{dt}(I\omega)$, where I engineers' units is the moment of inertia of the body about the axis of rotation and t sec is the time. If there is no couple, i.e. if $C = 0$, then $\dfrac{d}{dt}(I\omega) = 0$, and integrating with respect to t, we have $I\omega =$ constant. The dimensions of $I\omega$ are ML^2T^{-1}, where M, L, T are the units of mass, length, time respectively; this is the product of MLT^{-1}, which is momentum, and L, which is length. By analogy with the moment of a force, i.e. force $\times$ distance, $I\omega$ is called the *moment of momentum* or *angular momentum* about the axis of rotation. Thus, if a body is spinning about an axis and there is no couple acting about that axis, the angular momentum remains constant. This is the principle of the *conservation of angular momentum*. In problems where there is a change in the value of I, this principle and that of the conservation of energy enable us to find solutions.

EXAMPLE 1

A flywheel weighing 1 000 lb with a radius of gyration of 3 ft is rotating on a frictionless axis at 100 rev/min. By means of a clutch the flywheel is instantaneously connected to a second flywheel weighing 500 lb with a radius of gyration of 1 ft 6 in., which is rotating on a frictionless axis at 40 rev/min. Find the common speed of the flywheels after connection.

For the first flywheel, $I = \dfrac{9\,000}{g}$ engineers' units, and the angular speed is $\dfrac{200\pi}{60} = \dfrac{10\pi}{3}$ radn/sec. For the second flywheel, $I = \dfrac{2\,250}{2g}$ engineers' units and the angular speed is $\dfrac{80\pi}{60} = \dfrac{4\pi}{3}$ radn/sec.

Total angular momentum $= \dfrac{90\,000\pi}{3g} + \dfrac{3\,000\pi}{2g} = \dfrac{31\,500\pi}{g}$

Let N rev/min be the common speed. The angular momentum at this speed is $\left(\dfrac{9\,000}{g} + \dfrac{2\,250}{2g}\right) \times \dfrac{2N\pi}{60} = \dfrac{337{\cdot}5\pi}{g}$

Since the angular momentum is conserved, we have

$$\frac{337{\cdot}5\pi}{g} = \frac{31\,500\pi}{g}$$

$$\therefore \qquad N = \frac{31\,500}{337{\cdot}5} = 93{\cdot}3$$

The common speed is 93·3 rev/min.

EXAMPLE 2

A uniform circular disc of mass M and radius a rotates about as mooth fixed vertical axis through its centre perpendicular to its plane, and carries a particle of mass kM, which is free to move along a smooth radial groove. Initially the disc rotates with angular speed ω and the particle is at rest at the centre. Prove that, when the particle, after being slightly disturbed, has moved a distance r along the radius, the angular velocity of the disc is

$$\frac{a^2\omega}{a^2 + 2kr^2}$$

and that the radial velocity of the particle is

$$\frac{a\omega r}{\sqrt{a^2 + 2kr^2}} \qquad \text{(U.L.)}$$

In this case there is no couple acting on the disc-particle system, so that the angular momentum is conserved. This gives us one relation between the variable quantities. As there are two unknowns, the radial velocity $\dot{r}$ of the particle and the angular velocity ω_1 of the disc at the given instant, we require another relation. This we obtain by using the conservation of energy principle. We have then

final angular momentum = initial angular momentum

i.e.
$$\left(M\frac{a^2}{2} + kMr^2\right)\omega_1 = M\frac{a^2}{2}\,\omega$$

from which
$$\omega_1 = \frac{a^2\omega}{a^2 + 2kr^2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The initial kinetic energy is $\frac{1}{2} \times M\frac{a^2}{2} \times \omega^2 = \frac{Ma^2\omega^2}{4}$. The final kinetic energy is $\frac{Ma^2\omega_1^2}{4}$ due to the disc, $\frac{1}{2}kM(\omega_1 r)^2 = \frac{kMr^2\omega_1^2}{2}$ due to the transverse velocity of the particle, and $\frac{kM\dot{r}^2}{2}$ due to its radial velocity. We have then

$$\text{initial energy} = \text{final energy}$$

i.e.
$$\frac{Ma^2\omega^2}{4} = \frac{Ma^2\omega_1^2}{4} + \frac{kMr^2\omega_1^2}{2} + \frac{kM\dot{r}^2}{2}$$

and by simplifying,
$$a^2\omega^2 = (a^2 + 2kr^2)\omega_1^2 + 2k\dot{r}^2$$

Using (1),
$$a^2\omega^2 = \frac{a^4\omega^2}{a^2 + 2kr^2} + 2k\dot{r}^2, \text{ which gives}$$
$$\dot{r}^2 = \frac{a^2\omega^2r^2}{a^2 + 2kr^2}$$

Hence,
$$\dot{r} = \frac{a\omega r}{\sqrt{a^2 + 2kr^2}}$$

which is the radial velocity of the particle.

92. **Fluid Motion. Bernouilli's Theorem.** Consider the motion of a frictionless fluid in three-dimensional space. Let OX, OY, OZ be rectangular axes (Fig. 76), the plane XOY being horizontal, P the

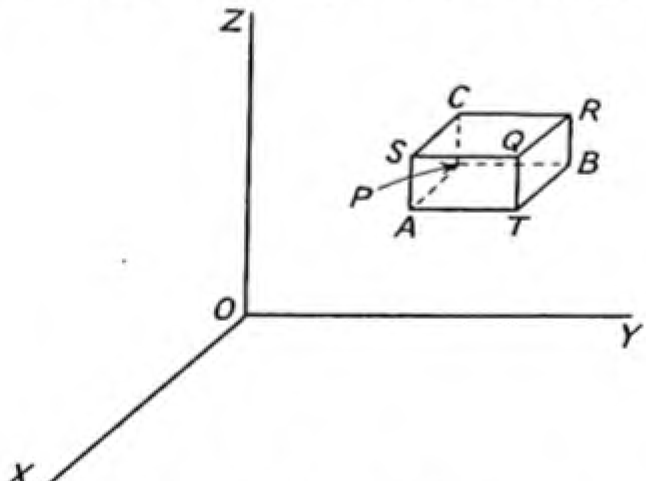


FIG. 76. FLUID MOTION

point (x, y, z), and Q the point $(x + \Delta x, y + \Delta y, z + \Delta z)$, all lengths being in feet, and let the small rectangular figure represent an element of fluid weighing w lb per cubic foot and having its faces

parallel to the co-ordinate planes. Consider the component motion of the element in the direction of OY. The acceleration in that direction is

$$\frac{d^2y}{dt^2} = \frac{d}{dt}(\dot{y}) = \frac{d\dot{y}}{dy} \cdot \frac{dy}{dt} = \dot{y}\frac{d\dot{y}}{dy} = \tfrac{1}{2}\frac{d}{dy}(\dot{y}^2)$$

t being the time in seconds. If p lb per square foot is the pressure at P, the force on the face $PCSA$ is $p\Delta x\Delta z$ and that on the opposite face $QRBT$ is $-[p\Delta x\Delta z + \frac{\partial}{\partial y}(p\Delta x\Delta z)\Delta y]$, so that the resultant force parallel to OY is $-\frac{\partial p}{\partial y}\Delta x\Delta y\Delta z$. The equation of motion is

force = mass × acceleration

i.e. $$-\frac{\partial p}{\partial y}\Delta x\Delta y\Delta z = \frac{w}{g}\Delta x\Delta y\Delta z \times \tfrac{1}{2}\frac{d}{dy}(\dot{y}^2) \quad . \quad . \quad \text{(IX.32)}$$

whence $$\frac{\partial p}{\partial y} = -\frac{w}{2g}\frac{d}{dy}(\dot{y}^2) \quad . \quad . \quad . \quad \text{(IX.33)}$$

In the same way, for motion in the direction OX, we have

$$\frac{\partial p}{\partial x} = -\frac{w}{2g}\frac{d}{dx}(\dot{x}^2) \quad . \quad . \quad . \quad \text{(IX.34)}$$

For motion in the vertical direction OZ the equation corresponding to (IX.32) will have an additional term $-w\Delta x\Delta y\Delta z$ on the left because of the force of gravity, and this will add a term w to the left side of the equation corresponding to (IX.34).

In this case, then

$$\frac{\partial p}{\partial z} + w = -\frac{w}{2g}\frac{d}{dz}(\dot{z}^2) \quad . \quad . \quad . \quad \text{(IX.35)}$$

The total variation Δp of p from P to Q is given by

$$\Delta p = \frac{\partial p}{\partial x}\Delta x + \frac{\partial p}{\partial y}\Delta y + \frac{\partial p}{\partial z}\Delta z \quad . \quad . \quad \text{(IX.36)}$$

as shown in Vol. I.

Substituting in (IX.36) from (IX.33), (IX.34), and (IX.35), we obtain

$$\Delta p = -\frac{w}{2g}\left[\frac{d}{dx}(\dot{x}^2)\Delta x + \frac{d}{dy}(\dot{y}^2)\Delta y + \frac{d}{dz}(\dot{z}^2)\Delta z\right] - w\Delta z \quad \text{(IX.37)}$$

If V is the velocity of the fluid at P, then

$V^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$, and $\dfrac{\partial}{\partial x}(V^2) = \dfrac{d}{dx}(\dot{x}^2)$, $\dfrac{\partial}{\partial y}(V^2) = \dfrac{d}{dy}(\dot{y}^2)$, and $\dfrac{\partial}{\partial z}(V^2) = \dfrac{d}{dz}(\dot{z}^2)$, so that (IX.37) can be written as

$$\Delta p = -\frac{w}{2g}\left[\frac{\partial}{\partial x}(V^2)\Delta x + \frac{\partial}{\partial y}(V^2)\Delta y + \frac{\partial}{\partial z}(V^2)\Delta z\right] - w\Delta z$$

i.e. $\Delta p = -\dfrac{w}{2g}\Delta(V^2) - w\Delta z$

On dividing through by w and integrating, we have

$$\int\frac{dp}{w} + \frac{V^2}{2g} + z = C \text{ (where } C \text{ is a constant)} \quad . \quad \text{(IX.38)}$$

This is known as *Bernouilli's Theorem*, and it expresses the fact that during the motion along the stream-lines the mechanical energy is conserved. The terms on the left of (IX.38) represent in order (1) the pressure energy, (2) the kinetic energy, and (3) the gravity potential energy, of 1 lb weight of the fluid, whether it is a gas or a liquid. If the fluid is incompressible, w is constant and $\int\dfrac{dp}{w} = \dfrac{p}{w}$. In this case (IX.38) becomes

$$\frac{p}{w} + \frac{V^2}{2g} + z = \text{constant} \quad . \quad . \quad \text{(IX.39)}$$

EXAMPLE 1

A gas in which the pressure p and the density ρ are connected by the adiabatic relation $p = k\rho^\gamma$, flows steadily along a pipe. Prove that the velocity q satisfies the relation

$$q^2 + \frac{2\gamma}{\gamma - 1}\cdot\frac{p}{\rho} = \text{constant}$$

external forces being neglected.

If the pipe diverges slightly in the direction of flow, show that the speed of any particle of the gas is increasing if $q > c$ and decreasing if $q < c$, where $c^2 = \dfrac{\gamma p}{\rho}$

(U.L.)

Writing q for V and $g\rho$ for w in (IX.38), we have

$$\int\frac{dp}{g\rho} + \frac{q^2}{2g} = \text{constant} \quad . \quad . \quad . \quad . \quad (1)$$

the term z being assumed constant.

Since $$p = k\rho^{\gamma}$$

then $$\rho = \left(\frac{p}{k}\right)^{\frac{1}{\gamma}}$$

so that $$\int \frac{dp}{g\rho} = \frac{k^{\frac{1}{\gamma}}}{g} \int p^{-\frac{1}{\gamma}}\, dp = \frac{\gamma}{g(\gamma-1)} \cdot \frac{p}{\rho}$$

(We leave the reader to prove this.)

The relation (1) then becomes

$$\frac{\gamma}{g(\gamma-1)} \cdot \frac{p}{\rho} + \frac{q^2}{2g} = \text{constant}$$

and, since g is constant, this is equivalent to

$$q^2 + \frac{2\gamma}{\gamma-1} \cdot \frac{p}{\rho} = \text{constant} = c_1 \quad . \quad . \quad . \quad (2)$$

Let A be the area of the cross-section of the pipe where the density is ρ. Then the quantity of gas flowing through any such section in a given time varies as $q\rho A$, and, since conditions are steady, the same quantity flows through all sections in the same time.

Thus, we have $$q\rho A = \text{constant} = c_2 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is known as the *equation of continuity*.

In terms of q and ρ, the relation (2) is

$$q^2 + \frac{2\gamma}{\gamma-1} k\rho^{\gamma-1} = c_1$$

From (3), $$\rho = \frac{c_2}{qA}$$

so that $$q^2 + \frac{2\gamma k}{\gamma-1} \cdot \frac{c_2^{\gamma-1}}{q^{\gamma-1}A^{\gamma-1}} = c_1$$

i.e. $$q^{\gamma+1} - c_1 q^{\gamma-1} + KA^{1-\gamma} = 0$$

where $$K = \frac{2\gamma k c_2^{\gamma-1}}{\gamma-1}$$

Differentiating with respect to A,

$$\{(\gamma+1)\, q^{\gamma} - c_1(\gamma-1)\, q^{\gamma-2}\} \frac{dq}{dA} + K(1-\gamma)A^{-\gamma} = 0$$

i.e. $$\left\{q^{\gamma} - \frac{c_1(\gamma-1)}{\gamma+1} q^{\gamma-2}\right\} \frac{dq}{dA} = \frac{K(\gamma-1)}{A^{\gamma}(\gamma+1)}$$

The sign of the quantity on the right is positive, and, therefore, since A is slowly increasing, q is increasing or decreasing according as the expression

$$q^{\gamma} - \frac{c_1(\gamma-1)}{\gamma+1} q^{\gamma-2}$$

is positive or negative,

i.e. according as $$q^2 > \text{or} < \frac{c_1(\gamma-1)}{\gamma+1}$$

Now, from (2), $$\frac{c_1(\gamma-1)}{\gamma+1} = \frac{\gamma-1}{\gamma+1}\left(q^2 + \frac{2\gamma}{\gamma-1}\cdot\frac{p}{\rho}\right)$$

$$= \frac{\gamma-1}{\gamma+1}q^2 + \frac{2\gamma}{\gamma+1}\cdot\frac{p}{\rho}$$

Hence, the condition becomes

$$q^2 > \text{ or } < \frac{\gamma-1}{\gamma+1}q^2 + \frac{2\gamma}{\gamma+1}\cdot\frac{p}{\rho}$$

which reduces to $$q^2 > \text{ or } < \frac{\gamma p}{\rho}$$

Thus, if $c^2 = \frac{\gamma p}{\rho}$, the speed q of any particle of the gas is increasing if $q > c$ and decreasing if $q < c$.

EXAMPLE 2

Prove Bernouilli's theorem in the case of the steady motion of a homogeneous incompressible inviscid fluid.

A channel has a horizontal bottom and vertical sides. At first the breadth is uniform and equal to b, then the bottom widens gradually and symmetrically until it is again of uniform breadth now equal to b_1. Water flows steadily along the channel. If u and h are the speed and depth where the breadth is b, and if h_1 is the depth where the breadth is b_1, prove that $h_1 < h$ if

$$u^2 > \frac{2gh_1^2}{h+h_1} \qquad \text{(U.L.)}$$

The theorem is proved above. u is constant, the pressure p varies with the depth below the free surface and, as the cross-section is a rectangle with horizontal and vertical sides, the mean pressure is $\frac{1}{2}wh$ where h is the depth. Similarly if we measure z from the horizontal bottom its mean value is $\frac{1}{2}h$.

Then from (IX.39) $$\frac{wh}{2w} + \frac{u^2}{2g} + \frac{h}{2} = \text{constant}$$

or $$\frac{u^2}{2g} + h = \text{constant}$$

No energy is lost so that, equating the energy of 1 lb of water as it passes two sections, one in each portion of the channel

$$\frac{u^2}{2g} + h = \frac{u_1^2}{2g} + h_1^2 \qquad . \quad . \quad . \quad . \quad . \quad (1)$$

The equation of continuity is $$buh = b_1u_1h_1 \qquad . \quad . \quad . \quad . \quad . \quad (2)$$

From (2) $u_1 = \frac{bh}{b_1h_1}u$ and substituting in (1) and

rearranging $$\frac{u^2}{2g}\left(\frac{b^2h^2}{b_1^2h_1^2} - 1\right) = h - h_1 \qquad . \quad . \quad . \quad . \quad . \quad (3)$$

The given inequality is $$\frac{u^2}{2g} > \frac{h_1^2}{h + h_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Combining (3) and (4) and assuming $h - h_1$ positive, we have

$$\frac{h_1^2}{h + h_1}\left(\frac{b^2h^2}{b_1^2h_1^2} - 1\right) < h - h_1 \quad . \quad . \quad . \quad . \quad (5)$$

i.e. $$b^2h^2 - b_1^2h_1^2 < b_1^2(h^2 - h_1^2)$$

i.e. $b < b_1$ as is given in the data.

If we assume that $h - h_1$ is negative instead of (5) we have

$$\frac{h_1^2}{h + h_1}\left(\frac{b^2h^2}{b_1^2h_1^2} - 1\right) > h - h_1$$

$$b^2h^2 - b_1^2h_1^2 > b_1^2(h^2 - h_1^2)$$

i.e. $b > b_1$ which is contrary to the data.

Hence it follows that $$h > h_1$$

VORTEX MOTION. Fig 77 shows the elevation $CDEF$ of a vertical circular cylinder rotating about its axis OZ with constant angular speed ω radians per sec. It contains a mass of incompressible fluid, the upper surface of which is shown in profile by $QO'PR$, O' being the

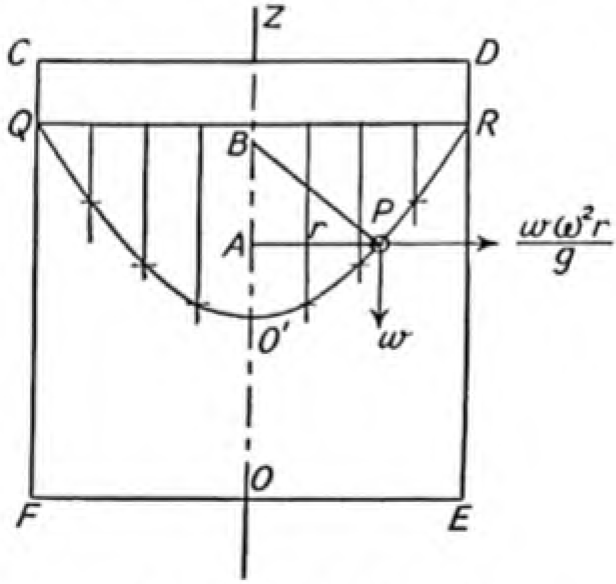


FIG. 77. FORCED VORTEX

lowest point. This fluid is rotating with the same angular speed as the cylinder without relative motion of its parts and the system is known as a *forced vortex*. P represents a particle of weight w lb. As P describes a circle of radius $\overline{AP} = r$ ft its acceleration is $\omega^2 r$ ft/sec^2

along PA and the reversed mass-acceleration is $\frac{w}{g}\,\omega^2 r$ lb weight acting along AP. The reaction of the air-fluid surface on the particle acts along the normal to the surface at P, i.e. along PB, where B lies on OZ. For equilibrium of the particle we have, taking moments about B

$$\frac{w}{g}\,\omega^2 r\overline{AB} = wr$$

or

$$\overline{AB} = \frac{g}{\omega^2} \qquad . \qquad . \qquad . \qquad . \qquad (IX.40)$$

Let z be the height of P above the base FE. Then since AB is the sub-normal

$$\overline{AB} = r\frac{dr}{dz}$$

Thus

$$\frac{g}{\omega^2} = r\frac{dr}{dz} \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (IX.41)$$

$$dz = \frac{\omega^2}{g}\,rdr$$

and integrating

$$z = \frac{\omega^2}{2g}\,r^2 + \text{constant} \qquad . \qquad . \qquad . \qquad (IX.42)$$

If we transfer the origin to the point O', $z = 0$ when $r = 0$ and

$$z = \frac{\omega^2}{2g}\,r^2 \qquad . \qquad . \qquad . \qquad . \qquad (IX.43)$$

which shows that the shape of the surface is the paraboloid of revolution produced by rotating about OZ the parabola $QO'R$ whose equation is this.

The above analysis applies equally well if the surface of profile $QO'R$ lies below the liquid-air surface being a surface of equal pressure, consequently all the surfaces of equal pressure are identical paraboloids of revolution, the pressure per unit area on any surface being the sum of the atmospheric pressure and ρh lb, where ρ is the weight of a cubic foot of the fluid and h ft the depth of the equal pressure surface below the fluid-air surface.

Now consider the case in which the fluid is not constrained to move as a solid body but is frictionless and is free in all its parts to take up any appropriate speeds. We shall also assume that the fluid

is escaping very slowly through a small hole downwards along the axis of z. Stream-line motion will take place and Bernouilli's theorem will apply. The curves in Fig. 78 show the intersection of the free surface with a plane through OZ. Fluid will flow along this surface, moving in spirals along the surface. If LL is an edge view

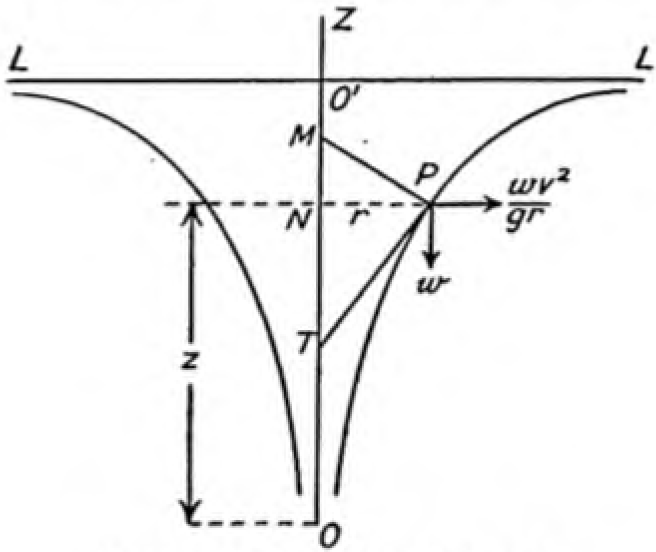


FIG. 78. FREE VORTEX MOTION

of the horizontal level of the water where it is at rest, then in accordance with Bernouilli's law

$$\frac{v^2}{2g} + z = \text{constant} \quad . \quad . \quad . \quad (IX.44)$$

We omit the term $\frac{p}{w}$ because we are dealing with surface flow and the pressure, being atmospheric, is constant. Thus at short distances below LL the velocity is small, but increases as z decreases. Since the downward flow is small, the fluid moves in spiral curves which are almost horizontal. Consider a particle of mass $\frac{w}{g}$ engineers' units at P. This particle is moving very nearly on a circle of radius $r = \overline{NP}$ ft with velocity v ft/sec. Its reversed mass-acceleration is $\frac{wv^2}{gr}$ lb, as shown. The particle may be treated as if it were in equilibrium under the action of three forces, (1) this mass-acceleration, (2) its weight w lb, and (3) the pressure of the fluid along PM the normal to its surface at P. PMN is a triangle of forces; $\overline{MN}$ represents w lb and

$\overline{NP}$ represents $\dfrac{wv^2}{gr}$ lb. PT is the tangent at P, meeting OZ at T. From the geometry of the figure

$$\frac{dr}{dz} = \tan \widehat{PTN} = \tan \widehat{MPN} = \frac{\overline{MN}}{NP}$$

i.e.

$$\frac{dr}{dz} = \frac{gr}{v^2} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (IX.45)$$

From (IX.44) and (IX.45)

$$\frac{r}{2}\frac{dz}{dr} + z = \text{constant } C$$

i.e.

$$\frac{2dr}{r} = \frac{dz}{C - z}$$

Integrating, $\log_e r^2 = \log_e A - \log_e (C - z)$, where A is an arbitrary constant.

Hence $$r^2 = \frac{A}{C - z},$$ which can be written in the form

$$z = C - \frac{A}{r^2} \qquad . \quad . \quad . \quad . \quad . \quad (IX.46)$$

At LL, $r = \infty$. Let z_0 be the value of z at LL. Then, from (IX.46), $z_0 = C$, and, therefore

$$z_0 - z = \frac{A}{r^2} \qquad . \quad . \quad . \quad . \quad . \quad (IX.47)$$

If the origin is changed to the point O' in which OZ meets LL, and z is measured positive downwards, then from (IX.47)

$$z = \frac{A}{r^2} \qquad . \quad . \quad . \quad . \quad . \quad (IX.48)$$

Each of the curves in Fig. 78 is the graph of r plotted horizontally against z plotted along $O'O$. An approximation to this kind of fluid motion can be seen when water in a full wash-basin is set spinning and the plug is withdrawn from the waste pipe. The motion is called *free vortex* motion.

93. **Line, Surface, and Volume Integrals.** In Art. 89 we met with integrals $\int_{r_1}^{r_2} \boldsymbol{F} \cdot d\boldsymbol{r}$ and $\int_{r=r_1}^{r=r_2} F_c ds$ representing the work done by

a force on a particle as the latter moves along a given curve. Integrals such as these, in which the integration is taken along a curve, are known as *line integrals*. In general, their values will depend upon the curves along which they are to be evaluated. The integrand may be any scalar function of the co-ordinates x, y of any point on the curve. For example, if $\phi(x,y)$ is the mass per unit length of curve, then $\int_{s_1}^{s_2} \phi(x, y)ds$ represents the total mass between the end points $s = s_1$ and $s = s_2$ of the curve.

In a conservative field of force, as we have seen, the work done in producing a displacement between two given points is independent of the path, so that in this case the value of $\int_{r_1}^{r_2} \mathbf{F} \,.\, d\mathbf{r}$ or $\int_{r=r_1}^{r=r_2} F_c ds$ will not depend on the particular curve along which the evaluation is effected so long as the curve is continuous and joins the two end points.

Instead of indicating the limits of integration as above, we usually denote a line integral by the notation $\int_C \phi(x, y)ds$, showing that the integral is to be evaluated along a curve C.

Line integrals often occur in the form

$$\int_C [M(x, y)dx + N(x, y)dy] \qquad . \qquad . \quad \text{(IX.49)}$$

For example, if the force $\mathbf{F}$ has horizontal and vertical components X and Y respectively, then

$$\int_C \mathbf{F} \,.\, d\mathbf{r} = \int_C (Xdx + Ydy)$$

EXAMPLE 1

Evaluate $\int_C [M(x, y)dx + N(x, y)dy]$ along the parabola $y = x^2$ between the points $x = 0$ and $x = 4$, where $M(x, y) = x + y$ and $N(x, y) = xy$.

Also evaluate the integral along the straight line joining the same end points.

We have $\int_C [(x + y)dx + xy\,dy] = \int_0^4 (x + x^2)dx + \int_0^{16} y^{\frac{3}{2}}dy$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^4 + \tfrac{2}{5}\left[y^{\frac{5}{2}}\right]_0^{16}$$

$$= 29\tfrac{1}{3} + 409\tfrac{3}{5} = 438{\cdot}9$$

The end points are (0, 0) and (4, 16), so that the equation of the straight line joining these end points is $y = 4x$.

The value of the integral along this path is

$$\int_C [(x + y)dx + xy\,dy] = \int_0^4 5x dx + \int_0^{16} \tfrac{1}{4}\,y^2 dy$$
$$= 40 + 341\tfrac{1}{3} = 381\cdot 3$$

Thus, in this case the value of the integral depends upon the path of integration. We can perform the integration with respect to either variable x or y, whichever is the more convenient. Thus, when $y = x^2$, the integral $\int_0^{16} xy\,dy$ can be transformed into $\int_0^4 2x^4 dx$ by the substitutions $y = x^2$ and $dy = 2x\,dx$, and its value is $409\frac{3}{5}$, as before.

EXAMPLE 2

Evaluate $I = \int_C [(x + x^2y)dx + (\frac{1}{3}x^3 + y)dy]$ between the points (1, 1) and (2, 4) along each of the following paths—(1) the straight line $y = 3x - 2$, (2) the parabola $y = x^2$, and (3) the straight line $y = 1$ from $x = 1$ to $x = 2$ followed by the straight line $x = 2$ from $y = 1$ to $y = 4$.

(1) Put $y = 3x - 2$ and $dy = 3dx$.

Then
$$I = \int_1^2 \{x + 3x^3 - 2x^2 + (\tfrac{1}{3}x^3 + 3x - 2)3\}dx$$
$$= \int_1^2 (4x^3 - 2x^2 + 10x - 6)dx$$
$$= \Big[x^4 - \tfrac{2}{3}x^3 + 5x^2 - 6x\Big]_1^2 = 19\tfrac{1}{3}$$

(2) Put $y = x^2$ and $dy = 2xdx$

Then
$$I = \int_1^2 \{x + x^4 + (\tfrac{1}{3}x^3 + x^2)2x\}dx$$
$$= \int_1^2 (\tfrac{5}{3}x^4 + 2x^3 + x)dx$$
$$= \Big[\tfrac{1}{3}x^5 + \tfrac{1}{2}x^4 + \tfrac{1}{2}x^2\Big]_1^2 = 19\tfrac{1}{3}$$

(3) Here $y = 1$ from $x = 1$ to $x = 2$. and $x = 2$ from $y = 1$ to $y = 4$. Then

$$I = \Big\{\int_1^2 (x + x^2)dx + \int_1^1 (\tfrac{1}{3}x^3 + 1)dy\Big\} + \Big\{\int_2^2 (2 + 4y)dx + \int_1^4 (\tfrac{8}{3} + y)dy\Big\}$$
$$= \Big[\tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\Big]_1^2 + 0 + 0 + \Big[\tfrac{8}{3}y + \tfrac{1}{2}y^2\Big]_1^4$$
$$= 3\tfrac{5}{6} + 15\tfrac{1}{2} = 19\tfrac{1}{3}$$

The value of I is the same along all three paths, and in this case will be found to be the same for any other path between the same points.

We saw in Vol. I, when solving exact differential equations, that $Pdx + Qdy$, in which P and Q are functions of x and y, is a perfect or complete differential if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. In this case $Pdx + Qdy$ can be written in the form du, where u is a function of x and y, and therefore

$$\int_C (Pdx + Qdy) = [u]_C \quad . \quad . \quad . \quad \text{(IX.50)}$$

With the necessary change of notation, this means that

$$\int_C (Mdx + Ndy) = [u]_C \quad . \quad . \quad . \quad \text{(IX.51)}$$

can be evaluated without assuming any particular path, the condition that the value of the line integral (IX.51) should be independent of the path of integration being

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad . \quad . \quad . \quad . \quad \text{(IX.52)}$$

In Ex. 1 above, $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = y$, so that (IX.52) is not satisfied and, therefore, the value of the line integral depends upon the path of integration. In Ex. 2, $\frac{\partial M}{\partial y} = x^2 = \frac{\partial N}{\partial x}$, so that here the condition (IX.52) is satisfied, and the value of the line integral is independent of the path. Now, by the method of Art. 133, Vol. I, if $u = \int (Mdx + Ndy)$ and M and N satisfy (IX.52), then $M = \frac{\partial u}{\partial x}$, so that in this second example $\frac{\partial u}{\partial x} = x + x^2y$, and by integration

$$u = \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3y + F(y) \quad . \quad . \quad \text{(IX.53)}$$

where $F(y)$ is an arbitrary function of y.

Similarly, $\qquad \frac{\partial u}{\partial y} = \frac{1}{3}x^3 + y$, and by integration

$$u = \tfrac{1}{3}x^3y + \tfrac{1}{2}y^2 + f(x) \quad . \quad . \quad . \quad \text{(IX.54)}$$

where $f(x)$ is an arbitrary function of x.

Since (IX.53) and (IX.54) must be identical, then $F(y) = \frac{1}{2}y^2$ and $f(x) = \frac{1}{2}x^2$, and

$$\int_C \left[(x + x^2y)dx + (\tfrac{1}{3}x^3 + y)dy\right] = \left[\tfrac{1}{3}x^3y + \tfrac{1}{2}y^2 + \tfrac{1}{2}x^2\right]_{1,1}^{2,4}$$
$$= (10\tfrac{2}{3} + 8 + 2) - (\tfrac{1}{3} + \tfrac{1}{2} + \tfrac{1}{2})$$
$$= 19\tfrac{1}{3}, \text{ as before.}$$

If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the value of $\int_C (Mdx + Ndy)$ round a closed path is zero. This is clearly true since, if we choose any two points on the path as end-points, the value of the line integral is the same along both parts. If we reverse the sense of integration along one of the paths, dx and dy will everywhere change sign and thus the integral along the reversed path will be numerically equal but opposite in sign to that along the direct path, and the total integral round the path will be zero. For example, the work done by a conservative force on a particle as it moves round a closed plane path in a conservative field is $\int_C (Xdx + Ydy) = 0$, where X and Y are the rectangular components of the force.

SURFACE AND VOLUME INTEGRALS. If ΔS is an element of area in a given surface whose equation is $z = f(x, y)$ or $\phi(x, y, z) = 0$, the integral $\int_S F(x, y, z)dS$, in which $F(x, y, z)$ is any function of the co-ordinates x, y, z, is known as a *surface integral*. Similarly, if ΔV is an element of volume enclosed by a surface $z = f(x, y)$ or $\phi(x, y, z) = 0$, the integral $\int_V F(x, y, z)dV$ is known as a *volume integral*. The reader should consult the sections of Vol. I which deal with multiple integrals, where he will find alternative forms for the above integrals and many examples of their applications.

EXAMPLES IX

(1) A horizontal square is formed by 4 uniform rods each weighing W lb and freely jointed at its corners. The square is supported by 4 strings whose lengths are twice that of a side of the square and which join the corners of the square to a fixed point. Find the force of tension in a string and that of compression in a rod.

(2) A uniform square lamina of 40 lb weight is acted upon by forces of magnitudes 10 lb, 20 lb, 30 lb, and 40 lb, along the edges of the square all pointing the same way round the square. If the side of the square is 1 ft long, find the angular acceleration of the lamina and the linear acceleration of its central point.

(3) A uniform rod AB, 4 ft long and 100 lb weight, moves so that its end travels along a vertical line $Y'OY$ whilst its end B travels along a horizontal line $X'OX$, each end having simple harmonic motion of amplitude 4 ft and frequency $20/2\pi$ vibrations per second. Prove that the mid-point C of the rod moves in a circle, centre O, of radius 2 ft with uniform speed of 40 ft/sec. Find the kinetic energy of the rod when it is inclined at 30° to $X'OX$.

(4) Three equal uniform rods AB, BC, CD each of length $2a$ ft are joined by hinges at B and C. They are in a straight line ABC and are moving with velocity v ft/sec in a direction perpendicular to ABC when (*a*) the point B is suddenly fixed, or (*b*) the mid-point of BC is suddenly fixed.

Find the angular speeds of the rods in each case just after impact.

(5) A uniform straight rod of length l ft rests on a smooth surface and is struck a blow at one end perpendicular to the rod. Show that a point on the rod $\frac{2}{3}l$ from the struck end will be initially at rest.

(6) A lever 500 lb weight with a radius of gyration about its mass-centre of 12 in. is pivoted about an axis 8 in. from the mass-centre. Find an equivalent dynamical system of two particles one of which is at the pivot. If the lever is rotating at 20 radn/sec, find (*a*) the kinetic energy, (*b*) the angular momentum of the two systems.

(7) Show that if a rigid lamina is mounted on a pivot, at any point O, perpendicular to the plane of the lamina, and is struck by an impulsive blow in the plane of the lamina through O' the centre of oscillation there will be no shock on the pivot. A much used swing door is prevented from opening too wide by stops against which it is continually banging. Assuming the mass of the door to be uniformly distributed, where should the stops be placed so as to prevent damage to the hinges?

(8) Assuming the area A of a triangle can be taken as concentrated in "particles" each of area $A/3$ at the mid-points of the sides when finding second moments, find the second moment of the area of a regular hexagon about a line joining (1) opposite corners, (2) the mid-points of opposite sides. (a = length of side.)

(9) Prove Clapeyron's theorem of three moments in the case of a heavy uniform rod with no concentrated loads assuming small differences of level at the supports. (U.L.)

(10) A uniform beam of length $3a$ and flexural rigidity EI rests on four supports at the same level so that there are three equal bays each of length a. It carries over its whole length a distributed load of intensity w. Find the central deflection and the reactions at each support. (U.L.)

(11) Assume the beam in the previous question to be of length $4l$ and to rest on 5 equally-spaced supports at the same level. Find the reactions on the supports.

(12) A uniform continuous beam length $2l$ is carried on three supports one at each end and one in the middle. The load $2wl$ is uniformly distributed and the end supports are above the level of the middle one by amounts $\dfrac{wl^4}{8EI}$ and $\dfrac{wl^4}{24EI}$ respectively. Assuming free ends, prove that the pressure on the middle support is three-fifths what it would be if the supports were at the same level.

(13) A uniform continuous beam PQR is built-in at P, where the slope is zero, and rests on supports at the same level at Q and R. There is a uniform distributed load of 100 lb/ft along PQ and one of 80 lb/ft along QR and there are concentrated loads of 500 lb, 5 ft from P and 600 lb, 4 ft from R. Find by the theorem of three moments the bending moments at P and Q.

(14) A uniform continuous beam ABC of two spans is supported at A, B and C, the level of B being 0·024 in. below that of A and C. AB is 9 ft and BC 12 ft long and there is a uniformly distributed load of 2 000 lb/ft. Given $E = 30 \times 10^6$ lb/in.^{2} and $I = 128$ (inch)^4 units, find by the theorem of three moments the bending moment at B and the pressure on the support there.

(15) A light uniform flexible inextensible string of length l is attached at its ends to the ends of a thin uniform rod of length $2a$ ($l > 2a$), and the string is supported by a smooth small peg. Show that the position of equilibrium in which the rod is horizontal is unstable. (U.L.)

(16) A point E is taken on one side of a square plate $ABCD$ of side a, and the triangular portion CDE is cut away. The remainder is immersed vertically in water with the side AB in the surface. Show that the depth of the centre of liquid pressure is $\frac{1}{2}\left(\frac{a^4 - b^4}{a^3 - b^3}\right)$, where $BE = b$. What does this result yield when $b = 0$, and when $b = a$? (U.L.)

(17) A cubical block of side $2a$ stands on a horizontal plane rough enough to prevent sliding. If the plane is suddenly given a horizontal velocity v parallel to two vertical faces of the block, determine the initial motion of the block and prove that the block will upset if $v^2 > \frac{16}{3} ag(\sqrt{2} - 1)$. (U.L.)

(18) A solid uniform hemisphere of weight W and radius a is held at rest on a horizontal plane with its plane face vertical. If it is released from rest find the initial values of the reaction of the plane and the angular acceleration (*a*) if the plane is perfectly smooth, (*b*) if it is rough enough to prevent slipping. (U.L.)

(19) Find the moment of inertia about its axes of a hollow cylinder of external radius a and internal radius b. If the cylinder is placed on an inclined plane of angle α with its axis perpendicular to the line of greatest slope of the plane, prove that it will roll down the plane without slipping if

$$\mu > \frac{a^2 + b^2}{3a^2 + b^2} \tan \alpha$$

If μ is less than this value, show that whilst the cylinder, starting from rest, makes one revolution about its axis it will move down the plane a distance greater by $\frac{\pi}{a\mu}\{(a^2 + b^2) \tan \alpha - (3a^2 + b^2)\mu\}$ than it does when μ is greater than this value. (U.L.)

(20) A uniform platform of weight W is in the form of a rectangle with sides a and b. The platform is supported in a horizontal plane by means of a smooth joint at one corner and ropes attached to the corners adjacent to the joint. If these ropes are attached to a point which is at a height h, vertically above the joint, show that the tensions in the ropes are $\frac{1}{2}W\sqrt{1 + \frac{a^2}{h^2}}$, $\frac{1}{2}W\sqrt{1 + \frac{b^2}{h^2}}$. Find the magnitude of the reaction at the joint. (U.L.)

(21) $ABCD$ is a rhombus formed by four equal thin uniform smoothly-jointed rods each of length $4\sqrt{3}a$. The system hangs over a smooth circular cylinder of radius a and centre O whose axis is horizontal. Find the angle BAO in the position of equilibrium in which AOC is a vertical straight line, and show that it is a stable position for displacements in which A and C move vertically. (U.L.)

(22) Show that the centre of gravity of one-half of a uniform right-circular

cone, cut off by an axial plane, is distant r/π from the plane of section, r being the radius of the plane base.

The half-cone is immersed in water so that the plane semi-circular base is in the free surface. Show that the line of action of the fluid pressure on the curved surface meets the plane base in its straight edge if the height of the cone is equal to the radius of the base. (U.L.)

(23) A cylinder has its axis parallel to the z-axis. Prove that the co-ordinates of the centre of gravity of the solid formed by cutting the cylinder by the planes $z = 0$, $z = px + qy + h$ are

$$\frac{pk_2^2}{h}, \frac{qk_1^2}{h}, \frac{p^2k_2^2 + q^2k_1^2 + h^2}{2h}$$

where the origin is the centre of gravity of the section $z = 0$, the axes of x and y are the principal axes at the origin, and k_1, k_2 are the principal radii of gyration.

If the cylinder is a right-circular cylinder of radius a, and the oblique section $z = 2x + na (n > 2)$, prove that if the body is suspended from the centre of the oblique section, its axes will make an angle $\tan^{-1}\dfrac{1}{n^2 - 1}$ with the vertical. (U.L.)

(24) A uniform rod AB, of mass m and length $2a$, can turn freely about a horizontal axis at A. An elastic string of unstretched length a and modulus λ is attached to B and to a point C vertically above A where $AC = 2a$. If $3\lambda > mg$ prove that there is a position of stable equilibrium given by $\sin\dfrac{\alpha}{2} = \dfrac{\lambda}{4\lambda - mg}$, where α is the angle the rod makes with the upward vertical. If the rod is slightly displaced from this position show that the period of small oscillations is

$$4\pi\left\{\frac{(4\lambda - mg)am}{3(5\lambda - mg)(3\lambda - mg)}\right\}^{\frac{1}{2}} \qquad \text{(U.L.)}$$

(25) Find the position of the centre of percussion of a uniform sphere rotating about an axis tangential to its surface. The cushion of a billiard table is at a height h in. above the horizontal surface and the diameter of the balls is 2 in. Find the best value of h to prevent excessive wear of the cloth due to slipping of the ball at its contact with the flat surface along a strip near to, and parallel to, the cushion.

(26) What is the chief characteristic of a conservative field of force? Prove that in such a field the sum of the kinetic energy and the potential energy is constant. Show that if a vertical plane lamina moves in its own plane its kinetic energy is the sum of (1) that of a particle of the same mass as the lamina moving with the centre of mass of the lamina, i.e. the kinetic energy of translation and (2) the kinetic energy of the lamina due to its rotation about its centre of mass, i.e. the kinetic energy of rotation.

(27) A circular cylinder of radius r has its centre of mass at a distance a from its axis and rests with its axis horizontal on an inclined plane making an angle α with the horizontal. Assuming no slipping, find the position of equilibrium and the condition for stability.

(28) A uniform square plate weighing W lb is held in a vertical plane with two sides vertical and is allowed to fall under gravity. When it has fallen through h ft one of the lower corners is suddenly fixed. If the side of the square is a ft, find the angular velocity of the plate immediately after impact, the impulse of the blow, and the loss of energy due to the impact.

(29) An elliptical cylinder of major axis a and minor axis b is fixed with its axis horizontal. A uniform plank of thickness $2h$ rests on it like a see-saw in a horizontal position. Show that the equilibrium is stable if $bh < a^2$.

(30) A framework consists of two vertical rods AB, DC each l ft long and a ft apart hinged at their lower ends and joined to a cross-bar BC by rigid joints at B and C. All bars are of the same material and of the same cross-section perpendicular to the plane of the figure. There is a load of W lb weight at the mid-point of BC. The vertical reactions at the hinges are each of $W/2$ lb; let the horizontal reactions there be F lb both acting inwards. Show that the strain energy due to bending in each of AB and DC is $F^2l^3/6EI$, using the formula

$$\text{Strain energy} = \int_0^l \frac{M^2}{2EI}\,dx$$

with the notation of Art. 88. Similarly show that the strain energy in the bar BC is $\dfrac{a}{96EI}(48F^2l^2 - 12alFW + Wa^2)$. Use the principle of least energy to find F.

(31) Twelve equal uniform bars are connected together by hinges in the form of a hexagonal framework, six of the rods joining the centre to the corners. Equal and opposite forces each of P lb are applied at opposite corners. Show that the forces in the two bars in line with the applied forces are each of $\frac{5}{6}P$ lb weight and those in the other bars are of $\frac{1}{6}P$ lb weight. [Assume R to be the force in each of the bars in line, determine the other forces and apply the principle of least energy.]

(32) Find the distance below the centre of gravity of the area of the centre of pressure of a plane area immersed vertically in a liquid and subject to fluid pressure on one side.

A sluice-gate in the form of a semi-circle of radius 2 ft hinged at its bounding diameter, is placed in the vertical side of the reservoir with its bounding diameter horizontal and uppermost and 10 ft below the level of the water; prove that the horizontal force which must be applied at the lowest point of the gate to keep it shut is approximately 1 863 lb weight. [1 ft^3 of water = 62·5 lb.] (U.L.)

(33) A sphere is half immersed in water and from a fixed point O, a line OP is drawn representing in magnitude and direction the thrust on any completely immersed portion of the spherical surface bounded by a circle of constant radius. Prove that the locus of P is part of a sphere. (U.L.)

(34) Prove Bernoulli's theorem in the case of the steady motion of a homogeneous incompressible fluid.

A water tap of diameter $\frac{1}{4}$ in. is 60 ft below the level of the reservoir which supplies water to a town. Find the maximum amount of water which could be delivered by the tap in gallons per hour.

(35) Prove Bernoulli's theorem in the case of steady motion of a homogeneous incompressible fluid.

Liquid of density ρ is flowing along a horizontal pipe of variable cross-section, and is connected with a differential pressure gauge at two points A and B. Show that if $p_1 - p_2$ is the pressure indicated by the gauge, the mass of fluid flowing through the pipe per second is

$$\sigma_1\sigma_2\left[\frac{2\rho(p_1 - p_2)}{\sigma_1{}^2 - \sigma_2{}^2}\right]^{\frac{1}{2}}$$

where σ_1; σ_2 are the areas of the cross-sections at A, B respectively. (U.L.)

(36) Obtain the expression I/V for the metacentric height of a uniform solid of revolution floating in equilibrium with its axis vertical.

A uniform *thin* hollow circular cylinder without ends made of material of specific gravity s floats in water with its axis vertical. Prove that the equilibrium will be stable if the ratio of the height to the radius of the base is less than $1/\sqrt{s - s^2}$. (U.L.)

(37) Show that if the cylinder in Ex. (36) is not thin and has an external radius a and an internal radius b, the equilibrium will be stable if $a^2 + b^2 > 2h^2s(1 - s)$, h being the height. If $b = 0$, i.e. the cylinder is solid, find the least value of a for stable equilibrium.

(38) Find the metacentric height in the case of a right solid hexagonal prism floating in water with its axis vertical, h = height, a = side of hexagon, s = density. Give the condition for stability.

(39) An elliptical cylinder of minor axis a and major axis b floats in water with its axis vertical. The specific gravity of the material is s. Show that for stability the length of the cylinder must be less than $\sqrt{\dfrac{a^3}{2b(s - s^2)}}$

(40) A square frame of 6 ft sides is made of bars of uniform material weighing 6 lb per foot length. Besides the four side bars there is one diagonal bar of the same uniform material. The joints are frictionless pivots. Find the force in the diagonal bar when the frame hangs vertically from its upper corner with that bar horizontal. Solve this example by the principle of virtual work and also by an alternative method.

(41) $ABCD$ is a framework of four light but rigid bars connected together by pivots at the corners. The length of each is l ft. The pivot at A is mounted on a vertical shaft with which it rotates at N r.p.m. and that at C is attached to a bush which can slide along the vertical shaft. Each of the pivots at B and D carries a heavy ball of W lb weight. Under the action of the gravity forces on the balls and the "centrifugal" forces on them, the framework takes up an equilibrium position whilst rotating. Using the method of virtual work find the distance AD in this position. (This is the mechanism of the Watt Governor.)

(42) A shaft with its attached rotating masses weighs 2 000 lb with a radius of gyration of 2 ft and is rotating at 120 r.p.m. A pinion of 20 teeth on this shaft, and moving with it, is suddenly engaged with a wheel of 80 teeth on a second shaft carrying masses weighing 800 lb with a radius of gyration of 2 ft 6 in. Find (*a*) the speed of the second shaft after connection, (*b*) the impulsive couple on the second shaft, (*c*) the loss of energy due to the impact.

(43) A circular horizontal disc of radius 4 ft is kept in rotation at 50 r.p.m. Steel balls of which 50 weigh 1 lb are fed continually to the centre of the disc down a vertical tube and are thrown outward by the "centrifugal" force being constrained to move along radial channels on the disc by radial plates secured to it. If the rate of feed is 1 000 balls per minute find the couple to keep the disc in motion. Neglect friction.

(44) A vertical hollow cylinder is open at the top. Its inside radius is 6 in. and it is full of water. The cylinder, which is deep, is set rotating at 120 r.p.m. and is kept rotating until all relative motion between the water and the cylinder is stopped. Find the equation of a section of the water surface by a plane through the axis of rotation and the amount of water which will be spilled from the cylinder.

(45) Explain the difference between a free vortex and a fixed vortex and prove that an axial plane intersects their surfaces in curves whose equations are

$z = \dfrac{\omega^2 r}{2g}$ and $z = \dfrac{A}{r^2}$ respectively where z is measured along the vertical and r along the horizontal from the axis of rotation.

(46) Evaluate $\int_C [(x^2 + y^2)dx + 2xy\,dy]$ along the arc of the parabola $y = x^2$ from $x = 0$ to $x = 4$. As the end-points of this arc are the points (0, 0) and (0, 16) the limits of integration can be shown by $\int_{(0,0)}^{(0,16)} [(x^2 + y^2)dx + 2xy\,dy]$. Evaluate $\int_{(1,2)}^{(2,8)} [(x + y)dx + (x - y)dy]$ along the parabola $y = 2x^2$.

(47) Evaluate $\int_C [(x + 2y)^2\,dx + (3x + y)^2\,dy]$ along the path $y = 2x + 1$ from $x = 0$ to $x = 2$. Evaluate $\int_{(1,1)}^{(4,2)} [(x + 2y)^2\,dx + (3x + y)^2\,dy]$ along the parabola $y = \sqrt{x}$.

(48) Evaluate $\int_{(0,0)}^{(1,1)} [(3x^2 + 4xy + 3y^2)dx + 2(x^2 + 3xy + 4y^2)dy]$ along the paths (*a*) the parabola $y^2 = x$, (*b*) the parabola $y = x^2$, (*c*) $y = 0$ from (0, 0) to (1, 0) and $x = 1$ from (1, 0) to (1, 1), (*d*) $x = 0$ from (0, 0) to (0, 1) and $y = 1$ from (0, 1) to (1, 1). Explain why the answers are not all different.

(49) When finding the area A enclosed by a loop of a closed curve in Vol. I we found that $A = \int_{x_1}^{x_2} (y_2 - y_1)dx$ where y_1 and y_2 are the ordinates of the lower and upper points in which a vertical line cuts the curve. Show that this is the same as $A = -\int_C y\,dx$ where the integration is taken along the curve, passing round in the counter-clockwise sense. Show also that $A = \int_C x\,dy$ in the same sense and that $A = \frac{1}{2}\int_C [-y\,dx + x\,dy]$.

(50) Using the last of the rules in (49) prove that the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab. Use the substitutions $x = a\cos\phi$, $y = b\sin\phi$, ϕ being the eccentric angle and substitute also for dx and dy in terms of ϕ.

(51) Find the area enclosed by the curves $y = x^2$ and $y^2 = 8x$. Use one of the rules in (49).

(52) Suppose a liquid to be flowing parallel to the plane XOY in a stream of unit thickness perpendicular to that plane. A closed curve is drawn on the xy-plane. Show that the volume of fluid crossing the boundary per second is $\int_C (-v\,dx + u\,dy)$ where u and v represent the component velocities of a particle of the fluid parallel, respectively to OX and OY. If the fluid is incompressible what is the value of the integral?

(53) Prove the formulae $s = c\tan\psi$ and $y^2 = s^2 + c^2$ for a uniform catenary. A heavy uniform flexible string has one end fixed and rests in equilibrium, passing over a smooth peg, with its other end hanging freely. The length of the vertical portion of the string is 2 ft. The fixed end and the lowest point of the catenary are respectively $2/\sqrt{3}$ ft and 1 ft above the free end. Find the angles which the string makes with the horizontal at the fixed end and at the peg, and also the total length of the string. (U.L.)

(54) A uniform chain of weight w lb/ft is suspended from two points at the same level, distant D ft apart. Prove that

$$D = 2c \log_e(\sec \phi + \tan \phi)$$

where wc is the tension at the lowest point and ϕ is the inclination of the tangent at either end to the horizontal. A chain 100 ft long is suspended from two points at the same level and the ends are inclined at 60° to the horizontal. Determine the span and the sag at the lowest point. (U.L.)

(55) One end of a uniform chain of length l is attached to a fixed point A; B is the rounded edge of a rough horizontal table, AB being a horizontal line of length $2a$. The chain lies partly on the table at right angles to the edge and partly hangs as a festoon between A and B. Prove that when the length on the table has the least value consistent with equilibrium the parameter c of the catenary is given by

$$\mu e^{\theta}\left(\frac{xl}{a} - 2 \sinh x\right) = \cosh x,\ \theta = \mu \tan^{-1} (\sinh x),$$

where $x = a/c$ and μ is the coefficient of friction. (U.L.)

(56) A uniform chain is stretched between two points at the same level 22 yd apart. If the sag in the centre is 3 ft, prove that the parameter of the catenary is approximately 182 ft. If the chain is tightened so that the sag is 1 ft, find the new parameter, and determine the ratio of the tensions in the two cases, and the ratio of the lengths of chain between the supports. (U.L.)

CHAPTER X

ANALYTICAL SOLID GEOMETRY

94. Rectangular Co-ordinates. Three planes (Fig. 79) mutually at right angles and intersecting along the lines XOX', YOY', ZOZ', divide space into eight compartments. We can determine the position of a point P in space if we know (1) the compartment in which P lies,

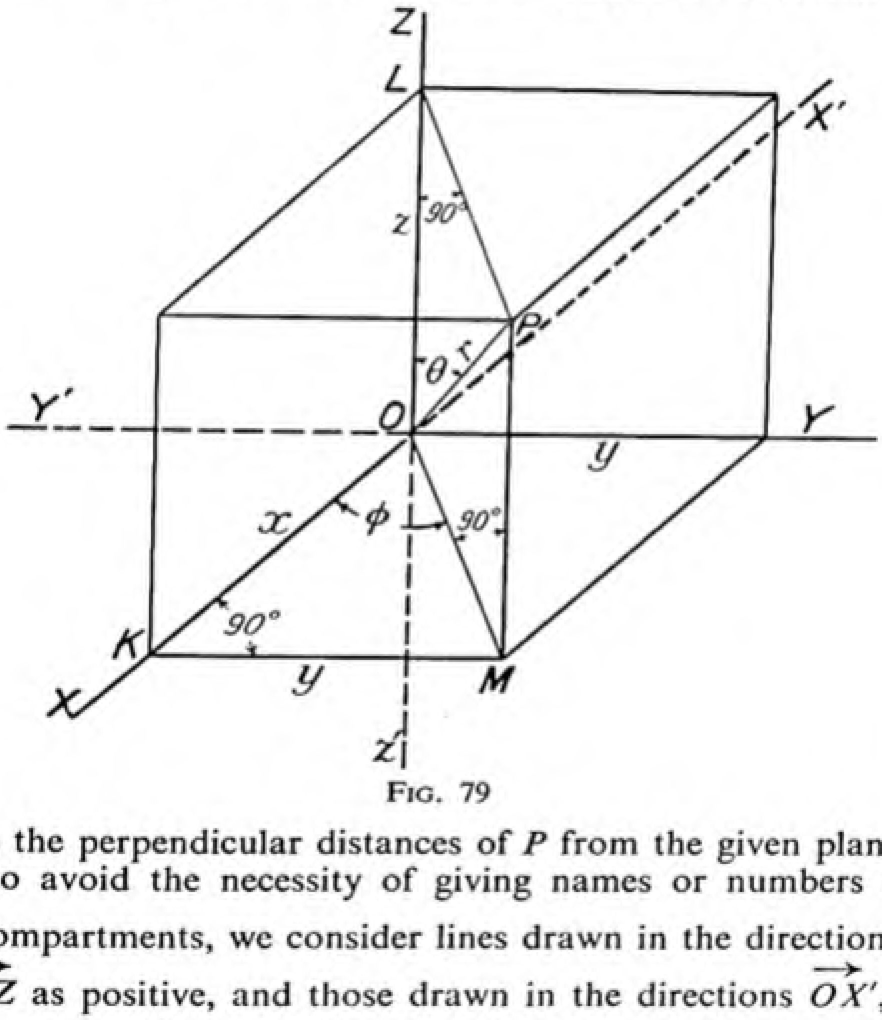


FIG. 79

and (2) the perpendicular distances of P from the given planes. In order to avoid the necessity of giving names or numbers to the eight compartments, we consider lines drawn in the directions $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ as positive, and those drawn in the directions $\overrightarrow{OX'}$, $\overrightarrow{OY'}$, $\overrightarrow{OZ'}$ as negative, so that, having their proper signs attached, the perpendicular distances of P from the three planes of reference fix the position of P uniquely. These perpendicular distances x, y, z of P from the planes YZ, ZX, XY respectively are termed the *rectangular*

co-ordinates of P, which is designated briefly as the point (x, y, z). In Fig. 80 we show the positions of the three points $P_1(3, -2, 4)$, $P_2(4, 2{\cdot}5, -3)$, $P_3(-4, 3{\cdot}5, 2{\cdot}5)$. The x of every point on the YZ-plane is zero, and accordingly the equation $x = 0$ represents that plane; similarly, the equations $y = 0$ and $z = 0$ represent the planes ZX and XY respectively.

95. **Polar Co-ordinates.** We can also determine the position of a point P in space if we know (see Fig. 79) (1) the length $OP(= r)$,

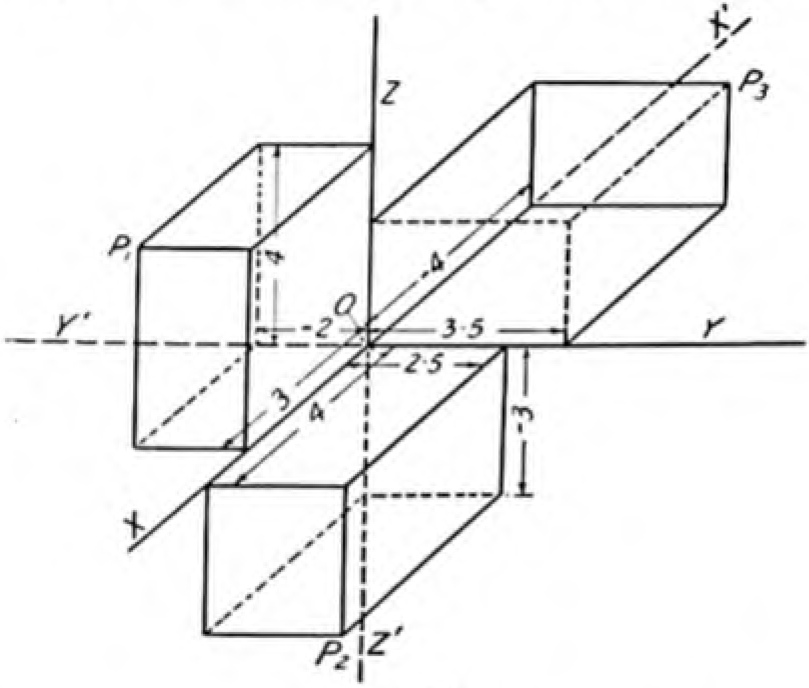


FIG. 80

(2) the angle $ZOP(=\theta)$, and (3) the angle $XOM(=\phi)$, where M is the projection of P on the XY-plane. These quantities r, θ, ϕ are called the *polar co-ordinates* of P, which is designated briefly as the point (r, θ, ϕ).

In the figure, the angles OKM, OMP, and OLP are right angles.

Hence, $OP^2 = OM^2 + MP^2$ and $OM^2 = OK^2 + KM^2$

$$\therefore \quad OP^2 = OK^2 + KM^2 + MP^2 \text{ or } r^2 = x^2 + y^2 + z^2$$

$$\therefore \quad r = \sqrt{x^2 + y^2 + z^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(X.1)}$$

$$\text{Again,} \quad \cos\theta = \frac{OL}{OP} = \frac{z}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(X.2)}$$

$$\text{and} \quad \tan\phi = \frac{KM}{OK} = \frac{y}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(X.3)}$$

The relations (X.1), (X.2), and (X.3) enable us to find r, θ, ϕ when we know x, y, z.

From (X.2), $z = r\cos\theta$ . . . . (X.4)

Also $x = OM\cos\phi = r\sin\theta\cos\phi$ . . . (X.5)

and $y = OM\sin\phi = r\sin\theta\sin\phi$ . . . (X.6)

The relations (X.4), (X.5), and (X.6) enable us to find x, y, z when we know r, θ, ϕ.

If r be assumed constant, the locus of P is a sphere of radius r and with centre O. By (X.1) the equation of this sphere is

$$x^2 + y^2 + z^2 = r^2 \quad . \quad . \quad . \quad (X.7)$$

EXAMPLE

Find (1) the polar co-ordinates of the point (3, 4, − 5), (2) the rectangular co-ordinates of the point (8, 30°, 120°).

(1) Here $r = \sqrt{3^2 + 4^2 + (-5)^2} = \sqrt{50} = 7{\cdot}071$

$$\cos\theta = \frac{-5}{7{\cdot}071} = -0{\cdot}7071$$

so that $\theta = 135°$

$$\tan\phi = \frac{4}{3} = 1{\cdot}3333, \text{ so that } \phi = 53°\,8'$$

(2) Here

$$\begin{aligned} x &= 8\sin 30°\cos 120° = 8\times 0{\cdot}5\times -0{\cdot}5 = -2 \\ y &= 8\sin 30°\sin 120° = 8\times 0{\cdot}5\times 0{\cdot}866 = 3{\cdot}464 \\ z &= 8\cos 30° = 8\times 0{\cdot}866 = 6{\cdot}928 \end{aligned}$$

96. **Direction-cosines of a Straight Line.** If any straight line OP through the origin O makes angles α, β, γ with OX, OY, and OZ respectively, the cosines of these angles, i.e. $\cos\alpha$, $\cos\beta$, $\cos\gamma$, are called the *direction-cosines* of the line OP and are frequently denoted by the letters l, m, n respectively. It is evident that l, m, n will also be the direction-cosines of any line in space parallel to OP.

Join PK, PL, PN (Fig. 81); then the angles PKO, PLO, PNO are right angles.

Hence

$$\left.\begin{aligned} l &= \cos\alpha = \cos POK = \frac{x}{r} \\ m &= \cos\beta = \cos PON = \frac{y}{r} \\ n &= \cos\gamma = \cos POL = \frac{z}{r} \end{aligned}\right\} \quad . \quad . \quad (X.8)$$

$$\therefore \qquad l^2 + m^2 + n^2 = \frac{x^2 + y^2 + z^2}{r^2}; \text{ but from (X.1)}$$

$$x^2 + y^2 + z^2 = r^2$$

so that $\quad l^2 + m^2 + n^2 = 1 \qquad . \quad . \quad . \quad . \quad . \qquad$ (X.9)

Again, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = (1 - l^2) + (1 - m^2) + (1 - n^2)$

so that $\qquad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2 \quad . \quad . \quad . \quad$ (X.10)

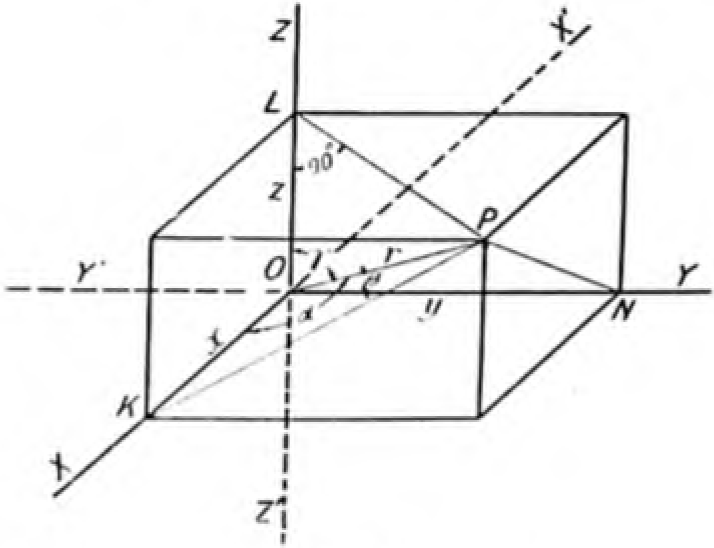


FIG. 81

If the direction-cosines of any line are proportional to f, g, h respectively, we can determine the actual direction-cosines l, m, n of the line as follows—

We have $\dfrac{l}{f} = \dfrac{m}{g} = \dfrac{n}{h}$

$$\therefore \qquad \frac{l^2}{f^2} = \frac{m^2}{g^2} = \frac{n^2}{h^2} = \frac{l^2 + m^2 + n^2}{f^2 + g^2 + h^2} = \frac{1}{f^2 + g^2 + h^2}$$

$$\therefore \qquad \left.\begin{aligned} l &= \frac{f}{\sqrt{f^2 + g^2 + h^2}} \\ m &= \frac{g}{\sqrt{f^2 + g^2 + h^2}} \\ n &= \frac{h}{\sqrt{f^2 + g^2 + h^2}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad \text{(X.11)}$$

EXAMPLE

(1) A straight line is inclined to the axes of y and z at angles 40° and 60°; find the inclination of the line to the x-axis.

(2) The direction-cosines of a straight line are proportional to 2, 5, -11. A point P on a parallel line through the origin is distant 15 from the origin. Find the rectangular co-ordinates of P.

(1) Here $\beta = 40°$, $\gamma = 60°$. The relation (X.9) gives

$$\cos^2 \alpha + \cos^2 40° + \cos^2 60° = 1$$

$$\therefore \qquad \cos \alpha = \sqrt{1 - (0{\cdot}7660)^2 - (0{\cdot}5000)^2} = \pm\, 0{\cdot}4042$$

$$\therefore \qquad \alpha = 66° \, 9' \text{ or } 113° \, 51'$$

(2) Let l, m, n, be the direction-cosines of the line, and x, y, z, the co-ordinates of P. Now $\sqrt{2^2 + 5^2 + (-11)^2} = \sqrt{150} = 5\sqrt{6}$.

By (X.11), $l = \dfrac{2}{5\sqrt{6}} = \dfrac{\sqrt{6}}{15}$; $m = \dfrac{5}{5\sqrt{6}} = \dfrac{\sqrt{6}}{6}$; $n = \dfrac{-11}{5\sqrt{6}} = -\dfrac{11\sqrt{6}}{30}$

From (X.8) we obtain $x = 15l = \sqrt{6}$; $y = 15m = \dfrac{5\sqrt{6}}{2}$; $z = 15n = -\dfrac{11\sqrt{6}}{2}$

97. **Co-ordinates of a Point Dividing a Given Line in a Given Ratio.** Let R (ξ, η, ζ) be a point which divides the straight line joining the two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $s{:}t$. It will be seen from a figure that if p, q, r are the projections of P, Q, R on the xy-plane and pP', qQ', rR' are drawn perpendicular to the x-axis, then, since Pp, Qq, Rr are three parallel lines cut by the two transversals PRQ and prq, $\dfrac{pr}{rq} = \dfrac{PR}{RQ} = \dfrac{s}{t}$; similarly, $\dfrac{P'R'}{R'Q'} = \dfrac{pr}{rq}$ so that $\dfrac{P'R'}{R'Q'} = \dfrac{s}{t}$. Now $OP' = x_1$, $OR' = \xi$, $OQ' = x_2$; hence, $P'R' = OR' - OP' = \xi - x_1$ and $R'Q' = OQ' - OR' = x_2 - \xi$.

$$\left.\begin{aligned} &\therefore \quad \frac{\xi - x_1}{x_2 - \xi} = \frac{s}{t}, \text{ which gives } \xi = \frac{tx_1 + sx_2}{s + t} \\ &\text{By similar arguments we prove } \eta = \frac{ty_1 + sy_2}{s + t} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \zeta = \frac{tz_1 + sz_2}{s + t} \end{aligned}\right\} \quad . \quad . \quad \text{(X.12)}$$

In particular, the co-ordinates of the mid-point of PQ are

$$\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}$$

98. **Equation to a Surface.** If in any equation $F(x, y, z) = 0$ involving the three variables x, y, z we put $z = 0$, we obtain an equation in x and y which we know represents a curve of some kind in the xy-plane. Let M be any point on this curve and substitute in the equation $F(x, y, z) = 0$ the values which x and y have at M. Solving the resulting equation for z, we obtain one or more values of z, and if P be a point on the line through M parallel to the z-axis such that MP is equal to one of those values of z, then the locus of P will, in general, be a surface. The equation $F(x, y, z) = 0$ represents then, in general, a surface of some kind. Two surfaces $F(x, y, z) = 0$ and $f(x, y, z) = 0$ will intersect along a curve, and these two equations will together represent this curve. If the two surfaces are plane, their intersection will be a straight line.

99. **Forms of the Equation of a Plane.** A surface is plane when the straight line joining any two points in it lies wholly in the surface. We shall show that the general equation of the first degree in x, y, z represents a plane. This equation is

$$ax + by + cz + d = 0 \text{ (where } a, b, c, d \text{ are constants)} \quad . \quad \text{(X.13)}$$

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points on the surface (X.13), and let $R(\xi, \eta, \zeta)$ be a point in the straight line PQ dividing that line in the ratio $s{:}t$.

We have $$ax_1 + by_1 + cz_1 + d = 0 \quad . \quad . \quad (1)$$

and $$ax_2 + by_2 + cz_2 + d = 0 \quad . \quad . \quad (2)$$

since P and Q are on the surface.

Multiplying (1) by $\dfrac{t}{s+t}$ and (2) by $\dfrac{s}{s+t}$ and adding, we obtain

$$a\frac{tx_1 + sx_2}{s+t} + b\frac{ty_1 + sy_2}{s+t} + c\frac{tz_1 + sz_2}{s+t} + d\frac{t+s}{s+t} = 0$$

i.e. $$a\xi + b\eta + c\zeta + d = 0 \text{ [by (X.12)]}$$

The point R therefore lies on the surface whatever be the actual value of the ratio $s{:}t$. The surface satisfies the definition of a plane, so that the equation (X.13) is the equation of a plane.

Let the plane cut the axes of x, y, z at distances a', b', c' respectively from the origin. Then a' = value of x obtained by putting $y = 0$, $z = 0$ in (X.13)

so that $$a' = -\frac{d}{a}$$

Hence $a = -\frac{d}{a'}$, and similarly $b = -\frac{d}{b'}$ and $c = -\frac{d}{c'}$

Substituting these values in (X.13) we obtain, after division by $-d$

$$\frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} = 1 \qquad \text{(X.14)}$$

This is the *intercept* form of the equation of a plane.

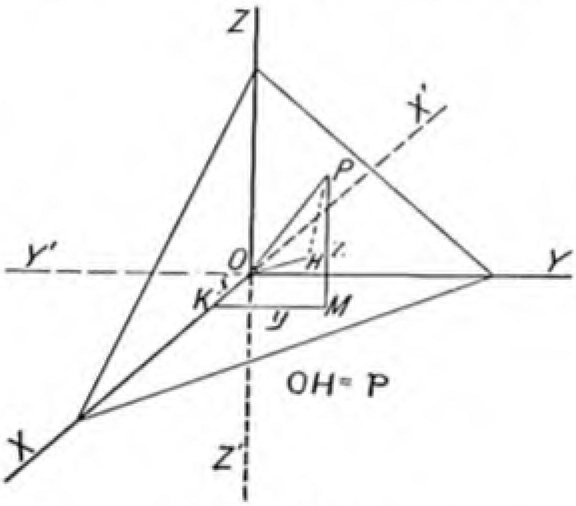


FIG. 82

In Fig. 82, P is any point (x, y, z) in a plane and $OH = p$ is the perpendicular from the origin on to the plane. Let l, m, n be the direction-cosines of the line OH. Then, if PM be drawn perpendicular to the xy-plane and MK perpendicular to the x-axis, $OK = x$, $KM = y$, and $MP = z$, and also $\cos HOK = l$, cos (angle between OH and KM) $= m$, and cos (angle between OH and MP) $= n$.

Now p = projection of OP on OH

= sum of projections of OK, KM, MP on OH

$= xl + ym + zn$

or $\quad lx + my + nz = p \qquad \text{(X.15)}$

This is the *perpendicular* form of the equation of a plane.

The equation of a plane can easily be transformed from one form to another. Comparing the forms (X.13) and (X.15), we see that l, m, n must be proportional to a, b, c, so that, by (X.11),

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

and therefore $p = -\dfrac{d}{\sqrt{a^2 + b^2 + c^2}}$. It follows that we can transform (X.13) into the form (X.15) by simply dividing each term by $\sqrt{a^2 + b^2 + c^2}$.

Again, comparing (X.14) and (X.15) we see that

$$la' = mb' = nc' = p$$

whence $$l^2 + m^2 + n^2 = \frac{p^2}{a'^2} + \frac{p^2}{b'^2} + \frac{p^2}{c'^2}$$

so that $$\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2} = \frac{1}{p^2} \quad . \quad . \quad . \quad . \quad (X.16)$$

EXAMPLE

Find the equation of the plane passing through the three points (2, 3, 4), (– 3, 5, 1), (4, – 1, 2), and find also the angles which the normal to this plane makes with the axes of reference.

Let the equation of the plane be

$$ax + by + cz + d = 0 \quad . \quad . \quad . \quad . \quad (1)$$

Then since the three points lie on the plane

$$2a + 3b + 4c + d = 0 \quad . \quad . \quad . \quad . \quad (2)$$

$$-3a + 5b + c + d = 0 \quad . \quad . \quad . \quad . \quad (3)$$

$$4a - b + 2c + d = 0 \quad . \quad . \quad . \quad . \quad (4)$$

Eliminating a, b, c, d, between (1), (2), (3), and (4), we obtain

$$\begin{vmatrix} x & y & z & 1 \\ 2 & 3 & 4 & 1 \\ -3 & 5 & 1 & 1 \\ 4 & -1 & 2 & 1 \end{vmatrix} = 0 \quad \text{(See Art. 5.)}$$

On expansion, this reduces to $x + y - z - 1 = 0$, which is the required equation of the plane.

Expressed in the "perpendicular" form, this equation becomes $\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y - \frac{1}{\sqrt{3}}z - \frac{1}{\sqrt{3}} = 0$; and if the normal to the plane makes angles α, β, γ with the x, y, and z axes respectively, then $\cos\alpha = \frac{1}{\sqrt{3}} = 0{\cdot}5774$, $\cos\beta = \frac{1}{\sqrt{3}} = 0{\cdot}5774$, $\cos\gamma = -\frac{1}{\sqrt{3}} = -0{\cdot}5774$.

Hence, $\alpha = 54° 44'$, $\beta = 54° 44'$, $\gamma = 125° 16'$.

Since two planes intersect along a straight line, the equations of these planes taken together will represent a straight line. Thus, $ax + by + cz + d = 0$ and $Ax + By + Cz + D = 0$, together represent their line of section. In the next article we deduce a more convenient form for the equations of a straight line.

100. **Distance Between Two Given Points and Equations of a Straight Line.** Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points in space. If the origin be transferred to P, the axes remaining parallel to their original positions, then the co-ordinates of Q become $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$. It follows from (X.1) that the length PQ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \qquad \text{(X.17)}$$

If $PQ = r$ is constant and P is fixed in position, the locus of Q is a sphere of radius r and centre P; the equation of this sphere is therefore

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2 \qquad \text{(X.18)}$$

the suffix 2 being dropped since Q is now any point satisfying the given condition.

Again, if l, m, n are the direction-cosines of the line PQ the relations (X.8) give

$$l = \frac{x_2 - x_1}{r}, m = \frac{y_2 - y_1}{r}, n = \frac{z_2 - z_1}{r}$$

or
$$\frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} = r \qquad \text{(X.19)}$$

Regarding the point Q as any point on the line through P we drop the suffix 2 and write

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \qquad \text{(X.20)}$$

which are the *standard* or *symmetrical* equations of a straight line.

The co-ordinates of any point on the line are given by

$$\left.\begin{aligned} x &= x_1 + rl \\ y &= y_1 + rm \\ z &= z_1 + rn \end{aligned}\right\} \qquad \text{(X.21)}$$

The equations of a straight line through the origin are, from (X.20)

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r \qquad \text{(X.22)}$$

EXAMPLE

The equations of a line are

$$x - 3y + 2z - 17 = 0 \qquad (1)$$
$$2x + 3y - 4z + 14 = 0 \qquad (2)$$

Express these in symmetrical form.

Adding (1) and (2) to eliminate y, we have

$$3x - 2z - 3 = 0 \text{ or } \frac{x-1}{2} = \frac{z}{3} \quad . \quad . \quad . \quad . \quad (3)$$

Multiplying (1) by 2 and subtracting (2) to eliminate x, we have

$$-9y + 8z - 48 = 0 \text{ or } \frac{z}{3} = \frac{y + 5\frac{1}{3}}{2\frac{2}{3}} \quad . \quad . \quad . \quad (4)$$

Combining (3) and (4), we obtain the required equations

$$\frac{x-1}{2} = \frac{y + 5\frac{1}{3}}{2\frac{2}{3}} = \frac{z}{3} \text{ or } \frac{x-1}{6} = \frac{y + 5\frac{1}{3}}{8} = \frac{z}{9}$$

The direction-cosines of the line are therefore proportional to 6, 8, 9, and their actual values are by (X.11), $\frac{6}{\sqrt{181}}, \frac{8}{\sqrt{181}}, \frac{9}{\sqrt{181}}$.

101. **Angle Between Two Lines whose Direction-cosines are Known.** Let OP, OQ (Fig. 83) be two lines through the origin

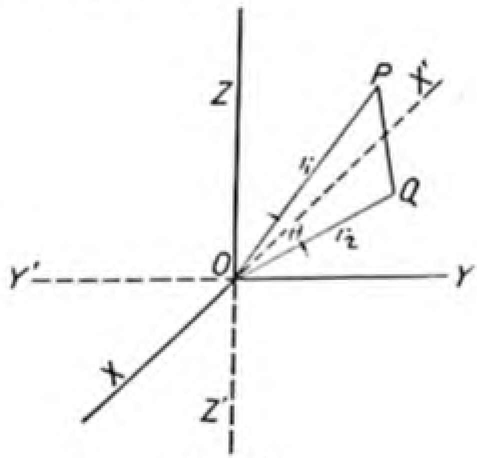


FIG. 83

parallel to the given lines whose direction-cosines are l_1, m_1, n_1 and l_2, m_2, n_2 respectively. Then the angle θ between the given lines is equal to the angle POQ. If $OP = r_1$, $OQ = r_2$, then, by (X.8), the rectangular co-ordinates of P are r_1l_1, r_1m_1, r_1n_1, and of Q, r_2l_2, r_2m_2, r_2n_2.

Hence, by (X.17)

$$\begin{aligned} PQ^2 &= (r_2l_2 - r_1l_1)^2 + (r_2m_2 - r_1m_1)^2 + (r_2n_2 - r_1n_1)^2 \\ &= r_2^2(l_2^2 + m_2^2 + n_2^2) + r_1^2(l_1^2 + m_1^2 + n_1^2) \\ &\qquad - 2r_1r_2(l_1l_2 + m_1m_2 + n_1n_2) \\ &= r_2^2 + r_1^2 - 2r_1r_2(l_1l_2 + m_1m_2 + n_1n_2) \end{aligned}$$

Now, from the triangle POQ

$$\cos \theta = \frac{OP^2 + OQ^2 - PQ^2}{2OP \cdot OQ}$$

$$= \frac{r_1^2 + r_2^2 - r_2^2 - r_1^2 + 2r_1r_2(l_1l_2 + m_1m_2 + n_1n_2)}{2r_1r_2}$$

i.e. $\cos \theta = l_1l_2 + m_1m_2 + n_1n_2$ . . . . . . (X.23)

The condition that the lines are at right angles is $\cos \theta = 0$ or

$$l_1l_2 + m_1m_2 + n_1n_2 = 0 \qquad \text{(X.24)}$$

An expression for $\sin^2 \theta$ is found as follows—

$$\begin{aligned}\sin^2 \theta &= 1 - \cos^2 \theta = 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2 \\ &= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2 \\ &= l_1^2m_2^2 + n_2^2l_1^2 + l_2^2m_1^2 + m_1^2n_2^2 + n_1^2l_2^2 + m_2^2n_1^2 \\ &\quad - 2l_1l_2m_1m_2 - 2m_1m_2n_1n_2 - 2n_1n_2l_1l_2\end{aligned}$$

i.e. $\sin^2 \theta = (l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2$ (X.25)

102. **Angle Between Two Given Planes.** Let the given planes be

$$a_1x + b_1y + c_1z + d_1 = 0 \qquad \text{(X.26)}$$

and

$$a_2x + b_2y + c_2z + d_2 = 0 \qquad \text{(X.27)}$$

The angle between two planes is equal to the angle between the normals to these planes. Now the normals to the planes (X.26) and (X.27) have direction-cosines proportional to a_1, b_1, c_1 and a_2, b_2, c_2 respectively. The actual direction-cosines of the normals are therefore

$$\frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$$

and

$$\frac{a_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \frac{b_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \frac{c_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

and, if θ is the angle between the planes, then

$$\cos \theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}} \qquad \text{(X.28)}$$

If the planes are at right angles

$$a_1a_2 + b_1b_2 + c_1c_2 = 0 \qquad \text{(X.29)}$$

103. **Equations of a Straight Line Through Two Given Points.** The equations of any straight line through the point (x_1, y_1, z_1) are, by (X.20),

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad . \quad . \quad (X.30)$$

If the point (x_2, y_2, z_2) lies on the line (X.30), then

$$\frac{x_2-x_1}{l}=\frac{y_2-y_1}{m}=\frac{z_2-z_1}{n} \quad . \quad . \quad (X.31)$$

Dividing corresponding ratios in (X.30) and (X.31), we obtain

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \quad . \quad . \quad (X.32)$$

These are the equations of the straight line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

EXAMPLE 1

Two places P and Q on the earth's surface are in north latitudes θ_1 and θ_2 respectively, and the difference of their longitudes is ϕ. Show that the angular distance between the earth's radii through P and Q is

$$\cos^{-1}(\cos\theta_1\cos\theta_2\cos\phi+\sin\theta_1\sin\theta_2) \qquad \text{(U.L.)}$$

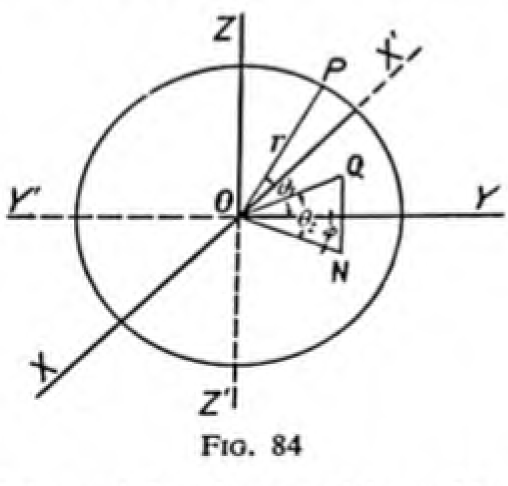


FIG. 84

Let the centre O of the earth be taken as origin, and let the equatorial plane be the xy-plane, P being assumed to lie on the zy-plane (Fig. 84). If l_1, m_1, n_1 are the direction-cosines of the line OP, then

$$l_1=\cos XOP=\cos 90^\circ=0$$

$$m_1=\cos YOP=\cos\theta_1$$

$$n_1=\cos ZOP=\cos(90^\circ-\theta_1)=\sin\theta_1$$

Let ON be the projection of OQ on the xy-plane. Then $\widehat{QON} = \theta_2$, $\widehat{YON} = \phi$, so that the polar co-ordinates of Q are r, $90° - \theta_2$, $90° - \phi$, and its rectangular co-ordinates are by (X.4, X.5, X.6)

$$x = r \sin (90° - \theta_2) \cos (90° - \phi) = r \cos \theta_2 \sin \phi$$
$$y = r \sin (90° - \theta_2) \sin (90° - \phi) = r \cos \theta_2 \cos \phi$$
$$z = r \sin \theta_2$$

Hence, the direction-cosines l_2, m_2, n_2 of OQ are given by

$$l_2 = \frac{x}{r} = \cos \theta_2 \sin \phi; \ m_2 = \frac{y}{r} = \cos \theta_2 \cos \phi; \ n_2 = \frac{z}{r} = \sin \theta$$

If θ is the angle POQ, then

$$\cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 + \cos \theta_1 \cos \theta_2 \cos \phi + \sin \theta_1 \sin \theta_2$$

or $$\theta = \cos^{-1} (\cos \theta_1 \cos \theta_2 \cos \phi + \sin \theta_1 \sin \theta_2)$$

EXAMPLE 2

Prove that the angle θ between two straight lines whose direction-cosines are l, m, n and l', m', n' respectively, is given by $\cos \theta = ll' + mm' + nn'$.

The co-ordinates of the angular points A, B, C, D of a tetrahedron are $(-2, 1, 3)$, $(3, -1, 2)$, $(2, 4, -1)$ and $(1, 2, 3)$ respectively. Calculate to the nearest minute the angle between the edges AC and BD. (U.L.)

For the first part of the question, see Art. 101.

By (X.32) the equations of the line AC are

$$\frac{x+2}{2+2} = \frac{y-1}{4-1} = \frac{z-3}{-1-3} \text{ or } \frac{x+2}{4} = \frac{y-1}{3} = \frac{z-3}{-4}$$

and the equations of the line BD are

$$\frac{x-3}{1-3} = \frac{y+1}{2+1} = \frac{z-2}{3-2} \text{ or } \frac{x-3}{-2} = \frac{y+1}{3} = \frac{z-2}{1}$$

The direction-cosines of AC are therefore proportional to 4, 3, -4, and those of BD to -2, 3, 1. The actual direction-cosines of AC and BD are $\frac{4}{\sqrt{41}}$, $\frac{3}{\sqrt{41}}$, $-\frac{4}{\sqrt{41}}$, and $-\frac{2}{\sqrt{14}}$, $\frac{3}{\sqrt{14}}$, $\frac{1}{\sqrt{14}}$. If θ is the angle between AC and BD, then by (X.23),

$$\cos \theta = \frac{4}{\sqrt{41}}\left(-\frac{2}{\sqrt{14}}\right) + \frac{3}{\sqrt{41}} \cdot \frac{3}{\sqrt{14}} + \left(-\frac{4}{\sqrt{41}}\right) \cdot \frac{1}{\sqrt{14}}$$

$$= -\frac{3}{\sqrt{574}} = -0{\cdot}1252$$

$$\therefore \quad \theta = 97° \ 12'$$

This is the obtuse angle between the lines; the acute angle between them is 82° 48′.

EXAMPLE 3

Find the equations of the line which passes through the point (2, − 6, 5) and which is perpendicular to the plane containing the points (2, − 3, − 4), (− 3, 2, 3·5), (2·5, 1, − 1).

As in Example, Art. 99, the equation of the plane through the three given points is

$$\begin{vmatrix} x & y & z & 1 \\ 2 & -3 & -4 & 1 \\ -3 & 2 & 3{\cdot}5 & 1 \\ 2{\cdot}5 & 1 & -1 & 1 \end{vmatrix} = 0$$

which reduces to $4x - 5y + 6z + 1 = 0$.

The direction-cosines of the normal to the plane are proportional to 4, − 5, 6. Let $\frac{x-2}{l} = \frac{y+6}{m} = \frac{z-5}{n}$ be the equations of the line through (2, − 6, 5); then if this line is perpendicular to the given plane, it is parallel to the normal to the plane, so that l, m, n are proportional to 4, − 5, 6. Hence, the equations of the line are

$$\frac{x-2}{4} = \frac{y+6}{-5} = \frac{z-5}{6}$$

EXAMPLE 4

Assuming the plane $4x - 3y + 7z = 0$ to be horizontal, find (1) the equations of the vertical line through the origin, (2) the direction-cosines of the line of greatest slope in the plane $2x + y - 5z = 0$.

(1) The equation of any straight line through the origin is $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$; if this line is vertical its direction-cosines are proportional to 4, − 3, 7, since the line is normal to the plane $4x - 3y + 7z = 0$. The equations of the vertical are then $\frac{x}{4} = \frac{y}{-3} = \frac{z}{7}$.

(2) Solving the equations

$$4x - 3y + 7z = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$2x + y - 5z = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

to obtain x and y in terms of z, we have

$$\frac{x}{15-7} = \frac{y}{14+20} = \frac{z}{4+6} \text{ or } \frac{x}{4} = \frac{y}{17} = \frac{z}{5} \quad . \quad . \quad (3)$$

The equations (3) are the equations of the line of intersection of the planes whose equations are (1) and (2). Let l', m', n', be the direction-cosines of the line of greatest slope in the plane (2). Since this line lies in the plane (2) it is perpendicular to the normal to that plane, so that we have

$$2l' + m' - 5n' = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The line of greatest slope is perpendicular to the line of intersection (3); hence

$$4l' + 17m' + 5n' = 0 \quad . \quad . \quad . \quad . \quad (5)$$

Solving (4) and (5), we obtain $l' : m' : n' = 3 : -1 : 1$; therefore $l' = \frac{3}{\sqrt{11}}$, $m' = -\frac{1}{\sqrt{11}}$, $n' = \frac{1}{\sqrt{11}}$.

104. **Length of Perpendicular from** (*a*) **a Given Point to a Given Plane;** (*b*) **a Given Point to a Given Straight Line.** (*a*) Let (x_1, y_1, z_1) be the given point and let the equation of the given plane, expressed in "perpendicular" form, be

$$lx + my + nz - p = 0 \qquad . \qquad . \qquad (1)$$

The equation of a plane parallel to the plane (1) is

$$lx + my + nz - p' = 0 \qquad . \qquad . \qquad (2)$$

since the normals to two parallel planes have the same direction-cosines.

If (x_1, y_1, z_1) lies on the plane (2), then $lx_1 + my_1 + nz_1 = p'$.

Now p = length of perpendicular from the origin on the plane (1), and p' = length of perpendicular from the origin on the plane (2).

$\therefore$ Length of perpendicular from (x_1, y_1, z_1) on the plane (1)

$$= p' - p = lx_1 + my_1 + nz_1 - p \qquad . \qquad (X.33)$$

If the equation of the given plane is expressed in the form $ax + by + cz + d = 0$, we transform it to the "perpendicular" form

$$\frac{ax + by + cz + d}{\sqrt{a^2 + b^2 + c^2}} = 0$$

and by substituting x_1, y_1, z_1 for x, y, z we obtain

$$\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \qquad . \qquad . \qquad . \qquad (X.34)$$

which is the length of perpendicular from (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$.

The equations of the perpendicular from (x_1, y_1, z_1) to the plane (1) are

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad . \qquad . \qquad (X.35)$$

(*b*) Let (x_1, y_1, z_1) be the given point and

$$\frac{x - u}{l} = \frac{y - v}{m} = \frac{z - w}{n} \qquad . \qquad . \qquad . \qquad (X.36)$$

the given straight line. The distance between the points (x_1, y_1, z_1) and (u, v, w) is $\sqrt{(x_1 - u)^2 + (y_1 - v)^2 + (z_1 - w)^2} = d$, say; and the equations of the line joining these points are by (X.32)

$$\frac{x - u}{x_1 - u} = \frac{y - v}{y_1 - v} = \frac{z - w}{z_1 - w} \qquad . \quad . \quad \text{(X.37)}$$

The direction-cosines of this line are $\frac{x_1 - u}{d}, \frac{y_1 - v}{d}, \frac{z_1 - w}{d}$ so that if θ is the angle between the lines (X.36) and (X.37), then, by (X.23)

$$\cos\theta = \frac{l(x_1 - u)}{d} + \frac{m(y_1 - v)}{d} + \frac{n(z_1 - w)}{d}, \text{ and, hence}$$

$$\sin\theta = \sqrt{1 - \frac{[l(x_1 - u) + m(y_1 - v) + n(z_1 - w)]^2}{d^2}}$$

Now the perpendicular distance p from (x_1, y_1, z_1) to the line (X.36)

$$= d\sin\theta = \sqrt{d^2 - \{l(x_1 - u) + m(y_1 - v) + n(z_1 - w)\}^2}$$

i.e. $$p = \sqrt{(x_1 - u)^2 + (y_1 - v)^2 + (z_1 - w)^2 - \{l(x_1 - u) + m(y_1 - v) + n(z_1 - w)\}^2} \quad . \quad \text{(X.38)}$$

105. **Shortest Distance Between Two Given Straight Lines.** Let the given straight lines be

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \qquad . \quad . \quad \text{(X.39)}$$

and $$\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2} \qquad . \quad . \quad \text{(X.40)}$$

The plane $lx + my + nz = p$ will contain the line (X.40) if

$$lx_2 + my_2 + nz_2 - p = 0 \qquad . \quad . \quad \text{(X.41)}$$

and $$ll_2 + mm_2 + nn_2 = 0 \qquad . \quad . \quad \text{(X.42)}$$

for the point (x_2, y_2, z_2) will lie on the plane, and the normal to the plane will be perpendicular to the line (X.40).

Again, the plane will be parallel to the line (X.39) if

$$ll_1 + mm_1 + nn_1 = 0 \qquad . \quad . \quad \text{(X.43)}$$

Eliminating l, m, n, $-p$ between the equation of the plane and the equations (X.41) to (X.43), we obtain

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ l_2 & m_2 & n_2 & 0 \\ l_1 & m_1 & n_1 & 0 \end{vmatrix} = 0, \text{ which reduces to}$$

$$\begin{vmatrix} x & y & z \\ l_2 & m_2 & n_2 \\ l_1 & m_1 & n_1 \end{vmatrix} - \begin{vmatrix} x_2 & y_2 & z_2 \\ l_2 & m_2 & n_2 \\ l_1 & m_1 & n_1 \end{vmatrix} = 0 \qquad . \qquad . \qquad (X.44)$$

This is the equation of the plane containing the line (X.40) and parallel to the line (X.39). The shortest distance δ between the lines (X.39) and (X.40) is equal to the perpendicular distance of the point (x_1, y_1, z_1) from the plane (X.44). By (X.34), δ is found by substituting x_1, y_1, z_1 for x, y, z in the left-hand side of (X.44) and dividing by the sum of the squares of the coefficients of x, y, z.

$$\therefore \quad \delta = \pm \frac{\begin{vmatrix} x_1 & y_1 & z_1 \\ l_2 & m_2 & n_2 \\ l_1 & m_1 & n_1 \end{vmatrix} - \begin{vmatrix} x_2 & y_2 & z_2 \\ l_2 & m_2 & n_2 \\ l_1 & m_1 & n_1 \end{vmatrix}}{\sqrt{\begin{vmatrix} m_2 & n_2 \\ m_1 & n_1 \end{vmatrix}^2 + \begin{vmatrix} n_2 & l_2 \\ n_1 & l_1 \end{vmatrix}^2 + \begin{vmatrix} l_2 & m_2 \\ l_1 & m_1 \end{vmatrix}^2}}, \text{ which reduces to}$$

$$\delta = \pm \frac{(x_1 - x_2)(m_2 n_1 - m_1 n_2) + (y_1 - y_2)(n_2 l_1 - n_1 l_2) + (z_1 - z_2)(l_2 m_1 - l_1 m_2)}{\sqrt{(m_2 n_1 - m_1 n_2)^2 + (n_2 l_1 - n_1 l_2)^2 + (l_2 m_1 - l_1 m_2)^2}} \qquad (X.45)$$

If l', m', n' are the direction-cosines of the line on which the shortest distance between the lines (X.39) and (X.40) is intercepted, then l', m', n' are also the direction-cosines of the normal to the plane (X.44), so that we have

$$l' : m' : n' = m_2 n_1 - m_1 n_2 : n_2 l_1 - n_1 l_2 : l_2 m_1 - l_1 m_2 \qquad (X.46)$$

EXAMPLE 1

Find the length of the common perpendicular to the two lines $x + 7 = 18 - 2y = 10z - 40$ and $x = 2y = 2(1 - z)$; and find also the equations of the line on which this perpendicular is intercepted and the points in which it intersects the given lines.

We can write the equations of the given lines as

$$\frac{x+7}{10} = \frac{y-9}{-5} = \frac{z-4}{1} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{x}{2} = \frac{y}{1} = \frac{z-1}{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Using the above method, we deduce the length δ of the common perpendicular to the two lines.

$$\delta = \pm \frac{(-7-0)(1-5)+(9-0)(-10-2)+(4-1)(-10-10)}{\sqrt{(1-5)^2+(-10-2)^2+(-10-10)^2}}$$

$$= \pm \frac{28-108-60}{\sqrt{16+144+400}}$$

whence $\delta = \dfrac{\sqrt{140}}{2} = \sqrt{35} = 5{\cdot}916$

Let the line on which the common perpendicular is intercepted be

$$\frac{x-\alpha}{l'} = \frac{y-\beta}{m'} = \frac{z-\gamma}{n'}$$

where we can assume (α, β, γ) to be the point in which this line intersects the line (1). By (X.46), l', m', n' are proportional to $1-5$, $-10-2$, $-10-10$, i.e. to 1, 3, 5, so that the equations of the line become

$$\frac{x-\alpha}{1} = \frac{y-\beta}{3} = \frac{z-\gamma}{5} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The equation of any plane through the point (α, β, γ) is

$$a(x-\alpha) + b(y-\beta) + c(z-\gamma) = 0 \quad . \quad . \quad . \quad (4)$$

If the plane (4) contains the lines (3) and (2), the normal to the plane will be perpendicular to these lines, so that we have

$$1 \,.\, a + 3 \,.\, b + 5 \,.\, c = 0 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$2 \,.\, a + 1 \,.\, b - 1 \,.\, c = 0 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The point (0, 0, 1) lies on the plane (4); hence,

$$a(-\alpha) + b(-\beta) + c(1-\gamma) = 0$$

or

$$\alpha\, a + \beta\, b + (\gamma - 1)c = 0 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Eliminating a, b, c between the equations (5), (6), (7), we have

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & -1 \\ \alpha & \beta & \gamma-1 \end{vmatrix} = 0, \text{ which reduces to } 8\alpha - 11\beta + 5\gamma = 5 \quad . \quad . \quad (8)$$

The point (α, β, γ) lies on the line (1), so that $\dfrac{\alpha+7}{10} = \dfrac{\beta-9}{-5} = \dfrac{\gamma-4}{1}$, or $\alpha = 10\gamma - 47$, $\beta = -5\gamma + 29$. Substituting these values of α, β, in (8), we obtain ultimately $\gamma = 5$, and therefore $\alpha = 3$, $\beta = 4$. The equations (3) become

$$\frac{x-3}{1} = \frac{y-4}{3} = \frac{z-5}{5} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Let $(\alpha', \beta', \gamma')$ be the point in which the lines (2) and (9) intersect; then from (2) $\frac{\alpha'}{2} = \frac{\beta'}{1} = \frac{\gamma' - 1}{-1}$, and from (9) $\frac{\alpha' - 3}{1} = \frac{\beta' - 4}{3} = \frac{\gamma' - 5}{5}$. Hence, $\alpha' = 2\beta' = \frac{\beta' - 4}{3} + 3$, so that $\beta' = 1$, $\alpha' = 2$, and therefore $\gamma' = 0$. The common perpendicular to the given two lines lies along the line $x - 3 = \frac{y - 4}{3} = \frac{z - 5}{5}$, and it meets the given lines at the points (3, 4, 5) and (2, 1, 0) respectively.

EXAMPLE 2

Prove that the planes $12x - 15y + 16z - 28 = 0$, $6x + 6y - 7z - 8 = 0$, and $2x + 35y - 39z + 12 = 0$ have a common line of intersection. Prove also that the point in which the line $\frac{x - 1}{3} = \frac{y}{-2} = \frac{z - 3}{1}$ meets the third plane is equidistant from the other two planes.

The equation of a plane through the line of intersection of the planes

$$12x - 15y + 16z - 28 = 0 \qquad . \quad . \quad . \quad (1)$$

and

$$6x + 6y - 7z - 8 = 0 \qquad . \quad . \quad . \quad (2)$$

is

$$12x - 15y + 16z - 28 + k(6x + 6y - 7z - 8) = 0 \qquad . \quad (3)$$

where k is some constant, for the equation (3) is of the first degree in x, y, z, and all values of x, y, z which satisfy (1) and (2) simultaneously also satisfy (3). The plane (3) will be identical with the plane $2x + 35y - 39z + 12 = 0$, provided that the ratios $\frac{12 + 6k}{2}, \frac{-15 + 6k}{35}, \frac{16 - 7k}{-39}, \frac{-28 - 8k}{12}$ have the same value for some particular value of k. The equation $6 + 3k = \frac{-15 + 6k}{35}$ gives $k = -\frac{25}{11}$, and for this value of k, $6 + 3k = \frac{-15 + 6k}{35} = -\frac{9}{11}$, and also $\frac{16 - 7k}{-39} = -\frac{9}{11}$ and $\frac{-28 - 8k}{12} = -\frac{9}{11}$. The three given planes have therefore a common line of intersection.

The co-ordinates of any point on the line $\frac{x - 1}{3} = \frac{y}{-2} = \frac{z - 3}{1} = r$ are given by $x = 1 + 3r$, $y = -2r$, $z = 3 + r$. Substituting these values in the equation $2x + 35y - 39z + 12 = 0$, we obtain ultimately $r = -1$, whence $x = -2$, $y = 2$, $z = 2$. The distance of the point $(-2, 2, 2)$ from the plane (1) above is

$$\frac{12(-2) - 15(2) + 16(2) - 28}{\sqrt{12^2 + (-15)^2 + 16^2}} = 2 \text{ numerically}$$

and from the plane (2),

$$\frac{6(-2) + 6(2) - 7(2) - 8}{\sqrt{6^2 + 6^2 + (-7)^2}} = 2 \text{ numerically}$$

106. **Projection of an Area on a Given Plane.** In Fig. 85, $PQRSTUV$ is any area S in a plane G, and from every point on the boundary of S perpendiculars Pp, Qq, Rr, etc., are drawn to a second plane H, the angle between the planes G and H being θ. The area S' enclosed by the curve $pqrstuv$ in the plane H is called the projection of the area S on the plane H. $PQTU$ is a narrow strip of breadth h in the area S perpendicular to the line of intersection of the planes G and

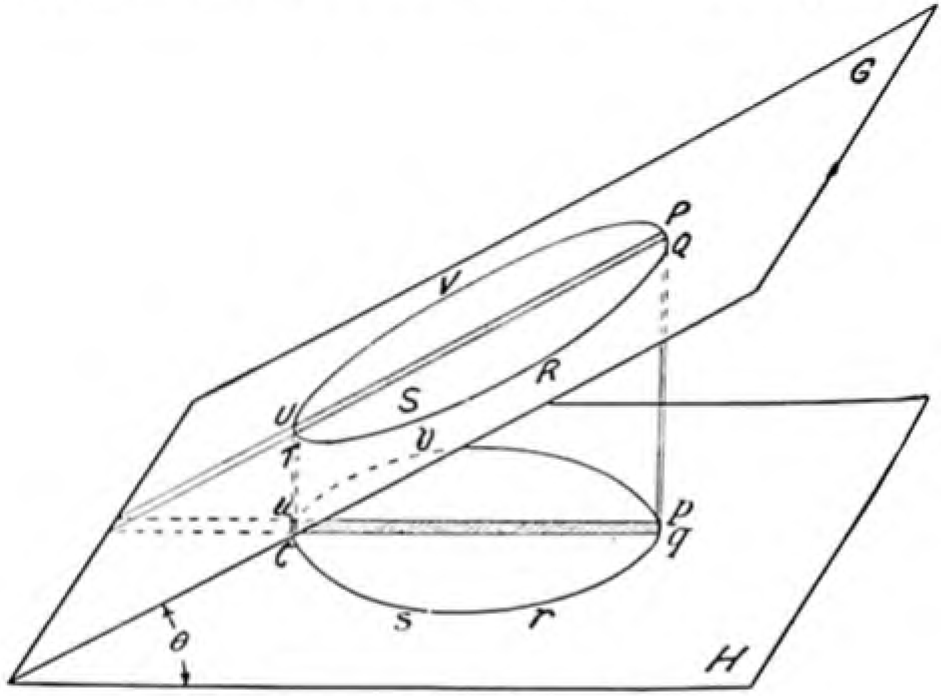


Fig. 85

H, and $pqtu$ is the projection of this strip on the plane H. Since h is small, the area $PQTU = PU \times h$ approximately, and the area $pqtu = pu \times h$ approximately.

$$\therefore \qquad \frac{\text{area } pqtu}{\text{area } PQTU} = \frac{pu \times h}{PU \times h} = \frac{pu}{PU} = \cos \theta$$

This relation will be true for every corresponding pair of such strips, so that

$$\frac{\Sigma \text{ area } pqtu}{\Sigma \text{ area } PQTU} = \cos \theta$$

Now $\underset{h \to 0}{Lt.}\ \Sigma$ area $pqtu = S'$ and $\underset{h \to 0}{Lt.}\ \Sigma$ area $PQTU = S$; hence, we have

$$\frac{S'}{S} = \cos \theta \text{ or } S' = S \cos \theta \qquad . \qquad . \qquad \text{(X.47)}$$

Thus, if the normal to the plane of any area S in space makes angles α, β, γ with the axes of reference OX, OY, OZ respectively, then the projections S_{yz}, S_{zx}, S_{xy} of S on the yz, zx, and xy planes are given by

$$\left.\begin{aligned} S_{yz} &= S\cos\alpha \\ S_{zx} &= S\cos\beta \\ S_{xy} &= S\cos\gamma \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(X.48)}$$

Squaring and adding, we obtain

$$S^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = S_{yz}{}^2 + S_{zx}{}^2 + S_{xy}{}^2$$

or, since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$S_{yz}{}^2 + S_{zx}{}^2 + S_{xy}{}^2 = S^2 \quad . \quad . \quad \text{(X.49)}$$

Again, if the plane of S makes an angle ϕ with another plane, the inclinations to OX, OY, OZ of the normal to which are α', β', γ', then, by (X.23)

$$\cos\phi = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'$$

Also, by (X.47), the projection of S on this plane

$$= S\cos\phi = S\cos\alpha\cos\alpha' + S\cos\beta\cos\beta' + S\cos\gamma\cos\gamma'$$

i.e. projection of S on the second plane

$$= S_{yz}\cos\alpha' + S_{zx}\cos\beta' + S_{xy}\cos\gamma'. \quad . \quad \text{(X.50)}$$

EXAMPLE 1

Prove that the area of the triangle whose angular points are (x_1, y_1), (x_2, y_2) (x_3, y_3) is equal to $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Find the volume of the tetrahedron whose angular points are (1, 2, 3), (2, 3, 5), (− 2, − 1, 2), and (3, 0, − 3).

The distance d between the points (x_2, y_2) and (x_3, y_3) is $\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}$, the equation of the line joining these points is $\dfrac{x - x_3}{x_2 - x_3} = \dfrac{y - y_3}{y_2 - y_3}$ or

$$(y_2 - y_3)x - (x_2 - x_3)y - x_3y_2 + x_2y_3 = 0 \quad . \quad . \quad . \quad (1)$$

The length of the perpendicular p from (x_1, y_1) to the line (1) is given by

$$p = \frac{(y_2 - y_3)x_1 - (x_2 - x_3)y_1 - x_3y_2 + x_2y_3}{\sqrt{(y_2 - y_3)^2 + (x_2 - x_3)^2}} = \frac{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}{d}$$

The area of the triangle $= \frac{1}{2}pd = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Let the angular points of the tetrahedron be A, B, C, D, in order.

The equation of the plane BCD is $\begin{vmatrix} x & y & z & 1 \\ 2 & 3 & 5 & 1 \\ -2 & -1 & 2 & 1 \\ 3 & 0 & -3 & 1 \end{vmatrix} = 0$

which reduces to $23x - 35y + 16z - 21 = 0$. [See Art. 99, Ex.]

By (X.34), the perpendicular p from the point (1, 2, 3,) on the plane BCD is given by

$$p = \frac{23(1) - 35(2) + 16(3) - 21}{\sqrt{23^2 + 35^2 + 16^2}} = \frac{20}{\sqrt{2\,010}}$$

Let S denote the area of the triangle BCD; then the projection of S on the yz-plane $= S_{yz} =$ area of the triangle whose angular points are ($y = 3$, $z = 5$), ($y = -1$, $z = 2$), ($y = 0$, $z = -3$).

Hence by above, $S_{yz} = \frac{1}{2}\begin{vmatrix} 3 & 5 & 1 \\ -1 & 2 & 1 \\ 0 & -3 & 1 \end{vmatrix} = \frac{23}{2}$

Similarly, $\qquad S_{zx} = \frac{1}{2}\begin{vmatrix} 2 & 5 & 1 \\ -2 & 2 & 1 \\ 3 & -3 & 1 \end{vmatrix} = \frac{35}{2}$

and $\qquad S_{xy} = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -2 & -1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = \frac{16}{2} = 8$

By (X.49), $S = \sqrt{S_{yz}^2 + S_{zx}^2 + S_{xy}^2} = \frac{1}{2}\sqrt{529 + 1\,225 + 256} = \frac{1}{2}\sqrt{2\,010}$

The volume of a tetrahedron is equal to one-third the area of its base multiplied by its perpendicular height.

$\therefore \qquad$ Volume of tetrahedron $= \frac{1}{3}Sp = \frac{1}{3}\cdot\frac{\sqrt{2\,010}}{2}\cdot\frac{20}{\sqrt{2\,010}} = 3\frac{1}{3}$

EXAMPLE 2

A tetrahedron is formed by the planes $3x + 4y + 5z = 30$, $3x + 4y = 0$, $4y + 5z = 0$, $5z + 3x = 0$; find its volume.

Solving the first, second, and third equations simultaneously, we obtain $x = 10$, $y = -7{\cdot}5$, $z = 6$. Similarly, from the first, second, and fourth, we obtain $x = -10$, $y = 7{\cdot}5$, $z = 6$; from the first, third, and fourth we obtain $x = 10$, $y = 7{\cdot}5$, $z = -6$; from the second, third, and fourth we obtain $x = 0$, $y = 0$, $z = 0$.

The vertices of the tetrahedron are therefore the points A (10, $-$ 7·5, 6), B($-$ 10, 7·5, 6), C (10, 7·5, $-$ 6), D (0, 0, 0).

The equation of the plane ABC is $3x + 4y + 5z = 30$, since the other three planes meet at the origin; and the perpendicular p on this plane from D is equal to $\dfrac{30}{\sqrt{3^2 + 4^2 + 5^2}}$ or $3\sqrt{2}$ numerically.

If S denotes the area ABC, then, as in Ex. 1,

$$S_{yz} = \frac{1}{2}\begin{vmatrix} -7\cdot5 & 6 & 1 \\ 7\cdot5 & 6 & 1 \\ 7\cdot5 & -6 & 1 \end{vmatrix} = \frac{45}{2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 90$$

$$S_{zx} = \frac{1}{2}\begin{vmatrix} 10 & 6 & 1 \\ -10 & 6 & 1 \\ 10 & -6 & 1 \end{vmatrix} = 30\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 120$$

$$S_{xy} = \frac{1}{2}\begin{vmatrix} 10 & -7\cdot5 & 1 \\ -10 & 7\cdot5 & 1 \\ 10 & 7\cdot5 & 1 \end{vmatrix} = \frac{75}{2}\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 150$$

$\therefore\ S = \sqrt{90^2 + 120^2 + 150^2} = 150\sqrt{2}$

$\therefore$ Volume of tetrahedron $= \frac{1}{3}\,Sp = \frac{1}{3} \times 150\sqrt{2} \times 3\sqrt{2} = 300$

107. **Surfaces of the Second Degree. Tangent Planes.** The most general equation of the second degree in x, y, z is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \qquad (X.51)$$

At the points where the surface represented by this is cut by the straight line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \qquad (X.52)$$

we have

$$\begin{aligned} a(x_1 + rl)^2 &+ b(y_1 + rm)^2 + c(z_1 + rn)^2 \\ &+ 2f(y_1 + rm)(z_1 + rn) + 2g(z_1 + rn)(x_1 + rl) \\ &+ 2h(x_1 + rl)(y_1 + rm) + 2u(x_1 + rl) \\ &+ 2v(y_1 + rm) + 2w(z_1 + rn) + d = 0 \end{aligned}$$

This equation can be written as

$$r^2(al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm) + r\left(l\frac{\partial\phi}{\partial x_1} + m\frac{\partial\phi}{\partial y_1} + n\frac{\partial\phi}{\partial z_1}\right) + \phi(x_1,y_1,z_1) = 0 \qquad (X.53)$$

where $\phi(x, y, z)$ denotes the expression on the left-hand side of (X.51) and $\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial y_1}, \frac{\partial\phi}{\partial z_1}$ are the values for $x = x_1$, $y = y_1$, $z = z_1$ of the partial differential coefficients of $\phi(x, y, z)$ with respect to x, y, z respectively.

The equation (X.53) is a quadratic in r, so that there are two points in which the straight line intersects the surface. If the point (x_1, y_1, z_1) lies on the surface, then $\phi(x_1, y_1, z_1) = 0$ and one root of the equation is zero. If, in addition, the other point of intersection coincides with (x_1, y_1, z_1), then both roots of the equation are zero, so that we must also have

$$l\frac{\partial\phi}{\partial x_1} + m\frac{\partial\phi}{\partial y_1} + n\frac{\partial\phi}{\partial z_1} = 0 \quad . \quad . \quad (X.54)$$

and in this case the straight line (X.52) is tangential to the surface. Eliminating l, m, n between (X.52) and (X.54), we obtain

$$(x - x_1)\frac{\partial\phi}{\partial x_1} + (y - y_1)\frac{\partial\phi}{\partial y_1} + (z - z_1)\frac{\partial\phi}{\partial z_1} = 0\ . \quad (X.55)$$

This is an equation of the first degree in x, y, z and therefore represents a plane. Now the co-ordinates of any point on any tangent line to the surface (X.51) through (x_1, y_1, z_1) satisfy the equation (X.55), so that this equation represents the tangent plane to the surface at the point (x_1, y_1, z_1).

EXAMPLE 1

Write down the equation of the normal at the point P whose co-ordinates are (u, v, w) to the surface $x^2 + y^2 - z^2 = 1$. If the normal at P meets the surface again at Q, show that $PQ = 2(u^2 + v^2 + w^2)^{\frac{3}{2}}$. (U.L.)

Let $\phi(x, y, z) = x^2 + y^2 - z^2 - 1$; then $\dfrac{\partial\phi}{\partial x} = 2x$, $\dfrac{\partial\phi}{\partial y} = 2y$, $\dfrac{\partial\phi}{\partial z} = -2z$.

The equation of the tangent plane at the point (u, v, w) to the surface $\phi(x, y, z) = 0$ is $(x - u)\,2u + (y - v)2v + (z - w)(-2w) = 0$ or

$$ux + vy - wz = u^2 + v^2 - w^2 = 1 \quad . \quad . \quad . \quad (1)$$

The direction-cosines of the normal at (u, v, w) are proportional to u, v, $-w$, the coefficients of x, y, z in the equation (1). The equations of the normal are therefore $\dfrac{x - x_1}{u} = \dfrac{y - y_1}{v} = \dfrac{z - z_1}{-w}$, where we can take (x_1, y_1, z_1) as the point Q in which the normal meets the surface again. Since (u, v, w) lies on the normal, we have

$$\frac{u - x_1}{u} = \frac{v - y_1}{v} = \frac{w - z_1}{-w} = \frac{\sqrt{(u - x_1)^2 + (v - y_1)^2 + (w - z_1)^2}}{\sqrt{u^2 + v^2 + w^2}}$$

$$= \frac{PQ}{\sqrt{u^2 + v^2 + w^2}} = k\text{, say}$$

Then $x_1 = u(1 - k)$; $y_1 = v(1 - k)$; $z_1 = w(1 + k)$; and since (x_1, y_1, z_1) lies on the given surface we have

$$x_1^2 + y_1^2 - z_1^2 = 1 \text{ or } u^2(1 - k)^2 + v^2(1 - k)^2 - w^2(1 + k)^2 = 1$$

$$\therefore \qquad (u^2 + v^2 - w^2) - 2k(u^2 + v^2 + w^2) + k^2(u^2 + v^2 - w^2) = 1$$

Now $u^2 + v^2 - w^2 = 1$, since (u, v, w) lies on the surface; hence,

$$-2k(u^2 + v^2 + w^2) + k^2 = 0$$

i.e. $k = 0$ or $k = 2(u^2 + v^2 + w^2)$.

The solution $k = 0$ is inadmissible here. Now, $k = \dfrac{PQ}{\sqrt{u^2 + v^2 + w^2}}$, so that $PQ = 2(u^2 + v^2 + w^2)^{\frac{3}{2}}$.

EXAMPLE 2

(1) Find the condition that the plane $ax + by + cz + d = 0$ touches the surface $px^2 + qy^2 + 2z = 0$.

(2) Find the inclination to the plane $z = 0$ of the tangent plane to the sphere $x^2 + y^2 + z^2 = 49$ at the point $(2, -3, 6)$.

(1) The equation of the tangent plane at the point (x_1, y_1, z_1) on the surface $px^2 + qy^2 + 2z = 0$ is $(x - x_1)\,2px_1 + (y - y_1)\,2qy_1 + (z - z_1)\,2 = 0$ or

$$pxx_1 + qyy_1 + z + z_1 = px_1^2 + qy_1^2 + 2z_1 = 0 \quad . \quad . \quad (1)$$

If the plane $ax + by + cz + d = 0$ is identical with the plane (1), then we must have $\dfrac{px_1}{a} = \dfrac{qy_1}{b} = \dfrac{1}{c} = \dfrac{z_1}{d}$ or $x_1 = \dfrac{a}{pc}$, $y_1 = \dfrac{b}{qc}$, $z_1 = \dfrac{d}{c}$

Now (x_1, y_1, z_1) lies on the surface, so that $px_1^2 + qy_1^2 + 2z_1 = 0$

Hence $p\left(\dfrac{a^2}{p^2c^2}\right) + q\left(\dfrac{b^2}{q^2c^2}\right) + \dfrac{2d}{c} = 0$, or $\dfrac{a^2}{p} + \dfrac{b^2}{q} + 2cd = 0$, which is the required condition.

(2) The equation of the tangent plane to the given sphere at the point $(2, -3, 6)$ is found to be

$$2x - 3y + 6z = 49 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If θ be the angle between the plane $z = 0$ and the plane (1), then by (X.28),

$$\cos\theta = \frac{2(0) - 3(0) + 6(1)}{\sqrt{2^2 + (-3)^2 + 6^2}\,\sqrt{0^2 + 0^2 + 1^2}} = \frac{6}{\sqrt{49}} = \frac{6}{7} = 0{\cdot}8571$$

$$\therefore \qquad \theta = 31^\circ$$

Otherwise, since the tangent plane at any point on a sphere is perpendicular to the radius through that point, then θ = angle between the radius through $(2, -3, 6)$ and the z-axis. Hence $\theta = \cos^{-1}\left(\dfrac{6}{\text{radius}}\right) = \cos^{-1}\dfrac{6}{7} = 31^\circ$.

108. **Plane Containing Chords which are Bisected at a Given Point.** If in Art. 107 the point (x_1, y_1, z_1) is the mid-point of the chord joining the points of intersection of the straight line with the surface,

then the two roots of the equation (X.53) will be equal and opposite, and accordingly we must have

$$l\frac{\partial\phi}{\partial x_1}+m\frac{\partial\phi}{\partial y_1}+n\frac{\partial\phi}{\partial z_1}=0 \qquad . \qquad . \qquad \text{(X.56)}$$

Eliminating l, m, n between (X.56) and (X.52) we have

$$(x-x_1)\frac{\partial\phi}{\partial x_1}+(y-y_1)\frac{\partial\phi}{\partial y_1}+(z-z_1)\frac{\partial\phi}{\partial z_1}=0 \qquad \text{(X.57)}$$

Since the co-ordinates of any point on any chord having (x_1, y_1, z_1) as mid-point satisfy the equation (X.57), this must be the equation of the plane containing all such chords. Although (X.57) is of the same form as (X.55), the reader should note that (x_1, y_1, z_1) is not on the surface in the case of (X.57).

109. **Locus of the Mid-points of a System of Parallel Chords.** The equation (X.56) of the last article gives the condition that the chord whose direction-cosines are l, m, n has the point (x_1, y_1, z_1) as its mid-point. If the chord moves parallel to itself the locus of its mid-point is therefore the plane

$$l\frac{\partial\phi}{\partial x}+m\frac{\partial\phi}{\partial y}+n\frac{\partial\phi}{\partial z}=0 \qquad . \qquad . \qquad \text{(X.58)}$$

Such a plane is called a *diametral plane*.

110. **Principal Planes.** A diametral plane of a surface of the second degree (or quadric surface) becomes a principal plane when it is perpendicular to the chords through whose mid-points it passes.

EXAMPLE 1

Find the equation of the plane containing the chords of the sphere

$$(x-3)^2+(y+2)^2+(z-1)^2=9$$

which have the point $(2, -1, 0{\cdot}5)$ as their common mid-point. Find also the radius of the circle in which the plane cuts the sphere.

Let $\phi(x, y, z)=(x-3)^2+(y+2)^2+(z-1)^2-9$; then $\dfrac{\partial\phi}{\partial x}=2(x-3)$; $\dfrac{\partial\phi}{\partial y}=2(y+2)$; $\dfrac{\partial\phi}{\partial z}=2(z-1)$. Hence, when $x=2$, $y=-1$, $z=0{\cdot}5$,

$$\frac{\partial\phi}{\partial x}=-2;\ \frac{\partial\phi}{\partial y}=2;\ \frac{\partial\phi}{\partial z}=-1$$

By (X.57) all chords of the sphere having the point $(2, -1, 0{\cdot}5)$ as mid-point lie in the plane

$$(x-2)(-2)+(y+1)(2)+(z-0{\cdot}5)(-1)=0$$

or

$$4x-4y+2z=13 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The centre of the sphere is the point $(3, -2, 1)$ and the radius is 3. [See Art. 100.] The perpendicular distance from the centre of the sphere to the plane (1) is $\dfrac{4(3)-4(-2)+2(1)-13}{\sqrt{4^2+(-4)^2+2^2}}$ or 1·5. The radius r of the circle in which the plane (1) cuts the sphere is then given by $r^2 = 3^2 - 1{\cdot}5^2 = 6{\cdot}75$; whence $r = 2{\cdot}60$ nearly.

EXAMPLE 2

Find the principal planes of the surface

$$4x^2+8y^2-12z^2-16yz+32zx-48xy-24y+16z+5=0$$

If the surface be denoted by $\phi(x, y, z) = 0$, we have $\dfrac{\partial\phi}{\partial x} = 8x+32z-48y$; $\dfrac{\partial\phi}{\partial y} = 16y-16z-48x-24$; $\dfrac{\partial\phi}{\partial z} = -24z-16y+32x+16$.

The equation of the diametral plane of chords parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is by (X.58)

$$\begin{aligned} l(8x+32z-48y)&+m(16y-16z-48x-24)\\ &+n(-24z-16y+32x+16)=0\end{aligned}$$

or

$$\begin{aligned} x(l-6m+4n)&+y(-6l+2m-2n)\\ &+z(4l-2m-3n)-3m+2n=0 \quad . \quad . \quad (2)\end{aligned}$$

If the plane (2) is perpendicular to the line (1), we have

$$\frac{l-6m+4n}{l}=\frac{-6l+2m-2n}{m}=\frac{4l-2m-3n}{n}=k, \text{ say}$$

$$\therefore \quad \left.\begin{aligned} (1-k)l-6m+4n&=0\\ -6l+(2-k)m-2n&=0\\ 4l-2m-(3+k)n&=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

Eliminating l, m, n between the equations (3), we obtain

$$\begin{vmatrix} 1-k & -6 & 4\\ -6 & 2-k & -2\\ 4 & -2 & -(3+k)\end{vmatrix} = 0, \text{ which reduces to } k^3-63k-162=0$$

This equation can be written

$$k^2(k+3)-3k(k+3)-54(k+3)=0$$

or

$$(k+3)(k-9)(k+6)=0$$

$$\therefore \quad k=-3 \text{ or } 9 \text{ or } -6$$

When $k = -3$, the relations (3) give $l : m : n = 1 : 2 : 2$

" $k = 9$, " " " $= 2 : -2 : 1$

" $k = -6$, " " " $= 2 : 1 : -2$

The equation (2) then gives as the equations of the principal planes

$3x + 6y + 6z + 2 = 0$, $18x - 18y + 9z + 8 = 0$, and $12x + 6y - 12z + 7 = 0$

111. **Surfaces of Revolution.** A curve $y = f(z)$ in the yz-plane is rotated about the z-axis (Fig. 86) and P is any point (x, y, z) on

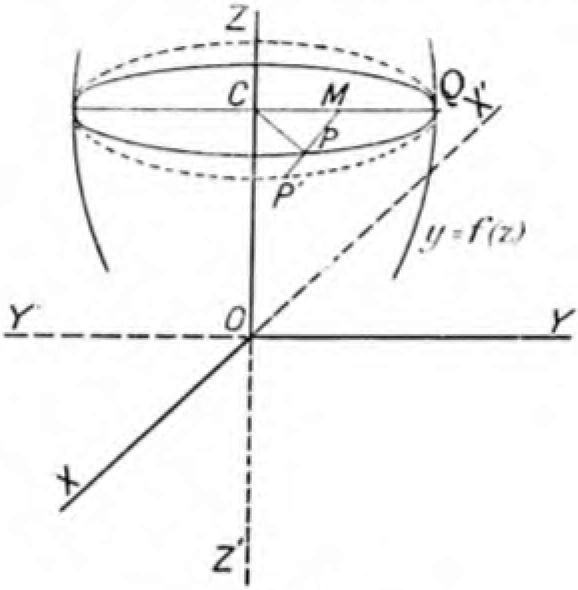


FIG. 86

the surface thus generated. The section of the surface made by a plane perpendicular to the z-axis and passing through P is a circle whose centre C lies on the z-axis. CMQ is a radius of this circle, M being the foot of the perpendicular from P on to the yz-plane.

For the point P, $x = MP$, $y = CM$, $z = OC$. Since $\widehat{CMP} = 90°$, $x^2 + y^2 = CP^2 = CQ^2$. Now Q is a point on the curve $y = f(z)$ in the yz-plane, so that $CQ = f(OC) = f(z)$.

$$\therefore \qquad x^2 + y^2 = [f(z)]^2 \qquad . \qquad . \qquad (X.59)$$

This is the equation of the surface of revolution generated by the rotation of the curve $y = f(z)$ about the z-axis.

Suppose now that P' is a point on MP produced such that $\frac{MP'}{MP} = \frac{a}{b}$, where we take a greater than b. This gives

$$x = MP = \frac{b}{a} MP'$$

and the equation (X.59) becomes $\frac{b^2}{a^2} MP'^2 + y^2 = [f(z)]^2$ or

$$\frac{MP'^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{b^2}[f(z)]^2$$

The locus of P' is then the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{b^2}[f(z)]^2 \quad . \quad . \quad . \quad (X.60)$$

This is the equation of a surface (not of revolution) which is cut by the plane $x = 0$ in the curves $y = \pm f(z)$ and whose sections by planes perpendicular to the z-axis are ellipses.

112. **The Cylinder.** Let $f(z) = a$. The equation (X.59) becomes

$$x^2 + y^2 = a^2 \quad . \quad . \quad . \quad (X.61)$$

This is the equation of a right-circular cylinder, the section by any plane parallel to the xy-plane being a circle of radius a.

Let $f(z) = b$. The equation (X.60) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad (X.62)$$

This is the equation of an elliptical cylinder, the section by any plane parallel to the xy-plane being an ellipse of major axis $2a$ and minor axis $2b$.

EXAMPLE

A right-circular cylinder is cut by a sphere whose centre is on one of the generators of the cylinder. Prove that the projection of the curve of intersection on the plane containing the axis of the cylinder and the centre of the sphere is a parabola whose latus-rectum is twice the radius of the cylinder. (U.L.)

Let the axis of the cylinder be taken as the z-axis, and let the plane containing the axis and the centre of the sphere be taken as the yz-plane. Then, if a be the radius of the cylinder, its equation is

$$x^2 + y^2 = a^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If r be the radius of the sphere, and if the centre be at a height h above the xy-plane, then the co-ordinates of the centre are $(0, a, h)$ and the equation of the sphere is therefore

$$x^2 + (y - a)^2 + (z - h)^2 = r^2 \quad . \quad . \quad . \quad . \quad (2)$$

[See Art. 100.]

The co-ordinates of any point on the curve of intersection must satisfy equations (1) and (2) simultaneously. Equating the values of x^2 from (1) and (2), we obtain

$$a^2 - y^2 = r^2 - (y - a)^2 - (z - h)^2$$

or

$$(z - h)^2 = 2a\left(y - a + \frac{r^2}{2a}\right)$$

Transferring the origin to the point $y = a - \dfrac{r^2}{2a}$, $z = h$, we obtain the equation

$$z^2 = 2ay \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is the projection of the curve of intersection on the yz-plane, and we see that (3) is a parabola with latus-rectum equal to $2a$ or twice the radius of the cylinder.

113. **The Cone.** Let $f(z) = az$. The equation (X.59) becomes

$$x^2 + y^2 = a^2z^2 \quad . \quad . \quad . \quad (X.63)$$

This is the equation of a right-circular cone with its vertex at the origin and its axis along the z-axis.

If $f(z) = bz$, the equation (X.60) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z^2 \quad . \quad . \quad . \quad (X.64)$$

This is the equation of a cone with its vertex at the origin whose sections by planes parallel to the xy-plane are ellipses.

EXAMPLE

Find the equation of the plane which touches the cone $x^2 + y^2 = z^2 \tan^2 \alpha$ along the generator $\dfrac{x}{\sin \alpha \cos \phi} = \dfrac{y}{\sin \alpha \sin \phi} = \dfrac{z}{\cos \alpha}$

Find the greatest value of α for which the cone can have two perpendicular tangent planes, and, supposing it to have two, determine the difference between the values of ϕ for the corresponding generators. (U.L.)

Let $\phi(x, y, z) = x^2 + y^2 - z^2 \tan^2 \alpha$; then $\dfrac{\partial \phi}{\partial x} = 2x, \dfrac{\partial \phi}{\partial y} = 2y, \dfrac{\partial \phi}{\partial z} = -2z \tan^2\alpha.$

The equation of the tangent plane at the point (x_1, y_1, z_1) on the cone is

$$(x - x_1)2x_1 + (y - y_1)2y_1 + (z - z_1)(-2z_1 \tan^2 \alpha) = 0$$

which reduces to

$$xx_1 + yy_1 - zz_1 \tan^2 \alpha = 0 \quad . \quad . \quad . \quad . \quad (1)$$

If the point (x_1, y_1, z_1) is on the given generator, then

$$x_1 : y_1 : z_1 = \sin \alpha \cos \phi : \sin \alpha \sin \phi : \cos \alpha = \cos \phi : \sin \phi : \cot \alpha$$

The equation (1) becomes

$$x \cos \phi + y \sin \phi - z \tan \alpha = 0 \quad . \quad . \quad . \quad . \quad (2)$$

which is the equation of the tangent plane required.

The equations of the tangent planes containing the generators corresponding to the values ϕ_1 and ϕ_2 of ϕ are by (2)

$$x \cos \phi_1 + y \sin \phi_1 - z \tan \alpha = 0 \quad . \quad . \quad . \quad (3)$$

$$x \cos \phi_2 + y \sin \phi_2 - z \tan \alpha = 0 \quad . \quad . \quad . \quad (4)$$

If the planes (3) and (4) are at right angles, then by (X.29) we have

$$\cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 + \tan^2 \alpha = 0$$

or

$$\tan^2 \alpha = -\cos (\phi_1 \sim \phi_2) \quad . \quad . \quad . \quad . \quad (5)$$

The left side of (5) is greatest when $\cos (\phi_1 \sim \phi_2) = -1$, i.e. when $\phi_1 \sim \phi_2 = \pi$. The greatest value of α is obtained from the relation $\tan^2 \alpha = 1$; whence $\alpha = \dfrac{\pi}{4}$

114. **The Sphere.** If in (X.59) we put $f(z) = \sqrt{a^2 - z^2}$, the equation becomes $x^2 + y^2 = a^2 - z^2$

or

$$x^2 + y^2 + z^2 = a^2 \text{ [See Art. 95]} \quad . \quad (X.65)$$

This is the equation of a sphere of radius a with its centre at the origin, the generating curve in the yz-plane being the circle $y^2 + z^2 = a^2$.

If the centre is at the point (x_1, y_1, z_1), the equation of the sphere is

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 - a^2 = 0 \text{ [See Art. 100]} \quad (X.66)$$

Expanding the left-hand side of this, we obtain

$$x^2 + y^2 + z^2 - 2x_1x - 2y_1y - 2z_1z + x_1^2 + y_1^2 + z_1^2 - a^2 = 0$$

or

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad . \quad (X.67)$$

where u, v, w and d are constants.

Comparing (X.67) with (X.51), we see that an equation of the second degree in x, y, z will represent a sphere, provided that (1) the coefficients of x^2, y^2, z^2 are equal, and (2) the terms in yz, zx, xy are absent.

If P is any point (ξ, η, ζ) outside the sphere (X.66), the square of the tangent from P to the sphere is equal to the square of the distance from P to the centre diminished by the square of the radius. This is easily seen from a figure. Hence (tangent from $P)^2 = (\xi - x_1)^2 + (\eta - y_1)^2 + (\zeta - z_1)^2 - a^2$.

It follows that if the co-ordinates ξ, η, ζ are substituted for x, y, z in the left-hand side of the equation (X.66) or (X.67), the result is equal to the square of the tangent from (ξ, η, ζ) to the sphere.

We leave the reader to deduce that the tangent planes at the point (x_1, y_1, z_1) on the spheres (X.65) and (X.67)

are
$$xx_1 + yy_1 + zz_1 = a^2 \qquad (X.68)$$

and
$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0 \qquad (X.69)$$

respectively.

EXAMPLE 1

Show that the spheres
$$x^2 + y^2 + z^2 + 6y + 2z + 8 = 0$$
and
$$x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$$
intersect at right angles.

Find the equation of the tangent plane to each sphere farthest from and parallel to the plane of intersection of the spheres. (U.L.)

Let (x_1, y_1, z_1) be a point on the curve of intersection of the two spheres
$$x^2 + y^2 + z^2 + 6y + 2z + 8 = 0 \qquad (1)$$
and
$$x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0 \qquad (2)$$

The equations of the tangent planes to the spheres (1) and (2) at the point (x_1, y_1, z_1) are by (X.69),
$$xx_1 + yy_1 + zz_1 + 3(y + y_1) + (z + z_1) + 8 = 0$$
and
$$xx_1 + yy_1 + zz_1 + 3(x + x_1) + 4(y + y_1) + 2(z + z_1) + 20 = 0$$
respectively. These equations can be written as
$$xx_1 + y(y_1 + 3) + z(z_1 + 1) + 3y_1 + z_1 + 8 = 0 \qquad (3)$$
and
$$x(x_1 + 3) + y(y_1 + 4) + z(z_1 + 2) + 3x_1 + 4y_1 + 2z_1 + 20 = 0 \qquad (4)$$

If the spheres (1) and (2) intersect at right angles, the planes (3) and (4) are at right angles. By (X.29) the required condition is
$$x_1(x_1 + 3) + (y_1 + 3)(y_1 + 4) + (z_1 + 1)(z_1 + 2) = 0$$
or
$$x_1^2 + y_1^2 + z_1^2 + 3x_1 + 7y_1 + 3z_1 + 14 = 0 . \qquad (5)$$

Now (x_1, y_1, z_1) lies on both spheres, so that from (1) and (2) we have
$$x_1^2 + y_1^2 + z_1^2 + 6y_1 + 2z_1 + 8 = 0. \qquad (6)$$
and
$$x_1^2 + y_1^2 + z_1^2 + 6x_1 + 8y_1 + 4z_1 + 20 = 0 \qquad (7)$$

Adding (6) and (7) and dividing throughout by 2, we obtain
$$x_1^2 + y_1^2 + z_1^2 + 3x_1 + 7y_1 + 3z_1 + 14 = 0$$
Thus, the condition (5) is satisfied and the spheres intersect at right angles.

The equation of the plane of intersection of the two spheres is

$$x^2 + y^2 + z^2 + 6y + 2z + 8 = x^2 + y^2 + z^2 + 6x + 8y + 4z + 20$$

i.e.

$$3x + y + z + 6 = 0 \quad . \quad . \quad . \quad . \quad (8)$$

for equation (8) is satisfied by the co-ordinates of any point which satisfy the equations (1) and (2) simultaneously. The equations of any plane parallel to (8) is

$$3x + y + z + d = 0 \quad . \quad . \quad . \quad . \quad (9)$$

where d is some constant. The equation (3) gives the tangent plane at any point (x_1, y_1, z_1) on the sphere (1), and if (3) and (9) are identical, we must have

$$\frac{x_1}{3} = \frac{y_1 + 3}{1} = \frac{z_1 + 1}{1} = \frac{3y_1 + z_1 + 8}{d} = k, \text{ say} \quad . \quad . \quad (10)$$

Substituting $x_1 = 3k$, $y_1 = k - 3$, $z_1 = k - 1$, in the equation (1) we obtain

$$9k^2 + k^2 - 6k + 9 + k^2 - 2k + 1 + 6k - 18 + 2k - 2 + 8 = 0$$

which gives $k^2 = \dfrac{2}{11}$ or $k = \pm \dfrac{\sqrt{22}}{11}$

From (10) we have: $d = \dfrac{3(k-3) + (k-1) + 8}{k} = 4 - \dfrac{2}{k} = 4 \mp \sqrt{22}$

Since $4 - \sqrt{22} = -0{\cdot}69$ and $4 + \sqrt{22} = 8{\cdot}69$, and we require the tangent plane farthest from the plane (8), we must take $d = 4 - \sqrt{22}$. Hence, the required tangent plane is from (9),

$$3x + y + z + 4 - \sqrt{22} = 0$$

Similarly, we prove that the required tangent plane to the sphere (2) is

$$3x + y + z + 15 + 3\sqrt{11} = 0$$

EXAMPLE 2

Find the equation of the sphere which passes through the four points (1, 2, 3), (0, − 2, 4), (4, − 4, 2), and (3, 1, 4).

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad . \quad . \quad (1)$$

Since the given points lie on the sphere (1), we have

$$1 + 4 + 9 + 2u + 4v + 6w + d = 0$$

or

$$14 + 2u + 4v + 6w + d = 0 \quad . \quad . \quad . \quad (2)$$

$$0 + 4 + 16 + 0\,.\,u - 4v + 8w + d = 0$$

or

$$20 + 0\,.\,u - 4v + 8w + d = 0 \quad . \quad . \quad . \quad (3)$$

$$16 + 16 + 4 + 8u - 8v + 4w + d = 0$$

or

$$36 + 8u - 8v + 4w + d = 0 \quad . \quad . \quad . \quad (4)$$

$$9 + 1 + 16 + 6u + 2v + 8w + d = 0$$

or

$$26 + 6u + 2v + 8w + d = 0 \quad . \quad . \quad . \quad (5)$$

Eliminating u, v, w, d, between the equations (1) to (5), we obtain

$$\begin{vmatrix} x^2+y^2+z^2 & 2x & 2y & 2z & 1 \\ 14 & 2 & 4 & 6 & 1 \\ 20 & 0 & -4 & 8 & 1 \\ 36 & 8 & -8 & 4 & 1 \\ 26 & 6 & 2 & 8 & 1 \end{vmatrix} = 0$$

On expansion this reduces to

$$x^2 + y^2 + z^2 - 4x + 2y - 2z = 8$$

which is the equation of the sphere through the four given points.

EXAMPLE 3

Prove that the equation of the sphere described on the line joining the points $(2, -1, 4)$ and $(-2, 2, -2)$ as diameter is

$$(x-2)(x+2) + (y+1)(y-2) + (z-4)(z+2) = 0$$

Find the area of the circle in which this sphere is intersected by the plane $2x + y - z = 3$. (U.L.)

By Art. 97, the mid-point of the straight line joining the points $(2, -1, 4)$ and $(-2, 2, -2)$ is the point $(0, \frac{1}{2}, 1)$ which must be the centre of the sphere. The radius of the sphere is equal to the distance between the points $(0, \frac{1}{2}, 1)$ and $(-2, 2, -2)$; whence the radius $= \sqrt{(-2)^2 + (2-\frac{1}{2})^2 + (-2-1)^2}$ $= \dfrac{\sqrt{61}}{2}$.

By (X.66) the equation of the sphere is

$$x^2 + (y-\tfrac{1}{2})^2 + (z-1)^2 = \frac{61}{4}$$

or

$$x^2 + y^2 + z^2 - y - 2z - 14 = 0 \qquad . \quad . \quad . \quad (1)$$

The equation $(x-2)(x+2) + (y+1)(y-2) + (z-4)(z+2) = 0$ reduces to (1).

Otherwise, let (x', y', z') be any point on the sphere; then the equations of the lines joining this point to the points $(2, -1, 4)$ and $(-2, 2, -2)$ are

$$\frac{x-2}{x'-2} = \frac{y+1}{y'+1} = \frac{z-4}{z'-4} \qquad . \quad . \quad . \quad (2)$$

and

$$\frac{x+2}{x'+2} = \frac{y-2}{y'-2} = \frac{z+2}{z'+2} \qquad . \quad . \quad . \quad (3)$$

By hypothesis the lines (2) and (3) are at right angles, so that we have

$$(x'-2)(x'+2) + (y'+1)(y'-2) + (z'-4)(z'+2) = 0$$

Hence, the locus of (x', y', z') is the sphere

$$(x-2)(x+2) + (y+1)(y-2) + (z-4)(z+2) = 0$$

The distance of the centre $(0, \frac{1}{2}, 1)$ of the sphere from the plane $2x + y - z - 3 = 0$ is $\dfrac{0 + \frac{1}{2} - 1 - 3}{\sqrt{2^2 + 1^2 + 1^2}} = \dfrac{7\sqrt{6}}{12}$

The radius r of the section of the sphere made by the given plane is given by

$$r^2 = (\text{radius of sphere})^2 - \left(\frac{7\sqrt{6}}{12}\right)^2 = \frac{61}{4} - \frac{49}{24} = \frac{317}{24}$$

$\therefore$
$$\text{Area of section} = \pi r^2 = \frac{317\pi}{24}$$

115. The Ellipsoid. In the equation (X.59) put

$$f(z) = b\sqrt{1 - \frac{z^2}{c^2}}$$

where c is a constant. The equation becomes

$$x^2 + y^2 = b^2\left(1 - \frac{z^2}{c^2}\right)$$

or
$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad . \quad . \quad . \quad . \quad (X.70)$$

This is the equation of an ellipsoid of revolution generated by the rotation of the ellipse $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ about the z-axis. If c is greater than b, the axis of rotation is the major axis and the surface is termed a *prolate spheroid*; if c is less than b the axis of rotation is the minor axis and the surface is termed an *oblate spheroid.*

Putting $f(z) = b\sqrt{1 - \frac{z^2}{c^2}}$ in equation (X.60) we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2}$$

or
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad . \quad . \quad . \quad (X.71)$$

The surface (X.71) is called an *ellipsoid.* Since

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}$$

and the sum of the squares of any number of quantities cannot be negative and can only be zero if each of the quantities is zero, it follows that x cannot exceed a numerically, and also that $y = 0$ and $z = 0$ simultaneously when $x = \pm a$. Similarly y cannot exceed b and z cannot exceed c; also $z = 0$, $x = 0$ when $y = \pm b$, and

$x = 0$, $y = 0$ when $z = \pm c$. If $a > b > c$, the greatest axis of the ellipsoid lies along the x-axis and the least along the z-axis. Any plane section of the ellipsoid is an ellipse.

If P is any point (x, y, z) on the ellipsoid and the semi-diameter $OP = r$ has direction-cosines l, m, n, then $x = rl$, $y = rm$, $z = rn$, and, substituting these values in (X.71), we obtain

$$\frac{r^2l^2}{a^2} + \frac{r^2m^2}{b^2} + \frac{r^2n^2}{c^2} = 1, \text{ or } \frac{1}{r^2} = \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \quad . \quad \text{(X.72)}$$

We deduce from (X.55) that the equation of the tangent plane at the point (x_1, y_1, z_1) on the ellipsoid is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1 \quad . \quad . \quad . \quad \text{(X.73)}$$

In Ex. 2 below, we prove that the equation of the diametral plane of chords of the ellipsoid having direction-cosines l, m, n is

$$\frac{lx}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2} = 0 \quad . \quad . \quad . \quad \text{(X.74)}$$

The normal to the ellipsoid at the point (x_1, y_1, z_1) is perpendicular to the plane (X.73); its equations are therefore

$$\frac{x - x_1}{\frac{x_1}{a^2}} = \frac{y - y_1}{\frac{y_1}{b^2}} = \frac{z - z_1}{\frac{z_1}{c^2}} \quad . \quad . \quad \text{(X.75)}$$

EXAMPLE 1

Find the equations of the tangent planes to the ellipsoid $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$ at the points in which it is met by the line $\frac{x}{2} = \frac{y}{2} = \frac{z}{3}$. Find the equation of the projection on the xy-plane of the section of the ellipsoid made by the plane through the centre parallel to the above tangent planes. (U.L.)

From the equations of the given line we have $y = x$ and $z = \frac{3}{2}x$. Substituting in the equation of the ellipsoid, we obtain

$$\frac{x^2}{4} + \frac{x^2}{16} + \frac{x^2}{4} = 1, \text{ or } x = \pm \tfrac{4}{3}, \text{ whence } y = \pm \tfrac{4}{3} \text{ and } z = \pm 2$$

By (X.73) the tangent plane at the point $(\frac{4}{3}, \frac{4}{3}, 2)$ is

$$\frac{x}{4}\left(\frac{4}{3}\right) + \frac{y}{16}\left(\frac{4}{3}\right) + \frac{z}{9}(2) = 1 \text{ or } 12x + 3y + 8z - 36 = 0 \quad . \quad (1)$$

and the tangent plane at the point $(-\frac{4}{3}, -\frac{4}{3}, -2)$ is

$$12x + 3y + 8z + 36 = 0 \quad . \quad . \quad . \quad . \quad (2)$$

Any plane parallel to the tangent planes (1) and (2) has an equation of the form $12x + 3y + 8z + d = 0$, and if this plane passes through the centre $(0, 0, 0)$, $d = 0$. Hence,

$$12x + 3y + 8z = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is the equation of the plane through the centre parallel to the tangent planes (1) and (2). The plane (3) meets the ellipsoid where

$$\frac{x^2}{4} + \frac{y^2}{16} = 1 - \frac{z^2}{9} = 1 - \frac{1}{9}\left[\frac{-3(4x + y)}{8}\right]^2 = 1 - \frac{(4x + y)^2}{64}$$

which reduces to $32x^2 + 8xy + 5y^2 = 64$. This is the equation of the projection on the xy-plane of the section of the ellipsoid made by the plane (3).

EXAMPLE 2

Show that the middle points of parallel chords of an ellipsoid lie on a plane. Obtain the equation of the diametral plane of chords having direction-cosines l, m, n in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

For the ellipsoid $x^2 + 3y^2 + 2z^2 = 1$, find the equations of the diameter in the plane $z = 0$ conjugate to the line $\frac{1}{4}x = 2y = z$, and also the equations of the third conjugate diameter. (U.L.)

We have proved in Art. 109 that the locus of the mid-points of a system of parallel chords of a quadric surface is the plane $l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y} + n\frac{\partial\phi}{\partial z} = 0$. Applied to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, this equation becomes

$$\frac{lx}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equation (1) gives the diametral plane of chords having direction-cosines l, m, n.

In the ellipsoid $x^2 + 3y^2 + 2z^2 = 1$, the equation (1) becomes

$$lx + 3my + 2nz = 0 \quad . \quad . \quad . \quad . \quad (2)$$

If the plane (2) is diametral to the line $\frac{1}{4}x = 2y = z$ or $\frac{x}{4} = \frac{y}{1} = \frac{z}{2}$, then l, m, n are proportional to 4, 1, 2, and the equation (2) becomes

$$4x + 3y + 4z = 0 \quad . \quad . \quad . \quad . \quad . \quad (3$$

If three diameters of an ellipsoid are such that the plane containing any two of them is diametral to the third, the three diameters are said to be conjugate.

The intersection of the plane (3) with the plane $z = 0$ is the line given by $4x + 3y = 0$, $z = 0$, or $\frac{x}{3} = \frac{y}{-4} = \frac{z}{0}$, and this line is the diameter in the plane $z = 0$ conjugate to the given line $\frac{1}{4}x = 2y = z$.

Let

$$Ax + By + Cz = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

be the plane containing the diameters $\frac{x}{4} = \frac{y}{1} = \frac{z}{2}$ and $\frac{x}{3} = \frac{y}{-4} = \frac{z}{0}$; then, since the normal to the plane (4) is perpendicular to both diameters, we have

$$4A + B + 2C = 0 \qquad . \quad . \quad . \quad . \quad (5)$$

and

$$3A - 4B + 0 \,.\, C = 0 \qquad . \quad . \quad . \quad . \quad (6)$$

Eliminating A, B, C between the equations (4), (5), (6), we obtain

$$\begin{vmatrix} x & y & z \\ 4 & 1 & 2 \\ 3 & -4 & 0 \end{vmatrix} = 0, \text{ which reduces to } 8x + 6y - 19z = 0 \qquad . \quad . \quad (7)$$

If the plane (7) is diametral to the line $\frac{x}{l'} = \frac{y}{m'} = \frac{z}{n'}$, then from (2) we see that (7) must be identical with the equation $l'x + 3m'y + 2n'z = 0$. Hence we have

$$\frac{l'}{8} = \frac{3m'}{6} = \frac{2n'}{-19} \text{ or } l' : m' : n' = 16 : 4 : -19$$

and the equations of the third conjugate diameter are therefore

$$\frac{x}{16} = \frac{y}{4} = -\frac{z}{19}$$

116. **The Hyperboloids.** In the equation (X.59) put

$$f(z) = b\sqrt{1 + \frac{z^2}{c^2}}$$

so that the generating curve in the yz-plane is the hyperbola

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

The equation (X.59) becomes $x^2 + y^2 = b^2\left(1 + \frac{z^2}{c^2}\right)$

or

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad . \quad . \quad . \quad (X.76)$$

This is the equation of a *hyperboloid of revolution of one sheet*, the sections by planes parallel to the xy-plane being circles. With the same value of $f(z)$ substituted in equation (X.60), we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad . \quad . \quad . \quad (X.77)$$

This is the equation of a *hyperboloid of one sheet*. When $x = 0$ and $y = 0$, z is imaginary, so that the z-axis does not cut the surface.

When $y = 0$ and $z = 0$, $x = \pm a$, and when $x = 0$ and $z = 0$, $y = \pm b$. Planes parallel to the yz- and zx-planes cut the surface in hyperbolas and planes parallel to the xy-plane cut the surface in ellipses.

Now, in equation (X.59), put

$$f(z) = b\sqrt{\frac{z^2}{c^2} - 1}$$

so that the generating curve in the yz-plane is the hyperbola

$$\frac{z^2}{c^2} - \frac{y^2}{b^2} = 1$$

which is conjugate to the hyperbola

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

The equation (X.59) becomes $x^2 + y^2 = b^2\left(\frac{z^2}{c^2} - 1\right)$

or

$$\frac{z^2}{c^2} - \frac{x^2}{b^2} - \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad (X.78)$$

This is the equation of a *hyperboloid of revolution of two sheets*, there being here two separate branches of the surface, the one the image of the other, with the xy-plane as the supposed mirror.

Again, substituting the same value for $f(z)$ in equation (X.60) we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} - 1$$

or

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad (X.79)$$

This is the equation of a *hyperboloid of two sheets*. If z is numerically less than c, $\frac{z^2}{c^2} - 1$ is negative and therefore $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ is negative, which is impossible. It follows that no part of the surface lies between the limits $z = -c$ to $z = +c$. The section of the surface by any plane parallel to the plane $y = 0$ or to the plane $x = 0$ is a hyperbola, and the section by the plane $z = k$, where k is numerically greater than c, is an ellipse.

117. **The Paraboloids.** In the equation (X.59) put $f(z) = \sqrt{kz}$, where k is a constant, the generating curve in the yz-plane being the parabola $y^2 = kz$. The equation (X.59) becomes

$$x^2 + y^2 = kz \qquad . \quad . \quad . \quad (X.80)$$

which is the equation of a *paraboloid of revolution.* The sections of the surface by the planes $x = 0$ and $y = 0$ are parabolas with the same axis, and the section by any plane $z = c$ (where c is positive) is a circle. The surface lies wholly on the positive side of the xy-plane.

With the substitution $f(z) = \sqrt{kz}$ in equation (X.60) we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{b^2} \cdot kz$$

or

$$\frac{x^2}{p} + \frac{y^2}{q} = 2z \quad . \quad . \quad . \quad . \quad (X.81)$$

where

$$p = \frac{ka^2}{2b^2}, q = \frac{k}{2}$$

This is the equation of an *elliptic paraboloid*, the sections by planes $z = c$ (where c is positive) being ellipses. The planes $x = 0$ and $y = 0$ cut the surface in parabolas.

If q is negative and equal to $-q'$ where q' is positive, the equation (X.81) becomes

$$\frac{x^2}{p} - \frac{y^2}{q'} = 2z \qquad . \quad . \quad . \quad (X.82)$$

This is the equation of a *hyperbolic paraboloid*, the sections by the planes $x = 0$ and $y = 0$ being parabolas with the same axis and vertex, the one concave downwards, the other concave upwards. The sections of the surface by the planes $z = \pm c$ are hyperbolas.

EXAMPLE

Find the equation of the tangent plane at the point (x_1, y_1, z_1) on the paraboloid $\frac{x^2}{4} + y^2 = 2z$. Deduce the equations of the normal at this point, and show that five normals can be drawn to the paraboloid from any point.

By (X.55), the equation of the tangent plane is

$$(x - x_1)\frac{x_1}{2} + (y - y_1)2y_1 + (z - z_1)(-2) = 0$$

or

$$xx_1 + 4yy_1 - 4(z + z_1) = 0 \quad . \quad . \quad (1)$$

The direction-cosines of the normal are proportional to x_1, $4y_1$, -4, so that the equations of the normal are

$$\frac{x-x_1}{x_1}=\frac{y-y_1}{4y_1}=\frac{z-z_1}{-4} \quad . \quad . \quad . \quad . \quad (2)$$

If the normal (2) passes through the point (ξ, η, ζ) we have

$$\frac{\xi-x_1}{x_1}=\frac{\eta-y_1}{4y_1}=\frac{\zeta-z_1}{-4}=k, \text{ say.}$$

so that $x_1=\dfrac{\xi}{k+1}$, $y_1=\dfrac{\eta}{4k+1}$, $z_1=\zeta+4k$. Substituting in the equation of the paraboloid, we have

$$\frac{\xi^2}{4(k+1)^2}+\frac{\eta^2}{(4k+1)^2}=2\zeta+8k \quad . \quad . \quad . \quad (3)$$

The equation (3) is of the fifth degree in k, so that five normals to the given paraboloid pass through the point (ξ, η, ζ).

EXAMPLES X

(1) Find the polar co-ordinates of the following points—

(i) (2, 5, 9). (ii) (− 3, − 5, 7). (iii) (4, 0, − 1). (iv) (5, − 3, 10).

(2) Find the rectangular co-ordinates of the following points—

(i) (12, 30°, 60°). (ii) (4, 50°, 150°). (iii) (2, 134°, 24°).

(3) The rectangular co-ordinates of a point P are (7, − 3, 4), and the polar co-ordinates of a point Q are (6, 64°, 70°); find the direction-cosines of the lines OP and OQ, where O is the origin of co-ordinates.

(4) The angular points of a triangle are (2, 4, 6), (3, 6, 9), and (− 4, 2, − 4) respectively. Find the lengths of the sides of the triangle and the angles which these sides make with the axes of reference.

(5) Find (i) the distance between the points $P(3, -4, 15)$ and $Q(8, 11, -5)$, and (ii) the co-ordinates of the point which divides PQ in the ratio 3 : 2.

(6) Find the equations of the straight line which passes through the two points P and Q in Question 5.

(7) The equation of a plane is $5x-3y+4z=4$. Find the length of the perpendicular on this plane from the origin and the angles which the perpendicular makes with the axes of reference.

(8) A plane passes through the three points whose rectangular co-ordinates are (8, − 2, 2), (2, 1, − 4), and (2, 4, − 6). Find the equation to this plane, the length of the perpendicular on it from the origin, and the direction-cosines of that perpendicular. (U.L.)

(9) Find the equations of the two planes which bisect the angles between the planes $3x-4y+5z=3$ and $5x+3y-4z=9$.

(10) Find the perpendicular distance from the point (− 3, 1, 5) to the plane $7x-y+2z+14=0$; find also the equations of this perpendicular and the co-ordinates of the point in which it meets the given plane.

(11) A plane passes through the point $(4, -1, 2)$, and is perpendicular to the line joining the points $(1, -5, 10)$ and $(2, 3, 4)$; find the equation of the plane and the angles which it makes with the co-ordinate planes.

(12) If θ be the angle between two straight lines whose direction-cosines are l, m, n and l', m', n', prove that $\cos\theta = ll' + mm' + nn'$.

The feet A, B, C of a tripod, having legs of equal length, are at the points $(0, 0)$, $(3, 9)$, $(7, 1)$ referred to rectangular axes in a horizontal plane. The apex P of the tripod is at a height 12 above the plane. Determine the cosines of the angles between (i) the lines AP and BC, (ii) the planes PAB and PAC. (U.L.)

(13) A tripod has its feet A, B, C on three walls, C being 2 ft higher than B and B 3 ft higher than A. In plan $ab = 18$ ft, $bc = 20$ ft, $ca = 21$ ft, and d, the plan of the apex of the tripod, is equidistant from a, b, and c. If D is 25 ft higher than C, find the lengths of the legs and the true values of the three plane angles at D. (U.L.)

(14) Define the *direction-cosines* of a line in space referred to three rectangular axes. Find the direction-cosines of the line of intersection of the planes $3x + 2y + z = 5$ and $x + y - 2z = 3$; and determine the angle this line makes with the line of intersection of $2x = y + z$ and $7x + 10y = 8z$. (U.L.)

(15) Find the equation of the plane which passes through the points $(2, 0, 3)$, $(0, 4, 1)$, $(5, 7, 10)$. Find also the equations of the straight line in the plane which intersects the z-axis, and whose inclination to the xy-plane is a maximum. (U.L.)

(16) Define the direction-cosines of a straight line with reference to three mutually perpendicular axes, and find the condition that two lines whose direction-cosines are given should be at right angles.

Taking the axis of z as vertical, find the direction-cosines of the line of greatest slope in the plane which passes through the points $(0, 0, 0)$, $(3, 5, -2)$, and $(4, 1, 1)$. (U.L.)

(17) With given rectangular axes, the line $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ is vertical. Find the direction-cosines of the line of greatest slope in the plane $3x - 2y + z = 5$ and the angle this line makes with the horizontal plane. (U.L.)

(18) Find the angle between (i) the planes $4x + y - z = 7$ and $x - 2y + 3z = 0$, (ii) the lines $\frac{x-2}{-1} = \frac{y+5}{4} = \frac{z-1}{2}$ and $\frac{x}{3} = \frac{y+2}{6} = \frac{z}{-2}$

(19) Find an expression for the cosine of the angle between two lines whose direction-cosines are given.

A right pyramid stands on a square base and the vertical angle of each of the isosceles triangular faces is α. Taking the vertex as origin, the axis of z along the axis of the pyramid and the axes of x and y parallel to the diagonals of the base, obtain the equations of the planes of the triangular faces and of their lines of intersection, and find the angle between the lines which join the vertex to a pair of opposite corners of the base. (U.L.)

(20) The vertical angle of each of the triangular faces of a right pyramid on a square base is 28°. A plane cuts the pyramid, the section being a quadrilateral $ABCD$ such that the distances of A, B, and C from the vertex are 4, 8, and 6 in. respectively. Find the distance of D from the vertex. If the pyramid is now placed with $ABCD$ on a horizontal plane, draw its elevation on a vertical plane perpendicular to AC. (U.L.)

(21) A straight line is inclined to the axes of x and z at angles of 58° and 134° respectively: find its inclination to the axis of y. Write down the equations of the line, given that it passes through the point $(3, -3, 7)$.

(22) The polar co-ordinates of two points P and Q are (8, 45°, 62°) and (10, 60°, 75°). Find the angle between the lines OP and OQ, O being the origin of co-ordinates.

(23) A plane passes through the point $(5, -5, 9)$ and is perpendicular to the line $\frac{x-3}{6} = \frac{y+1}{5} = \frac{z-7}{-2}$. Find the perpendicular distance of the point $(4, -2, 8)$ from this plane, and the co-ordinates of the point at which this perpendicular meets the plane.

(24) Show that the plane $5x - 10y - 6z - 29 = 0$ contains the line $\frac{x-3}{4} = \frac{y+2}{-1} = \frac{z-1}{5}$. Find the angle between this line and the line of greatest slope in the given plane, the xy-plane being assumed horizontal.

(25) Show that the line perpendicular to both of the lines whose direction-cosines are proportional to l, m, n; l', m', n', has direction-cosines proportional to $mn' - m'n$, $nl' - n'l$, $lm' - l'm$. A plane parallel to the line $x - 1 = 2y - 5 = 2z$ and to the line $3x = 4y - 11 = 3z - 4$ passes through the point (2, 3, 3). Find its equation. (U.L.)

(26) Find the angle between the line of intersection of the planes $2x + 2y - z + 15 = 0$ and $4y + z + 29 = 0$, and the line $\frac{x+4}{4} = \frac{y-3}{-3} = \frac{z+2}{1}$. What is the shortest distance between the two lines?

(27) Find the shortest distance between the lines $x = y + 4 = \frac{z}{2}$ and $\frac{x-1}{3} = \frac{y}{2} = z$.

(28) Define *direction-cosines*, and find the cosine of the angle between two straight lines whose direction-cosines are given.

The plane of a square $ABCD$ and thed iagonal AC make angles of 60° and 45° with the horizontal. The axis of z is vertical, and the axis of x is horizontal and in the plane of the square. Prove that the positive senses of the axes can be chosen so that the direction-cosines of AC are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}$, and that, when this is done, the direction-cosines of AB, AD are $\frac{1 \mp \sqrt{2}}{\sqrt{6}}, \frac{\sqrt{2} \pm 1}{2\sqrt{6}}, \frac{\sqrt{2} \pm 1}{2\sqrt{2}}$, where all the upper signs are to be taken for one, and all the lower signs for the other. (U.L.)

(29) If $u = 0$, $v = 0$ be the equations of two planes, show that $u + kv = 0$ is the equation of a plane containing their line of intersection, k being any constant.

Find the length of the common perpendicular to the two lines

$$\text{(i) } 3x = 5y = z - 2$$

and

$$\text{(ii) } x + 3 = y = 4z$$

(U.L.)

(30) Prove that the three planes $2x - 5y + 11z = 33$, $8x + 9y - 3z = 31$, $14x + 23y - 17z = 29$, intersect along a straight line. Find the direction-cosines of this line.

(31) Prove that the two lines $x - 3 = \frac{y + 4}{-3} = \frac{z - 5}{3}$ and $x - 4 = \frac{y - 5}{3} = \frac{z + 6}{-4}$ intersect, and find the co-ordinates of their point of intersection. Find also the perpendicular distance from the point $(3, -2, 3)$ to the plane containing the lines.

(32) The co-ordinates of the corners of the plane sloping roof of a shelter open on all sides are $(6, 6, 8)$, $(-6, 6, 8)$, $(-6, -6, 10)$, $(6, -6, 10)$. The axes of x and y are horizontal and their positive directions point due west and south respectively. The axis of z is drawn vertically upwards, the origin is on the ground, and the unit of length is 1 ft. Find the area of the shadow cast by the roof upon the ground when the sun's altitude is 60° and its direction south-east. (U.L.)

(33) Show that if the projections of an area S on the co-ordinate planes are S_{yz}, S_{zx}, S_{xy} respectively, then $S = \sqrt{S_{yz}^2 + S_{zx}^2 + S_{xy}^2}$

Find the area of the projection on the xy-plane of the triangle whose angular points are $(3, -4, 7)$, $(2, 1, 5)$, $(4, 2, 6)$.

(34) Prove that the volume of a tetrahedron the co-ordinates of whose angular points are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4), is equal to

$$\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

Find the volume of the tetrahedron whose angular points are $(0, 1, 2)$, $(2, 3, -5)$, $(-1, -2, -4)$, $(3, 5, 7)$.

(35) A tetrahedron has its vertices at the points $(1, 0, 0)$, $(0, 0, 1)$, $(0, 0, 2)$, $(1, 2, 3)$ respectively. Find the lengths and direction-cosines of the six edges, the equations of the four faces and the volume of the tetrahedron. (U.L.)

(36) Find the volume of the tetrahedron formed by the planes $2x + 3y + z = 6$, $2x + 3y = 0$, $3y + z = 0$, and $z + 2x = 0$.

(37) Write down the equation of the sphere whose radius is 4 and whose centre is at the point $(-6, 1, 3)$. Find the area of the section in which the sphere is cut by the plane $x - y + 2z + 5 = 0$.

(38) Find the equation of the tangent plane to the sphere in Question 37 at the point $(-2, 1, 3)$. Deduce the equations of the normal to the sphere at that point.

(39) A sphere passes through the four points $(2, 0, 4)$, $(-2, 3, 1)$, $(0, -4, 2)$, $(4, 3, -1)$; find its equation.

(40) Determine the centre and radius of the section of the sphere $x^2 + y^2 + z^2 = 16$ made by the plane $2x + y + 2z = 9$.

Find the orthogonal projection of this section on the xy-plane. (U.L.)

(41) Find the equation of the tangent plane to the surface $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 49$ at the point $(2, 6, 18)$. Also find the equations to the normal to the surface at the same point. (U.L.)

(42) A circular cylinder of radius 2 has for its axis the line $(x - 1)/2 = y = 3 - z$. Find (i) its equation, (ii) the lengths of the axes of the ellipse in which it is cut by the plane $x = y$. (U.L.)

(43) Show that the plane $4x - 6y - z + 14 = 0$ touches the surface $x^2 + 3y^2 + 2z = 0$, and find the co-ordinates of the point of contact.

(44) Find the equation of the plane containing all the chords of the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$ which are bisected at the point $(1, -1, \frac{1}{4})$. Find also the equations of the tangent planes to the ellipsoid parallel to this plane.

(45) Prove that the condition that the line $\frac{x-p}{l} = \frac{y-q}{m} = \frac{z-r}{n}$ may touch the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is

$$\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)\left(\frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} - 1\right) = \left(\frac{pl}{a^2} + \frac{qm}{b^2} + \frac{rn}{c^2}\right)^2$$

Find the equation of the cone with vertex at the origin and generators parallel to tangent lines from (p, q, r) to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. (U.L.)

(46) Find the locus of the mid-points of chords of the ellipsoid $\frac{x^2}{25} + \frac{y^2}{4} + \frac{z^2}{9} = 1$ which are parallel to the line $\frac{x}{5} = \frac{y}{2} = \frac{z}{-3}$

(47) Find the principal planes of the surface

$$28x^2 + 24y^2 + 20z^2 - 16yz - 16xy - 36x + 48y - 48z + 21 = 0$$

(48) Show that the equation of the tangent plane to the surface $f(x, y, z) = 0$ at the point (a, b, c) on the surface is

$$(x-a)f_a(a, b, c) + (y-b)f_b(a, b, c) + (z-c)f_c(a, b, c) = 0$$

Find the tangent plane to the surface $xy + yz + zx + 17 = 0$ which is parallel to the plane $2x - y + z = 0$. (U.L.)

[*Note*. $f_a(a, b, c), f_b(a, b, c), f_c(a, b, c)$ denote here the values of $\frac{\partial}{\partial x}f(x, y, z)$, $\frac{\partial}{\partial y}f(x, y, z)$, $\frac{\partial}{\partial z}f(x, y, z)$ respectively, when $x = a$, $y = b$, $z = c$, i.e. at the point (a, b, c) on the surface.]

(49) Find the co-ordinates of the centre and the lengths of the principal axes of the ellipsoid

$$9x^2 + 4y^2 + 36z^2 - 54x + 8y - 72z + 13 = 0$$

(50) Find the equations of the tangent planes to the hyperboloid $\frac{x^2}{4} + y^2 - 2z^2 = 1$ which are perpendicular to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-4}$

(51) Show that the projection on the xy-plane of the section of the cone $x^2 + y^2 - 5z^2 = 0$ made by the plane $2y - 5z + 1 = 0$ is an ellipse. Find the eccentricity of the ellipse.

(52) Find the equation of the tangent plane to the surface $3x^2 + y^2 + z^2 = 21$ which is equally inclined to the three axes of reference and touches the surface in the first octant. Prove that the volume of the tetrahedron whose faces are the planes $x = 0$, $y = 0$, $z = 0$, and this tangent plane is $57\frac{1}{6}$.

(53) Find the equations of the line of intersection of the tangent planes to the paraboloid $2x^2 + 3y^2 = 12z$ at the points $(3, 4, 5{\cdot}5)$ and $(6, -2, 7)$. Show that the plane passing through the line of intersection of the tangent planes and the mid-point of the line joining the points of contact is parallel to the z-axis.

(54) The points $(3, -2, 5)$ and $(5, 4, 8)$, are the ends of a diameter of a sphere; find the equation of the sphere and the equations of the two tangent planes to the sphere which are perpendicular to the line joining the centre of the sphere to the origin.

(55) Find the equation of the tangent plane to the sphere

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2$$

at the point (x_1, y_1, z_1).

Find the equation of the sphere which touches the sphere $x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$ at the point $(1, 2, -2)$ and passes through the origin. (U.L.)

(56) Find the equations of the diameter of the sphere $x^2 + y^2 + z^2 = 29$ such that a rotation about it will transfer the point $(4, -3, 2)$ to the point $(5, 0, -2)$ along a great circle of the sphere.

Find also the angle through which the sphere must be so rotated. (U.L.)

(57) Find the equations of the planes which contain the line of intersection of the planes

$$x - y + 2z = 0$$
$$3x + y + z = 5$$

and are inclined at 45° to the z-axis.

Find also the equation of the plane which contains this line and is parallel to the z-axis, and deduce the shortest distance between the line and the z-axis. (U.L.)

(58) Find the perpendicular distance of the point (f, g, h) from the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

Find the equation of the cone, whose whole vertical angle is 90°, which has its vertex at the origin and its axis along the line $x = -2y = z$, and show that the plane $z = 0$ cuts the cone in two straight lines inclined at an angle $\cos^{-1} \frac{1}{5}$. (U.L.)

(59) Show that the sphere

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

is cut by the plane

$$Ax + By + Cz + D = 0$$

in a circle of radius

$$\left\{r^2 - \frac{(Aa + Bb + Cc + D)^2}{A^2 + B^2 + C^2}\right\}^{\frac{1}{2}}$$

Find the equation of a sphere which passes through the circle $x^2 + y^2 = 4$, $z = 0$, and is cut by the plane $x + 2y + 2z = 0$ in a circle of radius 3. (U.L.)

(60) Prove that the surface $x^2 + y^2 - 4z^2 = c^2$ is met by the plane $x \cos \theta + y \sin \theta = 2z$ in two parallel straight lines whose shortest distance from the axis of z is constant, as also their inclination to the axis of z, and that the surface could be generated by the revolution of either of these lines about the axis of z. (U.L.)

CHAPTER XI

SPHERICAL TRIGONOMETRY

118. **The Spherical Triangle.** The section of a sphere made by any plane is a circle. If the plane passes through the centre of the sphere the circle is termed a *great circle*, and if the plane does not pass through the centre the circle is termed a *small circle*. The *poles* of any circular section of the sphere are the extremities of the diameter perpendicular to the section. The planes of two great circles intersect along a diameter of the sphere and cut off on the surface of the sphere two pairs of congruent areas, called *lunes*. The figure included by the arcs of three great circles is a *spherical triangle*.

In Fig. 87 ABC is a spherical triangle formed by the arcs of the three great circles $ABA'B'$, $BCB'C'$, $CAC'A'$. The sides $a(= BC)$, $b(= CA)$, $c(= AB)$ of the triangle are measured by the angles BOC, COA, AOB, which they subtend at the centre O of the sphere. The angles A, B, C of the triangle are the angles between the planes of the three great circles.

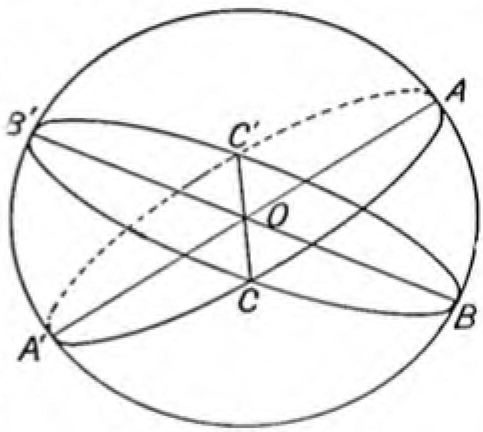


FIG. 87

119. **Area of a Lune and Area of a Spherical Triangle.** The surface $CAC'BC$ (Fig. 87) is a lune. It is evident that the area of a lune is proportional to the angle between the planes of the two great circles which cut it off on the surface of the sphere. Hence, the angles being measured in radians, we have

$$\frac{\text{Area of lune } CAC'BC}{\text{Surface of sphere}} = \frac{C}{2\pi}$$

$$\therefore \qquad \text{Area of lune } CAC'BC = \frac{C}{2\pi} \times 4\pi r^2 = 2Cr^2 \quad . \qquad \text{(XI.1)}$$

where r is the radius of the sphere.

We have then

$$\text{Area of lune } CAC'BC = 2Cr^2$$

$$\text{Area of lune } ACA'BA = 2Ar^2$$

$$\text{Area of lune } BCB'AB = 2Br^2$$

Now

$$\text{Area of lune } CAC'BC = \text{area of triangle } ABC + \text{area of triangle } ABC'$$

$$\text{Area of lune } ACA'BA = \text{area of triangle } ABC + \text{area of triangle } A'BC$$

$$\text{Area of lune } BCB'AB = \text{area of triangle } ABC + \text{area of triangle } ACB'$$

$$\therefore \text{Sum of areas of the three lunes} = 2(A + B + C)r^2$$

$$= 3(\text{area of triangle } ABC) + \text{sum of areas of triangles } ABC', A'BC, ACB'$$

$$= 3(\text{area of triangle } ABC) + \text{sum of areas of triangles } A'B'C, A'BC, ACB'$$

$$= 2(\text{area of triangle } ABC) + \text{half surface of whole sphere}$$

For the triangle $A'B'C$ is equal to the triangle ABC' and the triangles ABC, $A'B'C$, $A'BC$, ACB' together form the surface of a hemisphere.

Thus, $2(A + B + C)r^2 = 2(\text{area of triangle } ABC) + 2\pi r^2$ or, area of triangle $ABC = (A + B + C - \pi)r^2$ . . (XI.2)

The amount by which the sum of the angles of any spherical polygon exceeds the sum of the angles of the corresponding plane figure is termed the *spherical excess* of the polygon. Thus, since the sum of the angles of a plane triangle is π, the spherical excess E of a spherical triangle is given by

$$E = A + B + C - \pi$$

and the area of a spherical triangle can be expressed as Er^2 or $\dfrac{E}{4\pi} \times$ (surface of sphere).

We can divide a spherical polygon of n sides into $n - 2$ spherical triangles by joining one angular point to the other opposite angular

points by means of arcs of great circles. Applying (XI.2) to each triangle and adding the results, we obtain

$$\text{Area of spherical polygon} = \{\text{sum of angles} - (n-2)\pi\}r^2$$
$$= Er^2 \text{ or } \frac{E}{4\pi} \times (\text{surface of sphere})$$

where E = spherical excess of polygon = sum of angles $-\ (n-2)\pi$ since $(n-2)\pi$ is the sum of the angles of a plane polygon of n sides.

120. **Formulae connecting Sides and Angles of a Spherical Triangle.** Corresponding to the formulae connecting the sides and angles of a plane triangle, there are formulae connecting the sides and angles of a spherical triangle. We prove the most important of these in the course of the present chapter.

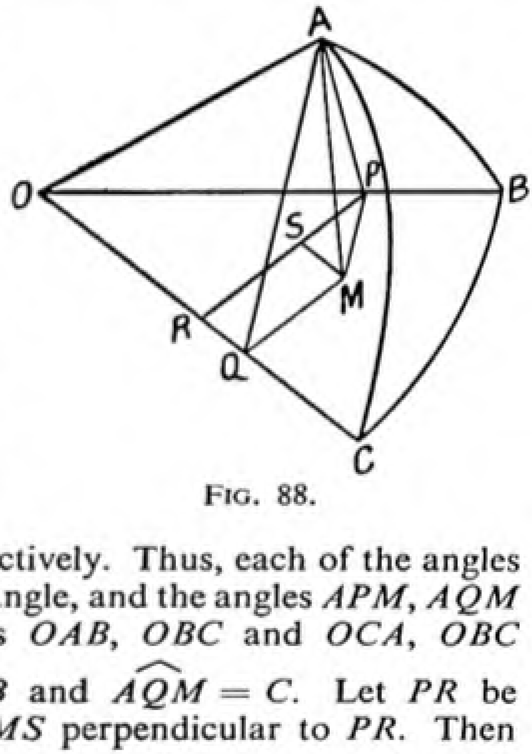


FIG. 88.

(i) COSINE FORMULAE. In Fig. 88, ABC is a spherical triangle on the surface of a sphere whose radius is r and centre is O. AM is drawn perpendicular to the plane OBC. The plane through AM perpendicular to OB cuts the planes OAB, OBC along the lines AP, MP respectively, and the plane through AM perpendicular to OC cuts the planes OCA, OBC along the lines AQ, MQ respectively. Thus, each of the angles OPA, OPM, OQA, OQM is a right angle, and the angles APM, AQM are the angles between the planes OAB, OBC and OCA, OBC respectively. Therefore, $\widehat{APM} = B$ and $\widehat{AQM} = C$. Let PR be drawn perpendicular to OC, and MS perpendicular to PR. Then $\widehat{SPM}$, being the angle between lines at right angles to OB, OC, is equal to $\widehat{BOC}$ or a. We have

$$\cos a = \cos BOC = \frac{OR}{OP} = \frac{OQ - RQ}{OP} = \frac{OQ - SM}{OP}$$

Now $\quad OQ = r\cos COA = r\cos b$

$$SM = MP\sin a = AP\cos B\sin a = r\sin c\cos B\sin a$$
$$OP = r\cos c$$

Hence

$$\cos a = \frac{r\cos b - r\sin c\sin a\cos B}{r\cos c} = \frac{\cos b - \sin c\sin a\cos B}{\cos c}$$

$$\left.\begin{aligned} &\text{or} \quad \cos b = \cos c\cos a + \sin c\sin a\cos B \\ &\text{Similarly} \\ &\cos c = \cos a\cos b + \sin a\sin b\cos C \\ &\cos a = \cos b\cos c + \sin b\sin c\cos A \end{aligned}\right\} \quad . \quad . \quad \text{(XI.3)}$$

(ii) SINE FORMULAE. From Fig. 88, we have

$$\sin c = \sin AOB = \frac{AP}{r} = \frac{1}{r}\cdot\frac{AM}{\sin B}$$

and $\sin b = \sin COA = \dfrac{AQ}{r} = \dfrac{1}{r}\cdot\dfrac{AM}{\sin C}$

By division, $\dfrac{\sin c}{\sin b} = \dfrac{\sin C}{\sin B}$ or $\dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$

We prove similarly that $\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b}$

Hence $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \quad . \quad . \quad . \quad . \quad \text{(XI.4)}$$

(iii) COTANGENT FORMULAE. From Fig. 88, we have

$$PR = PS + SR = PS + MQ$$

$\therefore$ $$OP\sin a = MP\cos a + AQ\cos C$$

$\therefore$ $$r\cos c\sin a = AP\cos B\cos a + r\sin b\cos C$$

$\therefore$ $$r\cos c\sin a = r\sin c\cos B\cos a + r\sin b\cos C$$

or $$\cos c\sin a = \sin c\cos a\cos B + \sin b\cos C$$

Dividing throughout by $\sin c$, we obtain

$$\cot c\sin a = \cos a\cos B + \frac{\sin b}{\sin c}\cos C$$

i.e. $$\cot c\sin a = \cos a\cos B + \frac{\sin B}{\sin C}\cos C$$

$$\left.\begin{aligned}
&\text{or} & \cot c \sin a &= \cos a \cos B + \sin B \cot C \\
&\text{Similarly} & \cot c \sin b &= \cos b \cos A + \sin A \cot C \\
&& \cot a \sin b &= \cos b \cos C + \sin C \cot A \\
&& \cot a \sin c &= \cos c \cos B + \sin B \cot A \\
&& \cot b \sin c &= \cos c \cos A + \sin A \cot B \\
&& \cot b \sin a &= \cos a \cos C + \sin C \cot B
\end{aligned}\right\} \quad . \quad \text{(XI.5)}$$

121. **Formulae Deduced from Polar Triangle.** In Fig. 89 ABC, $A_1B_1C_1$ are two spherical triangles on the surface of a sphere so related that the point A_1 is the pole of the great circle BC on the same side of this circle as A, the point B_1 is the pole of the great circle CA on the same side of this circle as B, and the point C_1 is the pole of the great circle AB on the same side of this circle as C. $A_1B_1C_1$ is called the *polar triangle* of the triangle ABC. We leave the reader to deduce from the figure that ABC is the polar triangle of $A_1B_1C_1$.

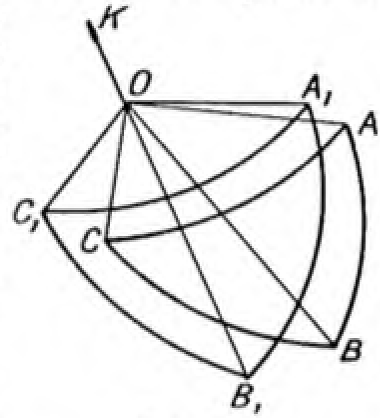


FIG. 89

Let B_1O be produced to any point K. Then, since OA_1 is perpendicular to the plane BOC and OK is perpendicular to the plane COA, the angle A_1OK = angle between the planes BOC and $COA = C$.

If a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$, then

$$c_1 = \text{angle } A_1OB_1 = \pi - \text{angle } A_1OK = \pi - C$$

We prove similarly, $a_1 = \pi - A$ and $b_1 = \pi - B$

Also, since ABC is the polar triangle of $A_1B_1C_1$, we have

$$c = \pi - C_1, \qquad a = \pi - A_1, \qquad b = \pi - B_1$$

or $$C_1 = \pi - c, \qquad A_1 = \pi - a, \qquad B_1 = \pi - b$$

Applying (XI.3) to the triangle $A_1B_1C_1$, we obtain

$$\cos b_1 = \cos c_1 \cos a_1 + \sin c_1 \sin a_1 \cos B_1$$

i.e. $$\cos(\pi - B) = \cos(\pi - C)\cos(\pi - A)$$
$$+ \sin(\pi - C)\sin(\pi - A)\cos(\pi - b)$$

Hence $\quad -\cos B = \cos C \cos A - \sin C \sin A \cos b$

$$\left.\begin{aligned} \text{or} \qquad \cos b &= \frac{\cos B + \cos C \cos A}{\sin C \sin A} \\ \text{Similarly} \qquad \cos c &= \frac{\cos C + \cos A \cos B}{\sin A \sin B} \\ \text{and} \qquad \cos a &= \frac{\cos A + \cos B \cos C}{\sin B \sin C} \end{aligned}\right\} \qquad \text{(XI.6)}$$

We note then that if, in any formula connecting the sides and angles of a spherical triangle ABC, we substitute $\pi - C$, $\pi - A$, $\pi - B$, $\pi - c$, $\pi - a$, $\pi - b$ for c, a, b, C, A, B respectively, we obtain another formula which is also true for the triangle ABC.

122. **Solution of Right-angled Triangles.** In the spherical triangle ABC let $A = 90°$. The formulae (XI.3) then give

$$\cos a = \cos b \cos c + \sin b \sin c \cos 90°$$

whence
$$\cos a = \cos b \cos c \qquad \text{(XI.7)}$$

Also
$$\cos b = \cos c \cos a + \sin c \sin a \cos B$$

and substituting from (XI.7) we have

$$\cos b = \cos c \cos b \cos c + \sin c \sin a \cos B$$

or
$$\cos B = \frac{\cos b\,(1 - \cos^2 c)}{\sin c \sin a} = \frac{\cos b \sin^2 c}{\sin c \sin a}$$

$$= \frac{\cos b \sin c}{\sin a} = \frac{\cos a}{\cos c} \cdot \frac{\sin c}{\sin a}$$

$$\therefore \qquad \cos B = \frac{\tan c}{\tan a} \qquad \text{(XI.8)}$$

Similarly
$$\cos C = \frac{\tan b}{\tan a} \qquad \text{(XI.9)}$$

The formulae (XI.4) give

$$\frac{\sin 90°}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

whence
$$\sin B = \frac{\sin b}{\sin a} \qquad \text{(XI.10)}$$

and
$$\sin C = \frac{\sin c}{\sin a} \qquad \text{(XI.11)}$$

Again $$\tan B = \frac{\sin B}{\cos B} = \frac{\sin b}{\sin a}\bigg/\frac{\tan c}{\tan a}$$

$$= \frac{\sin b}{\sin a}\cdot\frac{\sin a}{\cos a}\cdot\frac{\cos c}{\sin c} = \frac{\sin b \cos c}{\cos b \cos c \sin c}$$

$$\therefore \quad \tan B = \frac{\tan b}{\sin c} \qquad \text{(XI.12)}$$

Similarly, $$\tan C = \frac{\tan c}{\sin b} \qquad \text{(XI.13)}$$

The formulae (XI.6) give

$$\cos b = \frac{\cos B + \cos C \cos 90^\circ}{\sin C \sin 90^\circ}$$

whence $$\cos B = \cos b \sin C \qquad \text{(XI.14)}$$

Similarly $$\cos C = \cos c \sin B \qquad \text{(XI.15)}$$

Also $$\cos a = \frac{\cos 90^\circ + \cos B \cos C}{\sin B \sin C}$$

whence $$\cos a = \cot B \cot C \qquad \text{(XI.16)}$$

The formulae to be used in any given case will depend on the data. For example, if we are given the hypotenuse a and the angle B, we find c from (XI.8), C from (XI.16), and b from (XI.10). An ambiguous case occurs when we are given one side and the angle opposite that side, say b and B. For, using the formula (XI.14) to obtain C, we have

$$\sin C = \frac{\cos B}{\cos b},$$

and since

$$\sin C = \sin(\pi - C),$$

we have two values for C and, therefore, in general, two possible triangles.

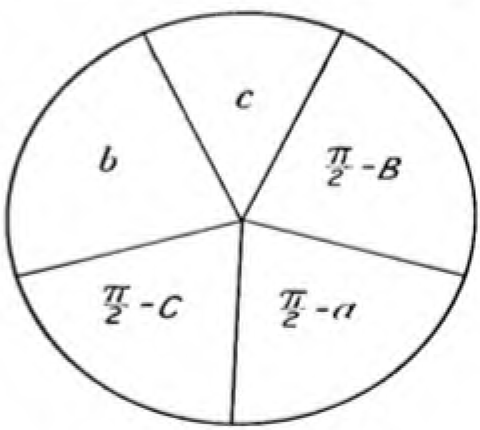


FIG. 90

123. **Napier's Rules.** Divide a circle (Fig. 90) into five sectors and write b, c, $\frac{\pi}{2} - B$, $\frac{\pi}{2} - a$, $\frac{\pi}{2} - C$ in order, one in each sector. Having selected any one sector, call that sector the *middle part*,

the sectors immediately next to it the *adjacent parts*, and the other two sectors the *opposite parts*. Then Napier's Rules state that for right-angled triangles—

(i) the sine of the middle part = the product of the tangents of the adjacent parts;

(ii) the sine of the middle part = the product of the cosines of the opposite parts.

For example, selecting $\frac{\pi}{2} - B$ as the middle part, we have, from Rule (i), $\sin\left(\frac{\pi}{2} - B\right) = \tan c \cdot \tan\left(\frac{\pi}{2} - a\right)$ or $\cos B = \frac{\tan c}{\tan a}$, which is formula (XI.8) above; and from Rule (ii), $\sin\left(\frac{\pi}{2} - B\right) = \cos b \cdot \cos\left(\frac{\pi}{2} - C\right)$ or $\cos B = \cos b \sin C$, which is formula (XI.14) above.

Any one of the formulae (XI.7) to (XI.16) can be obtained from Napier's Rules, which serve simply as a memory aid to the reader.

EXAMPLE 1

ABC is a spherical triangle in which $A = 90°$, $B = 135°$, $c = 45°$. Find a, b, C, and find also the area of the triangle, given that the radius of the sphere on which it is drawn is 4 000 miles.

From (XI.8),

$$\tan a = \frac{\tan c}{\cos B} = \frac{\tan 45°}{\cos 135°} = \frac{1}{\left(-\frac{1}{\sqrt{2}}\right)} = -\sqrt{2} = -1{\cdot}4142$$

$\therefore \quad a = 180° - 54°\,44' = 125°\,16'$

From (XI.12),

$$\tan b = \tan B \sin c = \tan 135° \sin 45° = (-1) \cdot \left(\frac{1}{\sqrt{2}}\right) = -0{\cdot}7071$$

$\therefore \quad b = 180° - 35°\,16' = 144°\,44'$

From (XI.15),

$$\cos C = \cos c \sin B = \cos 45° \sin 135° = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 0{\cdot}5$$

$\therefore \quad C = 60°$

The sum of the angles $A, B, C = 90° + 135° + 60° = 285°$

Therefore spherical excess, $E = 285° - 180° = 105° = 1{\cdot}8326$ radians

By (XI.2), area of triangle $ABC = Er^2 = 1{\cdot}8326 \times 4\,000^2$

$= 2{\cdot}932 \times 10^7$ square miles

EXAMPLE 2

Prove that in a spherical triangle ABC, right-angled at C,

$$\cos c = \cos a \cos b$$

and

$$\cos A = \tan b/\tan c$$

A ship takes a great circle course from a point in longitude 125° W., latitude 50° N., to a point on the equator at longitude 155° W. Find the length of the voyage and the directions of departure and arrival. [Radius of the earth = 3 960 miles.]

(U.L.)

The formulae in the first part of this question are proved by the methods of Art. 122, C being now a right angle in place of A. In Fig. 91, NS is the axis of the earth, O being its centre. The section of the earth's surface by any plane through the axis NS is termed a *meridian*. That meridian which passes through the Observatory at Greenwich is taken as the *prime meridian*, and the angle between the planes of this prime meridian and the meridian through any place on the earth's surface is the *longitude* of that place. The great circle whose plane is at right angles to the axis NS is termed the *equator*. The *latitude* of any place on the earth's surface is the angle subtended at the centre O by that part of the meridian of the place between the place and the equator. In the figure A is the point whose latitude is 50° N. and longitude 125° W., and B is the point on the equator whose longitude is 155° W.

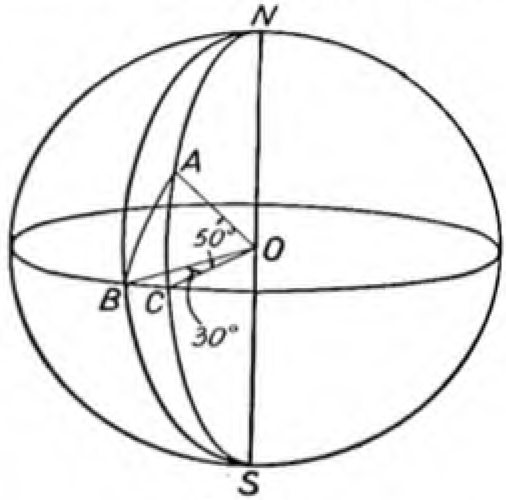


FIG. 91

Thus, $\widehat{AOC} = 50°$, and $\widehat{BOC} = 30°$, which is the difference of the longitudes of A and B. AB is the arc of the great circle along which the ship travels. Considering the spherical triangle ABC, we have $b = \widehat{AOC} = 50°$, $a = \widehat{BOC} = 30°$, $C = 90°$, since C is the angle between the planes of the meridian through A and the equator.

Using the formula $\cos c = \cos a \cos b$, we obtain

$$\cos c = \cos 30° \cos 50°$$

$\therefore \quad \log \cos c = \log \cos 30° + \log \cos 50° = \bar{1}\cdot9375 + \bar{1}\cdot8081 = \bar{1}\cdot7456$

i.e. $\quad \log \cos c = \log \cos 56°\ 10'$; whence $c = 56°\ 10'$

Again, $\quad \cos A = \tan b/\tan c = \tan 50°/\tan 56°\ 10'$

$\therefore \quad \log \cos A = \log \tan 50° - \log \tan 56°\ 10' = 0\cdot0762 - 0\cdot1737 = \bar{1}\cdot9025$

$\therefore \quad \log \cos A = \log \cos 36°\ 58'$; whence $A = 36°\ 58'$

Also $\quad \cos B = \tan a/\tan c = \tan 30°/\tan 56°\ 10'$

$\therefore \quad \log \cos B = \log \tan 30° - \log \tan 56°\ 10' = \bar{1}\cdot7614 - 0\cdot1737 = \bar{1}\cdot5877$

$\therefore \quad \log \cos B = \log \cos 67°\ 14'$; whence $B = 67°\ 14'$

The ship leaves A in the direction 36° 58′ W. of S., and arrives at B from the direction 22° 46′ (i.e. 90° − 67° 14′) E. of N.

The length of the voyage = length of arc AB

= 3 960 × number of radians in 56° 10′

i.e. length of voyage = 3 960 × 0·9803 = 3 882 miles

EXAMPLE 3

Show that the area of a spherical triangle is equal to the excess of the sum of its angles over two right angles multiplied by the square of the radius of the sphere.

A square on the surface of a sphere of radius R has each of its sides of length l and all its angles equal. By dividing the square into four right-angled spherical triangles by its diagonals, or otherwise, show that its area is equal to

$$4R^2 \sin^{-1}\left\{\tan^2 \frac{l}{2R}\right\} \qquad \text{(U.L.)}$$

For the first part, see Art. 119.

Let the diagonals of the square intersect at A and let ABC be one of the four right-angled triangles into which the square is divided. Then in triangle ABC, $B = C$, $A = \frac{\pi}{2}$, $a = \frac{l}{R}$

The formula (XI.16) gives $\cos a = \cot B \cot C$, whence $\cos \frac{l}{R} = \cot^2 B$

$$\therefore \quad 2\cos^2 \frac{l}{2R} - 1 = \operatorname{cosec}^2 B - 1, \text{ or } 2\sin^2 B = \sec^2 \frac{l}{2R}$$

$$\therefore \quad 1 - \cos 2B = 1 + \tan^2 \frac{l}{2R}$$

$$\therefore \quad -\cos 2B = \tan^2 \frac{l}{2R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now the area of the triangle ABC

$$= (A + B + C - \pi)R^2 = \left(\frac{\pi}{2} + 2B - \pi\right) R^2$$

i.e. area of triangle $ABC = \left(2B - \frac{\pi}{2}\right) R^2$

Also $\sin\left(2B - \frac{\pi}{2}\right) = -\cos 2B$, so that from (1), $\sin\left(2B - \frac{\pi}{2}\right) = \tan^2 \frac{l}{2R}$

$$\therefore \quad 2B - \frac{\pi}{2} = \sin^{-1}\left\{\tan^2 \frac{l}{2R}\right\}$$

$\therefore$ Area of triangle $ABC = R^2 \sin^{-1}\left\{\tan^2 \frac{l}{2R}\right\}$; and, hence, the area of the square $= 4R^2 \sin^{-1}\left\{\tan^2 \frac{l}{2R}\right\}$

124. Napier's Analogies. From (XI.6) we have

$$\cos A = \cos a \sin B \sin C - \cos B \cos C \quad . \quad \text{(XI.17)}$$

and

$$\cos B = \cos b \sin C \sin A - \cos C \cos A \quad . \quad \text{(XI.18)}$$

Adding (XI.17) and (XI.18)

$$\cos A + \cos B = \sin C (\sin A \cos b + \cos a \sin B) - \cos C (\cos A + \cos B)$$

or

$$\frac{(\cos A + \cos B)(1 + \cos C)}{\sin C} = \sin A \cos b + \cos a \sin B$$

$$= \sin A \cos b + \cos a \frac{\sin b}{\sin a} \sin A$$

$$= \frac{\sin A}{\sin a} (\sin a \cos b + \cos a \sin b)$$

i.e.

$$\frac{(\cos A + \cos B)(1 + \cos C)}{\sin C} = \frac{\sin A}{\sin a} \sin (a + b) \quad \text{(XI.19)}$$

Again

$$\sin A + \sin B = \sin A + \frac{\sin b}{\sin a} \sin A = \frac{\sin A}{\sin a} (\sin a + \sin b) \quad \text{(XI.20)}$$

Dividing corresponding sides of the last two relations, we obtain

$$\frac{\sin A + \sin B}{\cos A + \cos B} \cdot \frac{\sin C}{1 + \cos C} = \frac{\sin a + \sin b}{\sin (a + b)}$$

i.e.

$$\frac{2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}}{2 \cos \dfrac{A+B}{2} \cos \dfrac{A-B}{2}} \cdot \frac{2 \sin \dfrac{C}{2} \cos \dfrac{C}{2}}{2 \cos^2 \dfrac{C}{2}} = \frac{2 \sin \dfrac{a+b}{2} \cos \dfrac{a-b}{2}}{2 \sin \dfrac{a+b}{2} \cos \dfrac{a+b}{2}}$$

i.e.

$$\tan \frac{A+B}{2} \tan \frac{C}{2} = \frac{\cos \dfrac{a-b}{2}}{\cos \dfrac{a+b}{2}}$$

or

$$\tan \frac{A+B}{2} = \frac{\cos \dfrac{a-b}{2}}{\cos \dfrac{a+b}{2}} \cot \frac{C}{2} \quad . \quad \text{(XI.21)}$$

We prove similarly

$$\tan\frac{A-B}{2}=\frac{\sin\dfrac{a-b}{2}}{\sin\dfrac{a+b}{2}}\cot\frac{C}{2} \quad . \quad . \quad \text{(XI.22)}$$

Substituting $\pi-a$, $\pi-b$, $\pi-A$, $\pi-B$, $\pi-c$ for A, B, a, b, C respectively in (XI.21), we obtain (See Art. 121)

$$\tan\frac{(\pi-a)+(\pi-b)}{2}=\frac{\cos\dfrac{(\pi-A)-(\pi-B)}{2}}{\cos\dfrac{(\pi-A)+(\pi-B)}{2}}\cot\frac{\pi-c}{2}$$

i.e.

$$\tan\left(\pi-\frac{a+b}{2}\right)=\frac{\cos\dfrac{B-A}{2}}{\cos\left(\pi-\dfrac{A+B}{2}\right)}\cot\left(\frac{\pi}{2}-\frac{c}{2}\right)$$

i.e.

$$-\tan\frac{a+b}{2}=\frac{\cos\dfrac{A-B}{2}}{-\cos\dfrac{A+B}{2}}\tan\frac{c}{2}$$

or

$$\tan\frac{a+b}{2}=\frac{\cos\dfrac{A-B}{2}}{\cos\dfrac{A+B}{2}}\tan\frac{c}{2}\,. \quad . \quad . \quad \text{(XI.23)}$$

With the same substitution in (XI.22) we obtain

$$\tan\frac{a-b}{2}=\frac{\sin\dfrac{A-B}{2}}{\sin\dfrac{A+B}{2}}\tan\frac{c}{2} \quad . \quad . \quad \text{(XI.24)}$$

The formulae (XI.21) to (XI.24) are known as *Napier's Analogies.*

125. **Formulae Involving Half-angles and Half-sides.** From (XI.3) we have

$$\cos A=\frac{\cos a-\cos b\cos c}{\sin b\sin c}$$

$$\therefore \qquad 1 + \cos A = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\sin b \sin c + \cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos a - \cos (b + c)}{\sin b \sin c}$$

i.e.
$$2 \cos^2 \frac{A}{2} = \frac{-2 \sin \dfrac{a+b+c}{2} \sin \dfrac{a-b-c}{2}}{\sin b \sin c}$$

$$\therefore \qquad \cos \frac{A}{2} = \sqrt{\frac{\sin \dfrac{a+b+c}{2} \sin \dfrac{b+c-a}{2}}{\sin b \sin c}}$$

Let $a + b + c = 2s$; then $b + c - a = 2s - 2a = 2(s - a)$

$$\therefore \qquad \cos \frac{A}{2} = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}} \quad . \quad . \quad . \text{(XI.25)}$$

Again, from the relation $1 - \cos A = 1 - \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$ we deduce similarly

$$\sin \frac{A}{2} = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}} \quad . \quad . \quad \text{(XI.26)}$$

Dividing (XI.26) by (XI.25), we obtain

$$\tan \frac{A}{2} = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}} \quad . \quad . \quad \text{(XI.27)}$$

Substitute $\pi - a$, $\pi - A$, $\pi - B$, $\pi - C$ for A, a, b, c respectively in (XI.25). Then, since

$$2s = a + b + c = (\pi - A) + (\pi - B) + (\pi - C)$$
$$= 3\pi - (A + B + C) = 3\pi - 2S$$

where $2S = A + B + C$, we obtain

$$\cos \frac{\pi - a}{2} = \sqrt{\frac{\sin \left(\dfrac{3\pi}{2} - S\right) \sin \left(\dfrac{\pi}{2} - \overline{S - A}\right)}{\sin (\pi - B) \sin (\pi - C)}}$$

i.e.
$$\sin \frac{a}{2} = \sqrt{\frac{-\cos S \cos (S - A)}{\sin B \sin C}} \quad . \quad . \quad . \quad \text{(XI.28)}$$

With the same substitution in (XI.26) we obtain

$$\cos\frac{a}{2} = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}} \quad . \qquad . \quad \text{(XI.29)}$$

Dividing (XI.28) by (XI.29), we have

$$\tan\frac{a}{2} = \sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}} \quad . \qquad . \quad \text{(XI.30)}$$

We have corresponding formulae for the sines, cosines, and tangents of $\frac{B}{2}$, $\frac{C}{2}$, $\frac{b}{2}$, $\frac{c}{2}$.

126. **Delambre's Analogies.** Since $\cos\frac{A+B}{2} = \cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}$ and, using the formulae of the last article, we have

$$\begin{aligned}
\cos\frac{A+B}{2} &= \sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}\sqrt{\frac{\sin s\sin(s-b)}{\sin c\sin a}} \\
&\quad - \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c\sin a}} \\
&= \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}\left\{\frac{\sin s}{\sin c} - \frac{\sin(s-c)}{\sin c}\right\} \\
&= \sin\frac{C}{2}\left\{\frac{\sin s - \sin(s-c)}{\sin c}\right\} \\
&= \sin\frac{C}{2}\left\{\frac{2\cos\left(s-\frac{c}{2}\right)\sin\frac{c}{2}}{2\sin\frac{c}{2}\cos\frac{c}{2}}\right\}
\end{aligned}$$

i.e.

$$\cos\frac{A+B}{2} = \sin\frac{C}{2} \times \frac{\cos\frac{a+b}{2}}{\cos\frac{c}{2}}$$

or

$$\cos\frac{A+B}{2}\cos\frac{c}{2} = \cos\frac{a+b}{2}\sin\frac{C}{2} \quad . \qquad . \quad \text{(XI.31)}$$

We prove similarly

$$\cos \frac{A-B}{2} \sin \frac{c}{2} = \sin \frac{a+b}{2} \sin \frac{C}{2} \quad . \quad . \quad (\text{XI.32})$$

$$\sin \frac{A+B}{2} \cos \frac{c}{2} = \cos \frac{a-b}{2} \cos \frac{C}{2} \quad . \quad . \quad (\text{XI.33})$$

$$\sin \frac{A-B}{2} \sin \frac{c}{2} = \sin \frac{a-b}{2} \cos \frac{C}{2} \quad . \quad . \quad (\text{XI.34})$$

The formulae (XI.31) to (XI.34) are known as *Delambre's Analogies*.

127. Solution of the General Spherical Triangle. A spherical triangle has six parts, three sides and three angles, and three of these six parts are in general sufficient to determine the triangle. There are six cases to consider.

(i) LET THE THREE ANGLES A, B, C BE GIVEN. We can find a from the formula

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$

but the formula

$$\tan \frac{a}{2} = \sqrt{\frac{-\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}}$$

is better adapted for logarithmic calculation. To find b and c we use the corresponding formulae for $\tan \frac{b}{2}$ and $\tan \frac{c}{2}$

(ii) LET TWO ANGLES AND THE ADJACENT SIDE BE GIVEN, SAY A, B, c, to find a and b we can use the cotangent formulae

$$\cot a \sin c = \cos c \cos B + \sin B \cot A$$

and

$$\cot b \sin c = \cos c \cos A + \sin A \cot B$$

or the formulae

$$\tan \frac{a+b}{2} = \frac{\cos \dfrac{A-B}{2}}{\cos \dfrac{A+B}{2}} \tan \frac{c}{2}$$

and
$$\tan\frac{a-b}{2}=\frac{\sin\dfrac{A-B}{2}}{\sin\dfrac{A+B}{2}}\tan\frac{c}{2}$$

the latter being preferable. To find C we use the formula

$$\cos\frac{A+B}{2}\cos\frac{c}{2}=\cos\frac{a+b}{2}\sin\frac{C}{2}$$

We can also use the sine formula to find C, but the reader should avoid this formula wherever possible. (See cases (iii) and (v).)

(iii) Let Two Angles and a Side Opposite One of Them be Given, say A, B, b. To find a, we use the formula

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}$$

Now $\sin a=\sin(\pi-a)$, so that there are two values of a and, in general, two possible triangles. Having found a we use the formula

$$\tan\frac{A+B}{2}=\frac{\cos\dfrac{a-b}{2}}{\cos\dfrac{a+b}{2}}\cot\frac{C}{2}$$

and
$$\tan\frac{a+b}{2}=\frac{\cos\dfrac{A-B}{2}}{\cos\dfrac{A+B}{2}}\tan\frac{c}{2}$$

to find C and c.

(iv) Let Two Sides and the Included Angle be Given, say a, b, C. To find c we can use the cosine formula

$$\cos c=\cos a\cos b+\sin a\sin b\cos C$$

and then find A and B from the formulae for $\tan\dfrac{A+B}{2}$ and $\tan\dfrac{A-B}{2}$. Otherwise, having first found A and B from these formulae, we can find c from the formula

$$\cos\frac{A+B}{2}\cos\frac{c}{2}=\cos\frac{a+b}{2}\sin\frac{C}{2}$$

(v) Let Two Sides and the Angle Opposite One of Them be Given, say a, b, B. We use

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b}$$

to find A, and, as in case (iii), there will in general be two possible solutions. Having found A, we find C and c from the formulae

$$\tan\frac{A+B}{2} = \frac{\cos\dfrac{a-b}{2}}{\cos\dfrac{a+b}{2}}\cot\frac{C}{2}$$

and
$$\tan\frac{a+b}{2} = \frac{\cos\dfrac{A-B}{2}}{\cos\dfrac{A+B}{2}}\tan\frac{c}{2} \text{ respectively}$$

(vi) Let the Three Sides a, b, c be Given. To find A, B, C we can use the cosine formulae or the formulae for $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$

EXAMPLE 1

In a spherical triangle ABC, $a = 68° 24'$, $b = 45° 14'$, $c = 56° 38'$; find the angles A, B, C.

We have from (XI.27), $\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}$

Here $s = \frac{68°\ 24' + 45°\ 14' + 56°\ 38'}{2} = 85°\ 8'$, so that $s-a = 16°\ 44'$, $s-b = 39°\ 54'$, $s-c = 28°\ 30'$.

$$\therefore \log\tan\frac{A}{2} = \tfrac{1}{2}[\log\sin 39° 54' + \log\sin 28° 30' - \log\sin 85° 8' - \log\sin 16° 44']$$
$$= \tfrac{1}{2}[\bar{1}\cdot8072 + \bar{1}\cdot6787 - \bar{1}\cdot9984 - \bar{1}\cdot4593]$$
$$= \tfrac{1}{2}[0\cdot0282] = 0\cdot0141$$

$$\therefore \qquad \frac{A}{2} = 45° 56' \text{ and } A = 91° 52'$$

Also $\qquad \tan\frac{B}{2} = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)}}$

$$\therefore \log\tan\frac{B}{2} = \tfrac{1}{2}[\bar{1}\cdot6787 + \bar{1}\cdot4593 - \bar{1}\cdot9984 - \bar{1}\cdot8072]$$
$$= \tfrac{1}{2}[\bar{1}\cdot3324] = \bar{1}\cdot6662$$

$$\therefore \qquad \frac{B}{2} = 24° 53' \text{ and } B = 49° 46'$$

Again $\tan \dfrac{C}{2} = \sqrt{\dfrac{\sin (s-a) \sin (s-b)}{\sin s \sin (s-c)}}$

$$\therefore \log \tan \frac{C}{2} = \tfrac{1}{2}[\bar{1}\cdot 4593 + \bar{1}\cdot 8072 - \bar{1}\cdot 9984 - \bar{1}\cdot 6787]$$

$$= \tfrac{1}{2}[\bar{1}\cdot 5894] = \bar{1}\cdot 7947$$

$$\therefore \quad \frac{C}{2} = 31° \, 56' \text{ and } C = 63° \, 52'$$

EXAMPLE 2

In a spherical triangle ABC, $a = 79° \, 26'$, $b = 60° \, 20'$, $C = 32° \, 44'$; find c, A, B.

We have $\tan \dfrac{A+B}{2} = \dfrac{\cos \dfrac{a-b}{2}}{\cos \dfrac{a+b}{2}} \cot \dfrac{C}{2} = \dfrac{\cos \dfrac{19° \, 6'}{2}}{\cos \dfrac{139° \, 46'}{2}} \cot 16° \, 22'$

$$\therefore \quad \log \tan \frac{A+B}{2} = \log \cos 9° \, 33' + \log \cot 16° \, 22' - \log \cos 69° \, 53'$$

$$= \bar{1}\cdot 9939 + 0\cdot 5321 - \bar{1}\cdot 5365 = 0\cdot 9895$$

$$\therefore \quad \frac{A+B}{2} = 84° \, 9' \quad \ldots \ldots \quad (1)$$

Again, $\tan \dfrac{A-B}{2} = \dfrac{\sin \dfrac{a-b}{2}}{\sin \dfrac{a+b}{2}} \cot \dfrac{C}{2} = \dfrac{\sin 9° \, 33'}{\sin 69° \, 53'} \times \cot 16° \, 22'$

$$\therefore \quad \log \tan \frac{A-B}{2} = \bar{1}\cdot 2199 + 0\cdot 5321 - \bar{1}\cdot 9727 = \bar{1}\cdot 7793$$

$$\therefore \quad \frac{A-B}{2} = 31° \, 2'. \quad \ldots \ldots \quad (2)$$

Adding and subtracting (1) and (2) in succession, we obtain

$$A = 115° \, 11' \text{ and } B = 53° \, 7'$$

Using the formula $\cos \dfrac{A+B}{2} \cos \dfrac{c}{2} = \cos \dfrac{a+b}{2} \sin \dfrac{C}{2}$, we have

$$\cos 84° \, 9' \cos \frac{c}{2} = \cos 69° \, 53' \sin 16° \, 22'$$

$$\therefore \quad \log \cos \frac{c}{2} = \bar{1}\cdot 5365 + \bar{1}\cdot 4499 - \bar{1}\cdot 0083 = \bar{1}\cdot 9781$$

$$\therefore \quad \frac{c}{2} = 18° \, 2' \text{ and } c = 36° \, 4'$$

EXAMPLE 3

Prove that in any spherical triangle $\cos a = \cos b \cos c + \sin b \sin c \cos A$.

A ship is proceeding uniformly along a great circle. At a given moment the latitude is observed to be l_1; after the ship has travelled distances s and $2s$ the latitudes are observed to be l_2 and l_3 respectively. Show that

$$\cos s = \sin \tfrac{1}{2}(l_1 + l_3) \cos \tfrac{1}{2}(l_1 - l_3)/\sin l_2$$

Also express the total change of longitude in terms of the three latitudes.

(U.L.)

For the proof of the cosine formula see Art. 120.

In Fig. 92, P, Q, R are the three points on the ship's course at which the latitudes are l_1, l_2, l_3 respectively. The meridians NP, NQ, NR are drawn. Let

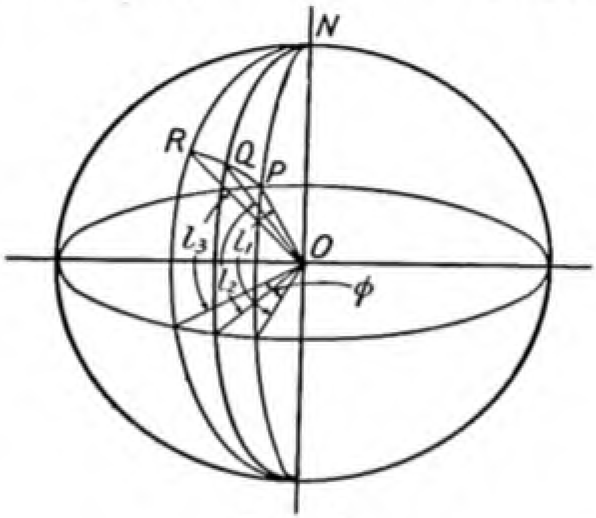


FIG. 92

the plane of the meridian NQ make angle α with the plane of the great circle PQR. Hence, for the triangle NPQ we have

$$\begin{aligned}\cos (\text{side } NP) &= \cos (\text{side } NQ) \cos (\text{side } PQ)\\ &\quad + \sin (\text{side } NQ) \sin (\text{side } PQ) \cos \alpha\end{aligned}$$

i.e. $$\cos\left(\frac{\pi}{2} - l_1\right) = \cos\left(\frac{\pi}{2} - l_2\right) \cos s + \sin\left(\frac{\pi}{2} - l_2\right) \sin s \cos \alpha$$

$$\therefore \qquad \cos \alpha = \frac{\sin l_1 - \sin l_2 \cos s}{\cos l_2 \sin s} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For the triangle NQR we have

$$\begin{aligned}\cos (\text{side } NR) &= \cos (\text{side } NQ) \cos (\text{side } QR)\\ &\quad + \sin (\text{side } NQ) \sin (\text{side } QR) \cos (\pi - \alpha)\end{aligned}$$

i.e. $$\cos\left(\frac{\pi}{2} - l_3\right) = \cos\left(\frac{\pi}{2} - l_2\right) \cos s - \sin\left(\frac{\pi}{2} - l_2\right) \sin s \cos \alpha$$

$$\therefore \qquad \cos \alpha = \frac{\sin l_2 \cos s - \sin l_3}{\cos l_2 \sin s} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equating the two values of $\cos\alpha$ in (1) and (2), we obtain

$$\frac{\sin l_1 - \sin l_2 \cos s}{\cos l_2 \sin s} = \frac{\sin l_2 \cos s - \sin l_3}{\cos l_2 \sin s}$$

$$\therefore \qquad \sin l_1 + \sin l_3 = 2\cos s \sin l_2 \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\cos s = \frac{2\sin\frac{l_1+l_3}{2}\cos\frac{l_1-l_3}{2}}{2\sin l_2} = \frac{\sin\frac{l_1+l_3}{2}\cos\frac{l_1-l_3}{2}}{\sin l_2}$$

The total change of longitude is equal to the angle between the planes of the meridians of P and R. Call this angle ϕ. Then, for the triangle NPR, we have from (XI.3),

$$\cos\phi = \frac{\cos(\text{side } PR) - \cos(\text{side } NP)\cos(\text{side } NR)}{\sin(\text{side } NP)\sin(\text{side } NR)}$$

i.e.

$$\cos\phi = \frac{\cos 2s - \sin l_1 \sin l_3}{\cos l_1 \cos l_3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now, $\quad \cos 2s = 2\cos^2 s - 1$; hence, from (1),

$$\cos 2s = 2\left[\frac{\sin l_1 + \sin l_3}{2\sin l_2}\right]^2 - 1 = \frac{\sin^2 l_1 + \sin^2 l_3 + 2\sin l_1 \sin l_3 - 2\sin^2 l_2}{2\sin^2 l_2}$$

Substituting this value of $\cos 2s$ in (2), we obtain ultimately

$$\cos\phi = \frac{\sin^2 l_1 + \sin^2 l_3 + 2\cos^2 l_2(\sin l_1 \sin l_3 + 1) - 2}{2\cos l_1 \sin^2 l_2 \cos l_3}$$

or

$$\cos\phi = \frac{2\cos^2 l_2(\sin l_1 \sin l_3 + 1) - \cos^2 l_1 - \cos^2 l_3}{2\cos l_1 \sin^2 l_2 \cos l_3}$$

This equation gives ϕ, the total change of longitude.

EXAMPLE 4

Two ports are in the same latitude l, their difference of longitude being 2λ. Show that the distance saved in sailing from one port to the other along a great circle, instead of due east or west, is $2r\{\lambda\cos l - \sin^{-1}(\sin\lambda\cos l)\}$, where r is the radius of the earth.

Calculate the distance thus saved if the latitude is 60° and the difference of longitude 90°, taking the radius of the earth as 3 960 miles. (U.L.)

Let O be the earth's centre and A and B the positions of the two ports (Fig. 93). We consider the triangle ABC formed by the meridians CA, CB of A, B respectively, and the arc AB of the great circle through A and B. In this triangle $a = b = \frac{\pi}{2} - l$, $C = 2\lambda$. Using the cosine formula, we have

$$\cos c = \cos\left(\frac{\pi}{2} - l\right)\cos\left(\frac{\pi}{2} - l\right) + \sin\left(\frac{\pi}{2} - l\right)\sin\left(\frac{\pi}{2} - l\right)\cos 2\lambda$$

i.e.
$$\cos c = \sin^2 l + \cos^2 l \cos 2\lambda$$

$$\therefore \quad 1 - 2\sin^2 \frac{c}{2} = 1 - \cos^2 l + \cos^2 l(1 - 2\sin^2 \lambda)$$

$$\therefore \quad \sin^2 \lambda \cos^2 l = \sin^2 \frac{c}{2}$$

or
$$c = 2\sin^{-1}(\sin \lambda \cos l)$$

Therefore length of arc AB of the great circle $= rc = 2r \sin^{-1}(\sin \lambda \cos l)$.

Let the small circle through A and B whose plane is perpendicular to OC cut OC at D. Then angle $ADB = 2\lambda$ and $AD = r\cos l$. Hence, length of

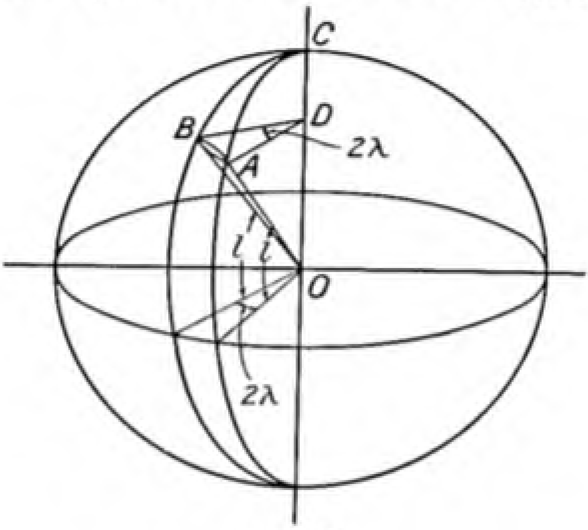


FIG. 93

arc AB of the small circle $= AD \times 2\lambda = 2r\lambda \cos l$, which is the distance the ship would have to cover from A to B if its direction remained due west.

Therefore distance saved $= 2r\lambda \cos l - 2r \sin^{-1}(\sin \lambda \cos l)$

$$= 2r\{\lambda \cos l - \sin^{-1}(\sin \lambda \cos l)\}$$

If $l = 60°$, $2\lambda = 90°$, or $\lambda = 45°$, $r = 3\,960$ miles, then

$$\text{the distance saved} = 2 \times 3\,960 \left\{\frac{\pi}{4}\cos 60° - \sin^{-1}(\sin 45° \cos 60°)\right\}$$
$$= 7\,920 \left\{\frac{\pi}{8} - \sin^{-1} 0{\cdot}3535\right\}$$
$$= 7\,920\,(0{\cdot}3927 - 0{\cdot}3613)$$
$$= 248{\cdot}7 \text{ miles}$$

EXAMPLES XI

(1) Prove that the area of a spherical triangle ABC is equal to Er^2, where E is the spherical excess of the triangle and r is the radius of the sphere on which the triangle is drawn. Deduce the formula which gives the area of a spherical polygon.

(2) Prove that in any spherical triangle ABC, $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$. Show that, if the triangle is equilateral, it is also equiangular.

(3) State Napier's Rules, and deduce from them that in a spherical triangle ABC in which $A = 90°$, $\cos a = \cos b \cos c$ and $\cot B = \cot b \sin c$.

Solve the triangle in which $A = 90°$, $b = 48° 12'$, $c = 53° 25'$.

Find the remaining parts of the triangle ABC in the following cases—

(4) $A = 90°$, $B = 84° 14'$, $C = 39° 21'$.

(5) $C = 90°$, $c = 67° 33'$, $B = 44° 44'$.

(6) $B = 90°$, $A = 73° 54'$, $a = 34° 25'$.

(7) $A = 90°$, $a = 85°$, $b = 70°$.

(8) $C = 90°$, $B = 46° 2'$, $a = 60° 47'$.

(9) Prove that in any spherical triangle ABC, $\sin A = \frac{2n}{\sin b \sin c}$, where $2n = \sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}$, s being equal to $\frac{a + b + c}{2}$

(10) Prove that, in any spherical triangle,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

Find the remaining parts of the triangle in which $A = 120°$, $b = 30°$, $c = 90°$, and find the ratio of its area to that of the sphere on which it is drawn. (U.L.)

(11) Prove the formula $(A + B + C - \pi)r^2$ for the area of a spherical triangle drawn on a sphere of radius r.

The sides of a spherical triangle are 90°, 70°, and 70°; show that its area is to that of the sphere on which it lies in the ratio 0·076 to 1, approximately. (U.L.)

(12) Prove that $\cos c = \cos a \cos b + \sin a \sin b \cos C$, and deduce that $\sin c \cos A = \cos a \sin b - \sin a \cos b \cos C$.

In a spherical triangle ABC, the side $a = 76° 35' 36''$, the side $b = 50° 10' 30''$, and the angle $C = 34° 15' 3''$; find the other side and the other angles. (U.L.)

(13) Prove that in any spherical triangle in which $A = 90°$, $\tan B \sin c = \tan b$.

From A, a point on the earth's surface in latitude 51°, a distance AB of length 20 000 feet is measured along the great circle perpendicular to the meridian through A. Calculate the angle which a great circle through B perpendicular to BA makes with the meridian through B. (Consider the earth as a sphere of radius 3 960 miles.) (U.L.)

(14) If θ_1 and θ_2 denote the north latitudes of two places on the earth's surface and δ the difference of their longitudes, show that the angle between the earth's radii through the two places is ϕ given by the equation

$$\cos \phi = \cos \theta_1 \cos \theta_2 \cos \delta + \sin \theta_1 \sin \theta_2$$

Use this to find the (solar) time of sunset at a place in latitude 52° north when the sun is $23\frac{1}{2}°$ south of the equator. (Assume that, at sunset, the line joining the earth's centre to the sun is perpendicular to the radius through the place.) (U.L.)

(15) Prove the formula of spherical trigonometry

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

The shortest distance between two points on the earth's surface is one-tenth of the earth's circumference at the equator, and their latitudes are 40° and 50°. Find the difference of their longitudes. (U.L.)

Find the remaining parts of the spherical triangle ABC in the following cases—

(16) $A = 80° 20'$, $B = 63° 46'$, $C = 55° 11'$.

(17) $B = 32° 11'$, $C = 87° 32'$, $a = 15° 23'$.

(18) $A = 46°$, $B = 64°$, $b = 90°$.

(19) $a = 53° 17'$, $b = 40° 51'$, $c = 37° 16'$.

(20) $C = 81° 30'$, $a = 50° 9'$, $b = 55° 19'$.

(21) $a = 70°$, $b = 60°$, $c = 90°$.

(22) $A = 65° 56'$, $b = 71° 29'$, $c = 84° 42'$.

(23) $A = 39° 8'$, $B = 69° 18'$, $C = 88° 50'$.

(24) $b = c = 59° 45'$, $C = 79° 27'$.

(25) Two points P and Q on the earth's surface are in north latitudes l_1 and l_2 and east longitudes λ_1 and λ_2 respectively. Assuming the earth to be a sphere of radius R, find the length of the shortest path which joins P to Q and lies on the earth's surface.

Use your result to find the shortest course a ship can take in travelling from latitude 34° 10′ N., longitude 12° 36′ E. to latitude 33° 26′ N., longitude 20° 14′ E. (Radius of earth = 3 960 miles.)

(26) A, B, and C are three places on the earth's surface; the shortest distances between the places are as follows: A to B, 4 146 miles; B to C, 2 073 miles; C to A, 3 110 miles.

Find the area of the spherical triangle ABC, assuming the earth to be a sphere of radius 3 960 miles.

(27) OA and OB are two radii of a sphere. The ends A and B move along great circles of the sphere, the angle between the planes of which is $\phi(\phi < 90°)$. In the initial position OA lies along the line of intersection of the planes. If the angle AOB is constant and equal to 90°, show that when OA has turned through an acute angle θ, the angle turned through by OB is $\tan^{-1}(\tan\theta\cos\phi)$.

For the application of this to the theory of the mechanism known as "Hooke's Joint," see *Theory of Machines* (Pitman).

(28) In a spherical triangle XYZ the angle XZY is a right angle.

Prove that
$$\cos YXZ = \cot XY \tan XZ$$

From the vertex A of the spherical triangle ABC a great circle arc is drawn to meet the side BC at right angles at the point P.

Prove that
$$\frac{\cos BAP}{\cos CAP} = \frac{\cot BA}{\cot CA}$$

Find an expression for $\sin AP$ in terms of the sides of triangle ABC. (U.L.)

(29) The most southerly latitude reached by the great circle joining a place P on the equator to a place Q in north latitude λ is ϕ. Prove that the difference of longitude between P and Q is $\sin^{-1}(\tan\lambda\cot\phi)$, and find the angle between the meridian through Q and the great circle PQ. (U.L.)

(30) A ship sails along a great circle. At noon on three successive days the latitudes are observed to be λ_1, λ_2, and λ_3, while the distance sailed each day is (in angular measure) δ.

Show that
$$\cos\delta = \sin\tfrac{1}{2}(\lambda_1 + \lambda_3)\cos\tfrac{1}{2}(\lambda_1 - \lambda_3)/\sin\lambda_2$$
and express the total difference of longitude in terms of the three given latitudes. (U.L.)

CHAPTER XII

NOMOGRAMS

128. In a nomogram the values of the variables in a formula are represented by graduations on axes, which may be straight or curved, and *any* index line cuts these axes in points the readings at which satisfy the given formula. In other words, corresponding values of the variables give collinear readings in the nomogram. For this reason a nomogram is often called a *line chart* or *alignment chart*. A nomogram, therefore, affords a means of finding very quickly and easily a result which may take minutes to obtain by calculation or by the use of a slide rule. Once a nomogram has been constructed, it can be used quite as readily whether the formula which it represents is complicated or otherwise. The degree of accuracy obtainable from a nomogram will depend on the overall size of the diagram, the disposition of the axes, and the degree of closeness of the graduations, but in any case it should compare very favourably with that obtained from a slide rule.

A nomogram can be constructed to represent a formula containing two or three or more variables, the actual form which it takes depending on the type of that formula.

In this chapter systematic methods of construction are established for nomograms which represent formulae of types likely to be encountered in scientific and, in particular, in engineering practice. The treatment is designed to generalize and unite the subject in such a way as to avoid the necessity of treating any formula *ab initio*. In any given case it is necessary merely to determine the particular type which the formula under consideration follows or can be made to follow by transformation, logarithmic or otherwise, and then to use the method shown for this type of formula.

By means of typical examples it is shown how the methods used in connection with three-variable formulae can be applied, frequently in combination, to the construction of nomograms representing formulae which contain more than three variables. It will be noted that the methods employed allow for a proper planning of the figure.

In actual practice the degree of closeness of the graduations would be greater than that shown in the diagrams drawn to illustrate the various methods of construction.

129. **Two-variable Formula** $X = Y$. Any formula of the type

$$X = Y \qquad \text{(XII.1)}$$

where X and Y are functions of two variables x and y respectively, can be represented by a single straight line graduated on both sides. If X_0, Y_0 and X_m, Y_m are the values of X, Y respectively corresponding to the extreme values of x, y, and L is the actual length of the nomogram between the graduations at these extreme values, then the scale-factors λ_1 on the x side and λ_2 on the y side are given by

$$\lambda_1(X_m - X_0) = L \text{ and } \lambda_2(Y_m - Y_0) = L$$

Now $X_m = Y_m$ and $X_0 = Y_0$; hence $\lambda_1 = \lambda_2$, i.e. the same scale-factor is used for both sets of graduations.

EXAMPLE

Construct nomograms to represent the following formulae, the overall length of each nomogram to be 9 in. approximately—

(*a*) $T = 2\pi\sqrt{\frac{l}{g}}$, where T (sec) is the time of swing of a simple pendulum of length l (ft), the length to vary from 1 ft to 4 ft.

(*b*) $pv^{1\cdot41} = 437$, where p (lb/in.^{2}) is the pressure of a mass of gas when its volume is v (in.^{3}), the volume to vary from 5 in.^{3} to 10 in.^{3}

(*a*) It is convenient to write the formula as

$$T^2 = \frac{4\pi^2 l}{g}$$

i.e.

$$T^2 = 1\cdot226l$$

The scale-factor λ is then given by

$$\lambda(1\cdot226 \times 4 - 1\cdot226 \times 1) = 9$$

i.e.

$$\lambda = \frac{9}{3\cdot678} = 2\cdot5, \text{ say}$$

If T_0 and T_m are the extreme values of T, then

$$T_0 = \sqrt{1\cdot226 \times 1} = 1\cdot107 \text{ and } T_m = \sqrt{1\cdot226 \times 4} = 2\cdot214$$

Distance (in inches) from base-line to any graduation l

$$= 2\cdot5(1\cdot226l - 1\cdot226 \times 1)$$
$$= 3\cdot065(l - 1)$$

Distance (in inches) from base-line to any graduation T

$$= 2\cdot5(T^2 - 1\cdot107^2)$$
$$= 2\cdot5(T^2 - 1\cdot226)$$

The straight line can now be graduated by projection from a uniform scale and a scale of squares, or by the use of a table of values built up from the above distance relations.

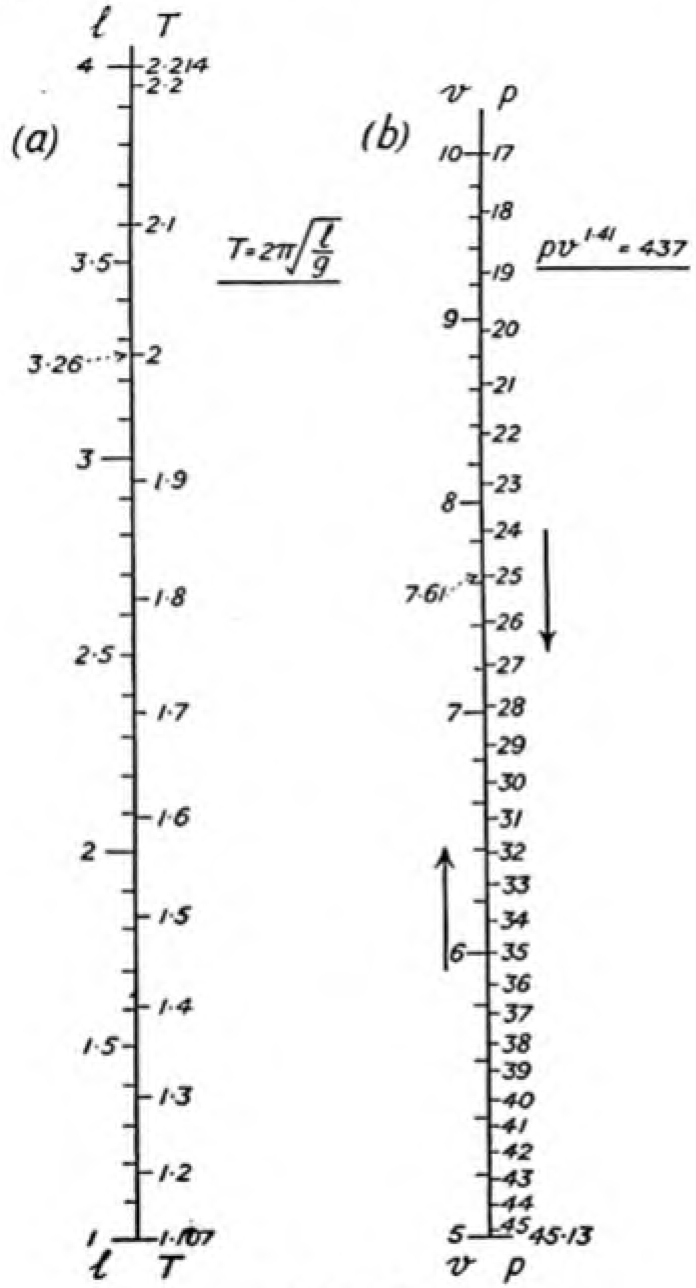


FIG. 94

From the nomogram, which is shown in Fig. 94 (*a*), it is seen that a pendulum of length 3·26 ft has time of swing 2 sec.

(*b*) In this case the formula is put into the form (XII.1) by logarithmic transformation; thus

$$1{\cdot}41 \log v = \log 437 - \log p$$

The scale-factor λ is given by

$$\lambda(1{\cdot}41 \log 10 - 1{\cdot}41 \log 5) = 9$$

i.e.
$$\lambda = \frac{9}{1{\cdot}41 \log 2} = 20, \text{ say}$$

When $v = 5$, $p = \dfrac{437}{5^{1\cdot41}} = 45{\cdot}17$, and when $v = 10$, $p = \dfrac{437}{10^{1\cdot41}} = 17{\cdot}00$

Distance (in inches) from base-line to any graduation v

$$= 20(1{\cdot}41 \log v - 1{\cdot}41 \log 5)$$

$$= 28{\cdot}2 \log \frac{v}{5}$$

Distance (in inches) from base-line to any graduation p

$$= 20[(\log 437 - \log p) - (\log 437 - \log 45{\cdot}17)]$$

$$= 20(1{\cdot}6549 - \log p)$$

The straight line can now be graduated by projection from a logarithmic scale, or by the use of tables of values calculated from the distance relations above.

The nomogram is shown in Fig. 94 (*b*).

The graduations on the p and v sides of the straight line are in opposite senses, the value of p decreasing as that of v increases. The volume is seen to be 7·61 in.^{3} when the pressure is 25 lb/in.^{2}

130. **Three-variable Formulae.** Before consideration is given to methods of construction it is convenient to determine types of formulae containing three variables which give rise to the various forms of nomograms listed below—

(*A*) Three curves.
(*B*) Two curves and a straight line.
(*C*) A curve and two non-parallel straight lines.
(*D*) A curve and two parallel straight lines.
(*E*) Three concurrent straight lines.
(*F*) Three non-concurrent straight lines no two of which are parallel.
(*G*) Two parallel straight lines and another straight line.
(*H*) Three parallel straight lines.

(*A*) Three Curves. In Fig. 95 a straight line cuts the three curves, the x-, y-, z-axes, at P, Q, R respectively. For *all* positions of this straight line, the readings at P, Q, R (when the axes are graduated) must satisfy the formula (type at present unknown) which the nomogram represents. Axes of reference—for convenience called here the u- and v-axes respectively—are taken in the plane

of the curves. In the figure these axes of reference are rectangular, but they may equally well be oblique. The co-ordinates of P will be functions of x, those of Q will be functions of y, and those of R will be functions of z. Thus, at P, $u = f_1(x)$ and $v = f_2(x)$, where $f_1(x)$ and $f_2(x)$ are functions of x; at Q, $u = \phi_1(y)$ and $v = \phi_2(y)$, where $\phi_1(y)$ and $\phi_2(y)$ are functions of y; and at R, $u = F_1(z)$ and $v = F_2(z)$, where $F_1(z)$ and $F_2(z)$ are functions of z.

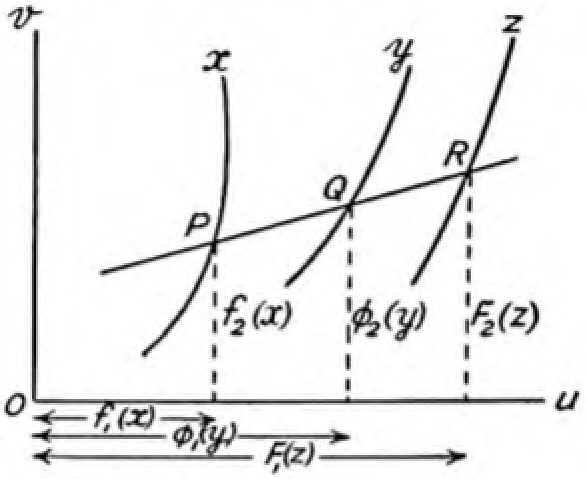


FIG. 95

The equation of the straight line PQR is of the form

$$au + bv + 1 = 0 \qquad \text{(XII.2)}$$

where a and b are constants.

Since the points P, Q, R lie on this line, their co-ordinates will satisfy (XII.2).

Hence
$$af_1(x) + bf_2(x) + 1 = 0$$
$$a\phi_1(y) + b\phi_2(y) + 1 = 0$$
and
$$aF_1(z) + bF_2(z) + 1 = 0$$

If a and b are eliminated from these three relations, then

$$f_1(x)[\phi_2(y) - F_2(z)] + \phi_1(y)[F_2(z) - f_2(x)] + F_1(z)[f_2(x) - \phi_2(y)] = 0 \qquad \text{(XII.3)}$$

or, in determinant form

$$\begin{vmatrix} f_1(x) & f_2(x) & 1 \\ \phi_1(y) & \phi_2(y) & 1 \\ F_1(z) & F_2(z) & 1 \end{vmatrix} = 0$$

If $f_1(x)$, $f_2(x)$, $\phi_1(y)$, $\phi_2(y)$, $F_1(z)$, $F_2(z)$ are denoted by X', X'', Y', Y'', Z', Z'' respectively, the equation (XII.3) becomes

$$X'(Y''-Z'')+Y'(Z''-X'')+Z'(X''-Y'')=0 \qquad \text{(XII.4)}$$

A formula of the type (XII.4) can, therefore, be represented by a nomogram consisting of three curves suitably graduated.

(*B*) TWO CURVES AND A STRAIGHT LINE. Suppose that one of the curves in Fig. 95 is a straight line—say the x-axis, which without loss of generality may be assumed to coincide with the v-axis.

In this case, $f_1(x) = 0$, and the relation (XII.3) becomes

$$\phi_1(y)[F_2(z)-f_2(x)]+F_1(z)[f_2(x)-\phi_2(y)]=0$$

i.e. $-f_2(x)[\phi_1(y)-F_1(z)]+\phi_1(y)\times F_2(z)+\phi_2(y)\times -F_1(z)=0$

If $X=-f_2(x), Y'=\phi_1(y)$, $Y''=\phi_2(y), Z'=-F_1(z)$, and $Z''=F_2(z)$ then this relation becomes

$$X(Y'+Z')+Y'Z''+Y''Z'=0 \qquad \text{(XII.5)}$$

Thus a formula of the type (XII.5) can be represented by a nomogram consisting of a straight line (the x-axis) and two curves (the y- and z-axes) [Fig. 96 (*a*)].

(*C*) A CURVE AND TWO NON-PARALLEL STRAIGHT LINES. Suppose that the x- and y-axes are non-parallel straight lines, the x-axis, as in (*B*), coinciding with the v-axis, the z-axis is a curve, and the u, v origin is taken at the intersection of the x- and y-axes. In this case, $f_1(x) = 0$, and, since the y-axis is a straight line passing through the origin, $\phi_2(y) = m\phi_1(y)$, where m is a constant. The relation (XII.3) now becomes

$$\phi_1(y)[F_2(z)-f_2(x)]+F_1(z)[f_2(x)-m\phi_1(y)]=0$$

i.e. $\quad \phi_1(y)[F_2(z)-mF_1(z)]+F_1(z)\times f_2(x)=f_2(x)\times\phi_1(y)$

i.e.
$$YZ'+Z''X=XY \qquad \text{(XII.6)}$$

where $X=f_2(x)$, $Y=\phi_1(y), Z'=F_2(z)-mF_1(z)$, and $Z''=F_1(z)$

On division by XY, (XII.6) becomes

$$\frac{Z'}{X}+\frac{Z''}{Y}=1 \qquad \text{(XII.7)}$$

When the y-axis is at right angles to the x-axis, $m = 0$. In this case $Z' = F_2(z)$ instead of $Z' = F_2(z) - mF_1(z)$, and the type of formula represented does not differ from (XII.6) or (XII.7).

It is seen that a formula of the type (XII.6) or (XII.7) gives rise to a nomogram consisting of a curve (the z-axis) and two non-parallel straight lines (the x- and y-axes) which may be taken at right angles to each other or not according to convenience [Fig. 96 (*b*)].

(*D*) A CURVE AND TWO PARALLEL STRAIGHT LINES. Suppose that the x- and z-axes are parallel straight lines at distance d apart,

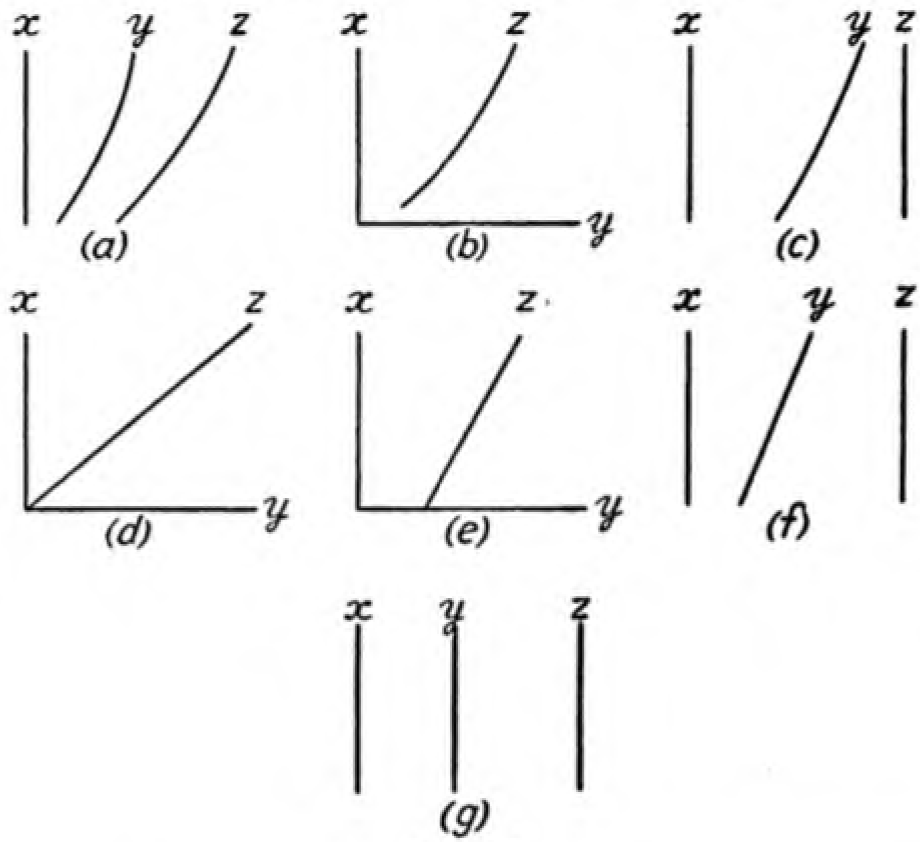


FIG. 96

the x-axis, as before, coinciding with the v-axis, and that the y-axis is a curve.

Here $f_1(x) = 0$ and $F_1(z) = d$, so that the relation (XII.3) becomes

$$\phi_1(y)[F_2(z) - f_2(x)] + d[f_2(x) - \phi_2(y)] = 0$$

which can be expressed as

$$f_2(x) \times \frac{d - \phi_1(y)}{\phi_1(y)} + \frac{-d\phi_2(y)}{\phi_1(y)} + F_2(z) = 0$$

i.e.

$$XY' + Y'' + Z = 0 \qquad . \qquad . \quad \text{(XII.8)}$$

where $X = f_2(x)$, $Y' = \dfrac{d - \phi_1(y)}{\phi_1(y)}$, $Y'' = -\dfrac{d\phi_2(y)}{\phi_1(y)}$ and $Z = F_2(z)$

Accordingly, a formula of the type (XII.8) can be represented by a nomogram consisting of two parallel straight lines (the x- and z-axes) and a curve (the y-axis) [Fig. 96 (c)].

(E) THREE CONCURRENT STRAIGHT LINES. Suppose that the x-, y-, z-axes are straight lines having a common point of intersection, the x-axis, as previously, coinciding with the v-axis, and the u, v origin being at the point of intersection of the straight lines. Here

$$f_1(x) = 0,\ \phi_2(y) = m \times \phi_1(y), \text{ and } F_2(z) = n \times F_1(z)$$

where m and n are constants.

The relation (XII.3) now becomes

$$\phi_1(y)[nF_1(z) - f_2(x)] + F_1(z)[f_2(x) - m\phi_1(y)] = 0$$

which leads to

$$\frac{f_2(x)}{n-m} \times \phi_1(y) = \phi_1(y) \times F_1(z) + F_1(z) \times \frac{f_2(x)}{n-m}$$

i.e.

$$XY = YZ + ZX \quad . \quad . \quad . \quad \text{(XII.9)}$$

or

$$Z = \frac{XY}{X+Y} \quad . \quad . \quad . \quad . \quad \text{(XII.10)}$$

or

$$\frac{1}{Z} = \frac{1}{X} + \frac{1}{Y} \quad . \quad . \quad . \quad \text{(XII.11)}$$

where $X = \dfrac{f_2(x)}{n-m}$, $Y = \phi_1(y)$, and $Z = F_1(z)$

Thus, any formula of the type (XII.9) or (XII.10) or (XII.11) can be represented by a nomogram consisting of three concurrent straight lines.

As in (b), two of the straight lines can be taken at right angles to each other, and usually this is found most convenient [Fig. 96 (d)].

(F) THREE NON-CONCURRENT STRAIGHT LINES NO TWO OF WHICH ARE PARALLEL. Suppose that the x-, y-, z-axes are three non-concurrent straight lines no two of which are parallel, the x-axis coinciding with the v-axis and the u, v origin being taken at the intersection of the x- and y-axes.

In this case, $f_1(x) = 0$, $\phi_2(y) = m\phi_1(y)$, and $F_2(z) = nF_1(z) + k$, where m, n, k are constants.

The relation (XII.3) becomes

$$\phi_1(y)[nF_1(z) + k - f_2(x)] + F_1(z)[f_2(x) - m\phi_1(y)] = 0$$

which may be expressed as

$$\frac{f_2(x)}{n-m}\,[F_1(z) - \phi_1(y)] + \phi_1(y)\left[F_1(z) + \frac{k}{n-m}\right] = 0$$

i.e. $X(Z - Y) + Y(Z + a) = 0$

or $$XY = Y(Z + a) + ZX \quad . \quad \text{(XII.12)}$$

or $$\frac{Z + a}{X} + \frac{Z}{Y} = 1 \quad . \quad . \quad . \quad . \quad \text{(XII.13)}$$

where $X = \dfrac{f_2(x)}{n-m}$, $Y = \phi_1(y)$, $Z = F_1(z)$, and $a = \dfrac{k}{n-m}$

The formula (XII.12) or (XII.13) is not altered in type when the x- and y-axes are at right angles to each other. Thus, any formula of the type (XII.12) or (XII.13) can be represented by a nomogram consisting of three non-concurrent straight lines no two of which are parallel [Fig. 96 (e)].

(G) Two Parallel Straight Lines and another Straight Line. Suppose that the x- and z-axes are parallel straight lines at distance d apart, the x-axis coinciding with the v-axis, and that the y-axis is a straight line not parallel to the other axes.

Here $f_1(x) = 0$, $F_1(z) = d$, and $\phi_2(y) = m\phi_1(y) + n$, where m and n are constants.

The relation (XII.3) now becomes

$$\phi_1(y)[F_2(z) - f_2(x)] + d[f_2(x) - m\phi_1(y) - n] = 0$$

which can be expressed as

$$[f_2(x) - n] \times \left[\frac{d - \phi_1(y)}{\phi_1(y)}\right] + [F_2(z) - n - dm] = 0$$

i.e. $$XY + Z = 0 \quad . \quad . \quad . \quad \text{(XII.14)}$$

where $X = f_2(x) - n$, $Y = \dfrac{d - \phi_1(y)}{\phi_1(y)}$, and $Z = F_2(z) - n - dm$

A formula of the type (XII.14) can be represented, therefore, by a nomogram consisting of two parallel straight lines (the x- and z-axes) and another straight line (the y-axis) [Fig. 96 (f)].

(*H*) THREE PARALLEL STRAIGHT LINES. If the x-, y-, and z-axes are parallel straight lines, the x-axis coinciding with the y-axis, and the distances of the z-axis from the x- and y-axes are $a + b$ and b respectively, then

$$f_1(x) = 0,\ \phi_1(y) = a, \text{ and } F_1(z) = a + b$$

so that the relation (XII.3) becomes in this case

$$a[F_2(z) - f_2(x)] + (a + b)[f_2(x) - \phi_2(y)] = 0$$

i.e.

$$bf_2(x) - (a + b)\phi_2(y) + aF_2(z) = 0$$

i.e.

$$X + Y + Z = 0 \quad . \quad . \quad . \quad \text{(XII.15)}$$

where $X = bf_2(x)$, $Y = -(a + b)\phi_2(y)$, and $Z = aF_2(z)$.

Thus, the formula (XII.15) can be represented by a nomogram consisting of three parallel straight lines [Fig. 96 (*g*)].

In the rest of this chapter methods of construction are shown for nomograms representing what are probably the most important of the above types of formulae, or at any rate the most frequently occurring, starting with the simplest case $X + Y + Z = 0$.

131. **Formula** $X + Y + Z = 0$. A formula of the type

$$X + Y + Z = 0 \quad . \quad . \quad . \quad \text{(XII.16)}$$

gives rise to a nomogram consisting of three parallel straight lines, as established in Art. 130. In Fig. 97 a base-line ABC cuts three

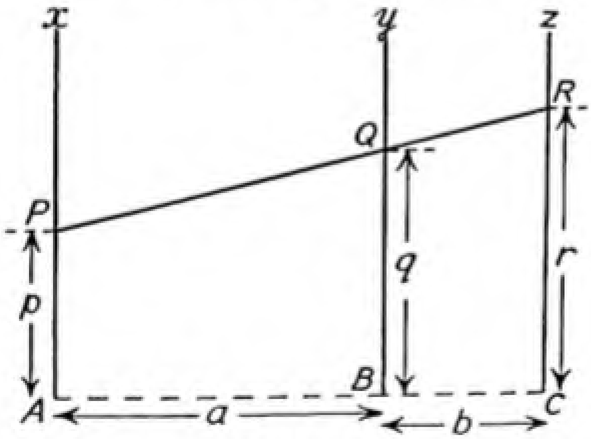


FIG. 97

parallel straight lines, the x-, y-, z-axes, in the points A, B, C respectively, the corresponding readings being x_0, y_0, z_0. This base-line is drawn perpendicular to the axes, but it may be drawn at any convenient angle to these axes.

A method of graduating these axes and of determining the ratio of their distances apart has now to be found. Let *any* straight line intersect the axes at P, Q, R respectively. Then the readings x, y, z at these points must satisfy the formula (XII.16). Let the actual lengths of AP, BQ, CR be p, q, r, and those of AB, BC be a, b, as shown in the diagram. It is easy to prove that

$$\frac{q-p}{r-p} = \frac{a}{a+b}$$

which reduces to

$$bp - (a+b)q + ar = 0 \quad . \quad . \quad \text{(XII.17)}$$

If X_0, Y_0, Z_0 are the values of X, Y, Z respectively at the base-line, i.e. where $x = x_0$, $y = y_0$, $z = z_0$, then by (XII.16)

$$X_0 + Y_0 + Z_0 = 0$$

By combining this relation with (XII.16)

$$(X - X_0) + (Y - Y_0) + (Z - Z_0) = 0 \quad . \quad \text{(XII.18)}$$

Now let λ_1, λ_2, λ_3 be scale-factors such that

$$p = \lambda_1(X - X_0),\ q = \lambda_2(Y - Y_0),\ \text{and } r = \lambda_3(Z - Z_0)$$

and substitute these values in (XII.17).

Then

$$b\lambda_1(X - X_0) - (a+b)\lambda_2(Y - Y_0) + a\lambda_3(Z - Z_0) = 0 \quad . \quad \text{(XII.19)}$$

(XII.18) and (XII.19) are equivalent relations, and the latter must be the former multiplied through by a common factor.

Hence $$b\lambda_1 = -(a+b)\lambda_2 = a\lambda_3$$

from which

$$\frac{a}{b} = \frac{\lambda_1}{\lambda_3} \quad . \quad . \quad . \quad . \quad . \quad \text{(XII.20)}$$

and

$$\lambda_2 = -\frac{\lambda_1\lambda_3}{\lambda_1 + \lambda_3} \quad . \quad . \quad . \quad \text{(XII.21)}$$

Suitable values for λ_1 and λ_3 are chosen in accordance with the required ranges over which x and z are to vary and with the length of paper on which the nomogram is to be constructed, but once these values have been fixed, the values of $\frac{a}{b}$ and λ_2 are determined accurately from (XII.20) and (XII.21) respectively.

The length $a + b$ is chosen to suit the breadth of paper and the ratio $\frac{a}{b}$. The scales are all graduated so that x, y, z increase upwards. If the ratio $\frac{\lambda_1}{\lambda_3} = \frac{a}{b}$ is positive, the y-axis falls between the other two; if the ratio lies between 0 and -1, the y-axis is on the left, and if the ratio lies between -1 and $-\infty$, the y-axis is on the right. If the ratio is -1, the y-axis is at infinity, and an adjustment of one of the values λ_1 and λ_3 is necessary to bring the ratio well away from -1.

A great variety of formulae can be expressed in the form (XII.16), frequently by logarithmic transformation.

EXAMPLE 1

The formula $k^2 = \frac{3r^2}{20} + \frac{h^2}{10}$ gives the radius of gyration k of a uniform circular cone, radius of base r, height h, about a diameter of the base. It is desired to construct on a sheet of paper, 30 in. by 22 in., a nomogram to represent this formula, r to vary from 2 in. to 9 in., and h to vary from 7 in. to 15 in.

The formula is written as

$$3r^2 + (-20k^2) + 2h^2 = 0$$

which agrees with (XII.16) if X, Y, Z are substituted for $3r^2$, $-20k^2$, $2h^2$ respectively.

λ_1, λ_2, λ_3 are the scale-factors along the r, k, h axes respectively. The length between the lowest and the highest graduations on the r-axis will be

$$\lambda_1(3 \times 9^2 - 3 \times 2^2), \text{ i.e. } 231\lambda_1 \text{ in.}$$

The length of the paper is 30 in., so that with a 2 in. allowance for borders,

$$231\lambda_1 = 28, \text{ and hence } \lambda_1 = \frac{28}{231} = 0{\cdot}12, \text{ say}$$

Similarly, $\lambda_3(2 \times 15^2 - 2 \times 7^2) = 28$, and hence $\lambda_3 = \frac{28}{352} = 0{\cdot}08$, say.

λ_2 is now determined accurately from the relation (XII.21).

$$\lambda_2 = -\frac{0{\cdot}12 \times 0{\cdot}08}{0{\cdot}12 + 0{\cdot}08} = -\frac{0{\cdot}0096}{0{\cdot}20} = -0{\cdot}048$$

From (XII.20), $$\frac{a}{b} = \frac{\lambda_1}{\lambda_3} = \frac{0{\cdot}12}{0{\cdot}08} = \frac{3}{2}$$

a and b being the distances between the axes, as in Fig. 97.

The breadth of the paper is 22 in., and here it is convenient to take $a + b = 20$ in., so that $a = 12$ in. and $b = 8$ in.

The extreme values k_0 and k_m of k occur when $r = 2$ in., $h = 7$ in. and when $r = 9$ in., $h = 15$ in. respectively.

Hence, $$k_0^2 = \frac{3 \times 2^2}{20} + \frac{7^2}{10} = 0{\cdot}6 + 4{\cdot}9 = 5{\cdot}5 \text{ in.}^2$$

and $$k_0 = \sqrt{5{\cdot}5} = 2{\cdot}345 \text{ in.}$$

$$k_m^2 = \frac{3 \times 9^2}{20} + \frac{15^2}{10} = 12{\cdot}15 + 22{\cdot}5 = 34{\cdot}65 \text{ in.}^2$$

and $$k_m = \sqrt{34{\cdot}65} = 5{\cdot}887 \text{ in.}$$

For the graduation of the r-axis, distance (in inches) from base-line

$$= \lambda_1(3 \times r^2 - 3 \times 2^2)$$
$$= 0{\cdot}36(r^2 - 4)$$

For the graduation of the h-axis, distance (in inches) from base-line

$$= \lambda_3(2 \times h^2 - 2 \times 7^2)$$
$$= 0{\cdot}16(h^2 - 49)$$

For the graduation of the k-axis, distance (in inches) from base-line

$$= \lambda_2[(-20 \times k^2) - (-20 \times 5{\cdot}5)]$$
$$= 0{\cdot}96(k^2 - 5{\cdot}5)$$

The axes can now be graduated, either by projection from a scale of squares, or from tables of values calculated from these distance relations.

The nomogram is shown reduced in scale in Fig. 98.

When $r = 7{\cdot}3$ in. and $h = 10{\cdot}5$ in., the nomogram gives $k = 4{\cdot}36$ in.

EXAMPLE 2

Construct a nomogram to represent the formula $T = \frac{\pi}{16} f d^3$, d to vary from 4·5 in. to 10 in. and f to vary from 2 000 lb/in.² to 9 000 lb/in.² Paper, 22 in. by 15 in.

Logarithmic transformation gives

$$\log f + \left(\log \frac{\pi}{16} - \log T\right) + 3 \log d = 0$$

which is in the form (XII.16) where $X = \log f$, $Y = \log \frac{\pi}{16} - \log T$, and $Z = 3 \log d$.

λ_1, λ_2, λ_3 are the scale-factors along the f-, T-, d-axes respectively, and the values of these are obtained as in the last Example.

$$\lambda_1(\log 9\,000 - \log 2\,000) = 21$$

so that $$\lambda_1 = \frac{21}{\log 4{\cdot}5} = \frac{21}{0{\cdot}6532} = 32, \text{ say}$$

$$\lambda_3(3 \log 10 - 3 \log 4{\cdot}5) = 21$$

so that $$\lambda_3 = \frac{7}{1 - 0{\cdot}6532} = \frac{7}{0{\cdot}3468} = 20, \text{ say}$$

Then $$\lambda_2 = -\frac{\lambda_1\lambda_3}{\lambda_1 + \lambda_3} = -\frac{32 \times 20}{32 + 20} = -\frac{640}{52}$$

i.e. $$\lambda_2 = -12{\cdot}31$$

The ratio of the distances a and b between the axes is given by

$$\frac{a}{b} = \frac{\lambda_1}{\lambda_3} = \frac{32}{20} = \frac{8}{5}$$

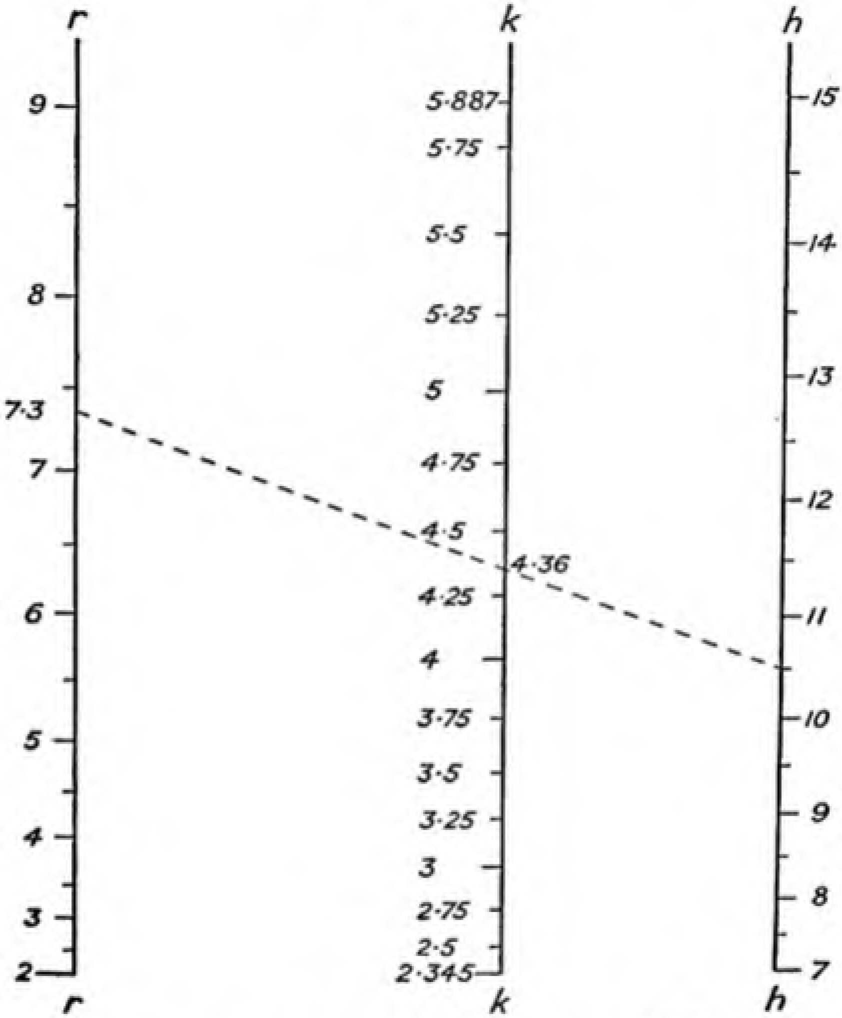


FIG. 98. NOMOGRAM FOR FORMULA $k^2 = \frac{3r^2}{20} + \frac{h^2}{10}$

A convenient value for $a + b$ is 13 in., so that $a = 8$ in. and $b = 5$ in. If T_0 and T_m are the extreme values of T, then

$$T_0 = \frac{\pi}{16} \times 2\,000 \times 4{\cdot}5^3 = 35\,780 \text{ lb-in.}$$

$$T_m = \frac{\pi}{16} \times 9\,000 \times 10^3 = 1\,767\,000 \text{ lb-in.}$$

For the graduation of the f-axis, distance (in inches) from base-line

$$= \lambda_1(\log f - \log 2\,000)$$

$$= 32 \log \frac{f}{2\,000}$$

For the graduation of the T-axis, distance (in inches) from base-line

$$= \lambda_2 \left[\left(\log \frac{\pi}{16} - \log T\right) - \left(\log \frac{\pi}{16} - \log 35\,780\right)\right]$$

$$= - 12{\cdot}31(- \log T + 4{\cdot}5536)$$

$$= 12{\cdot}31(\log T - 4{\cdot}5536)$$

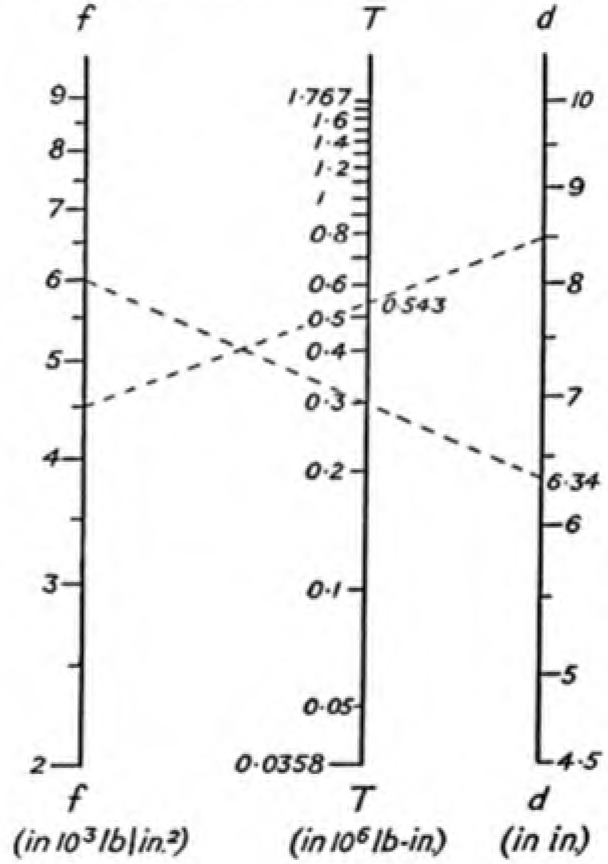


FIG. 99. NOMOGRAM FOR FORMULA $T = \frac{\pi}{16} f d^3$

For the graduation of the d-axis, distance (in inches) from base-line

$$= \lambda_3(3 \log d - 3 \log 4{\cdot}5)$$

$$= 60(\log d - 0{\cdot}6532)$$

The axes can now be graduated by projection from a logarithmic scale, or from tables of values calculated from these distance relations.

Fig. 99 shows the nomogram on a reduced scale.

From the diagram, $T = 543\,000$ lb-in. when $f = 4\,500$ lb/in.^{2} and $d = 8{\cdot}5$ in., and also $d = 6{\cdot}34$ in. when $T = 300\,000$ lb-in. and $f = 6\,000$ lb/in.^{2}

132. **Formula** $XY' + Y'' + Z = 0$. A formula of the type

$$XY' + Y'' + Z = 0 \quad . \quad . \quad . \quad \text{(XII.22)}$$

can be represented by a nomogram consisting of two parallel straight lines (the x- and z-axes) and a curve (the y-axis).

In Fig. 100, A and C are the points on the x- and z-axes respectively corresponding to the lower extreme values $x = x_0$ and $z = z_0$, and the point B in which the straight line AC cuts the y-axis will correspond accordingly to the value $y = y_0$. PQR is *any* straight line cutting the x-, y-, z-axes at P, Q, R respectively. Then, when the axes are graduated, the readings at P, Q, R must satisfy the relation (XII.22).

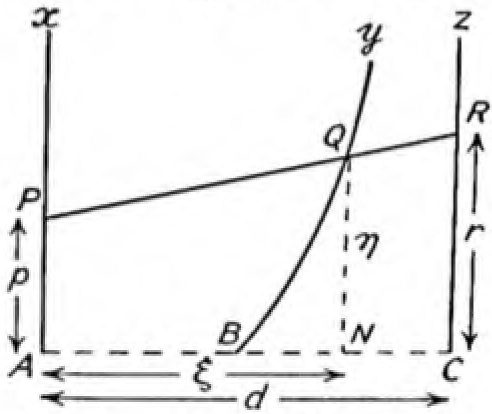


FIG. 100

Let the actual length AP and CR be p and r respectively, and let $AN = \xi$ and $NQ = \eta$, where N is the point in which the line through Q parallel to the x- and z-axes meets AC. [This base-line AC can have any inclination to the parallel axes, but it is usually convenient to have AC perpendicular to these axes, as in Fig. 100.]

It can easily be proved that

$$\frac{\eta - p}{r - p} = \frac{\xi}{d}, \text{ where } d = \text{length } AC$$

Hence

$$p(d - \xi) - d\eta + r\xi = 0 \quad . \quad . \quad . \quad \text{(XII.23)}$$

If X_0, Z_0 are the values of X, Z respectively at the base-line, i.e. where $x = x_0$, $z = z_0$, then AP represents the function $X - X_0$ and CR represents the function $Z - Z_0$, so that

$$p = \lambda_1(X - X_0) \text{ and } r = \lambda_3(Z - Z_0)$$

where λ_1 and λ_3 are scale-factors chosen to suit the ranges of values of x and z and the required overall length of the nomogram. If these values of p and r are substituted in (XII.23), then, after division by $\lambda_3\xi$ (XII.23) becomes

$$(X - X_0) \times \frac{\lambda_1(d - \xi)}{\lambda_3\xi} - \frac{d\eta}{\lambda_3\xi} + (Z - Z_0) = 0 \quad . \quad \text{(XII.24)}$$

Now the relation (XII.22) may be written as

$$(X - X_0)Y' + X_0Y' + Y'' + (Z - Z_0) + Z_0 = 0$$

or $$(X - X_0)Y' + (X_0Y' + Y'' + Z_0) + (Z - Z_0) = 0 \qquad \text{(XII.25)}$$

The relations (XII.24) and (XII.25) hold for all positions of the straight lines PQR, and these relations are identical, so that

$$\frac{\lambda_1(d - \xi)}{\lambda_3\xi} = Y' \text{ and } -\frac{d\eta}{\lambda_3\xi} = X_0Y' + Y'' + Z_0$$

From the former of these equations

$$\xi = \frac{\lambda_1 d}{\lambda_1 + \lambda_3 Y'} \quad . \quad . \quad . \text{(XII.26)}$$

and from a combination of both equations

$$\eta = -\frac{\lambda_1\lambda_3(X_0Y' + Y'' + Z_0)}{\lambda_1 + \lambda_3 Y'} \quad . \quad . \text{(XII.27)}$$

Thus, ξ and η are functions of y, and, as y varies, the corresponding values of ξ and η determine points on a curve, the y-axis.

If the starting values of x and z are such that

$$X_0 = 0 \text{ and } Z_0 = 0$$

then in this case the relation (XII.27) becomes

$$\eta = -\frac{\lambda_1\lambda_3 Y''}{\lambda_1 + \lambda_3 Y'} \quad . \quad . \quad . \text{(XII.28)}$$

the relation (XII.26) remaining unaltered.

Exs. 1 and 2 below illustrate the method of construction.

EXAMPLE 1

On a sheet of paper, 14 in. by 10 in., construct a nomogram to represent the formula $T_e = M + \sqrt{M^2 + T^2}$, M and T both to vary from 2 ton-in. to 10 ton-in.

The formula is written as

$$2 \times M \times T_e - T_e^2 + T^2 = 0$$

which is of the type (XII.22), where $X = 2 \times M$, $Y' = T_e$, $Y'' = -T_e^2$, and $Z = T^2$. If the overall length of the nomogram is taken as 12 in., the scale-factors λ_1 and λ_3 on the M- and T-axes are given by

$$\lambda_1(2 \times 10 - 2 \times 2) = 12 \text{ and } \lambda_3(10^2 - 2^2) = 12$$

so that $$\lambda_1 = 0{\cdot}75 \text{ and } \lambda_3 = 0{\cdot}125$$

Hence, distance (in inches) from base-line to any graduation on M-axis

$$= 0{\cdot}75(2 \times M - 2 \times 2)$$
$$= 1{\cdot}5(M - 2)$$

and distance (in inches) from base-line to any graduation on T-axis

$$= 0{\cdot}125(T^2 - 4)$$

The M- and T-axes, drawn a convenient distance apart, say 8 in., can now be graduated from these two distance relations.

From (XII.26), $$\xi = \frac{0{\cdot}75 \times 8}{0{\cdot}75 + 0{\cdot}125 \times T_e} = \frac{48}{6 + T_e}$$

and from (XII.27), $$\eta = -\frac{0{\cdot}75 \times 0{\cdot}125\,(2 \times 2 \times T_e - T_e^2 + 2^2)}{0{\cdot}75 + 0{\cdot}125 \times T_e}$$

i.e. $$\eta = \frac{3[(T_e - 2)^2 - 8]}{4(6 + T_e)}$$

The extreme values of T_e are

$$2 + \sqrt{4 + 4}, \text{ i.e. } 4{\cdot}828 \text{ ton-in.,}$$

and $$10 + \sqrt{100 + 100}, \text{ i.e. } 24{\cdot}14 \text{ ton-in.}$$

Values of T_e ranging from 4·828 to 24·14 are substituted in these expressions for ξ and η, and the T_e-axis plotted and graduated accordingly.

The nomogram is shown (on reduced scale) in Fig. 101.

The broken lines in the figure give the following results—

When $M = 5{\cdot}5$ ton-in. and $T = 6{\cdot}2$ ton-in., then $T_e = 13{\cdot}8$ ton-in.

When $T_e = 18$ ton-in. and $M = 7$ ton-in., then $T = 8{\cdot}49$ ton-in.

EXAMPLE 2

In the formula $\alpha = \theta - e \sin \theta$, α and θ are angles in radian measure and e is the eccentricity of an orbit. Assuming that both α and θ vary from 0 to π, and that e varies from 0 to 0·5, construct a nomogram to represent this formula.

Expressed as

$$e \sin \theta - \theta + \alpha = 0$$

the formula is in the form (XII.22), where $X = e$, $Y' = \sin \theta$, $Y'' = -\theta$, and $Z = \alpha$.

If 8 in. is chosen as the overall length of the nomogram, then the scale-factors λ_1 and λ_3 on the e- and α-axes respectively, are given by

$$\lambda_1(0{\cdot}5 - 0) = 8 \text{ and } \lambda_3(\pi - 0) = 8$$

so that $$\lambda_1 = 16 \text{ and } \lambda_3 = \frac{8}{\pi}$$

Thus the distance from the base-line of any graduation on the e-axis is $16e$, and the corresponding distance on the α-axis is $\frac{8}{\pi}\alpha$. These axes can now be drawn at a suitable distance apart, say 5 in., and graduated.

From (XII.26), $$\xi = \frac{\lambda_1 d}{\lambda_1 + \lambda_3 \sin \theta}, \text{ and since } d = 5,$$

$$\xi = \frac{16 \times 5}{16 + \frac{8}{\pi} \sin \theta}$$

i.e. $$\xi = \frac{5\pi}{\pi + 0{\cdot}5 \sin \theta}$$

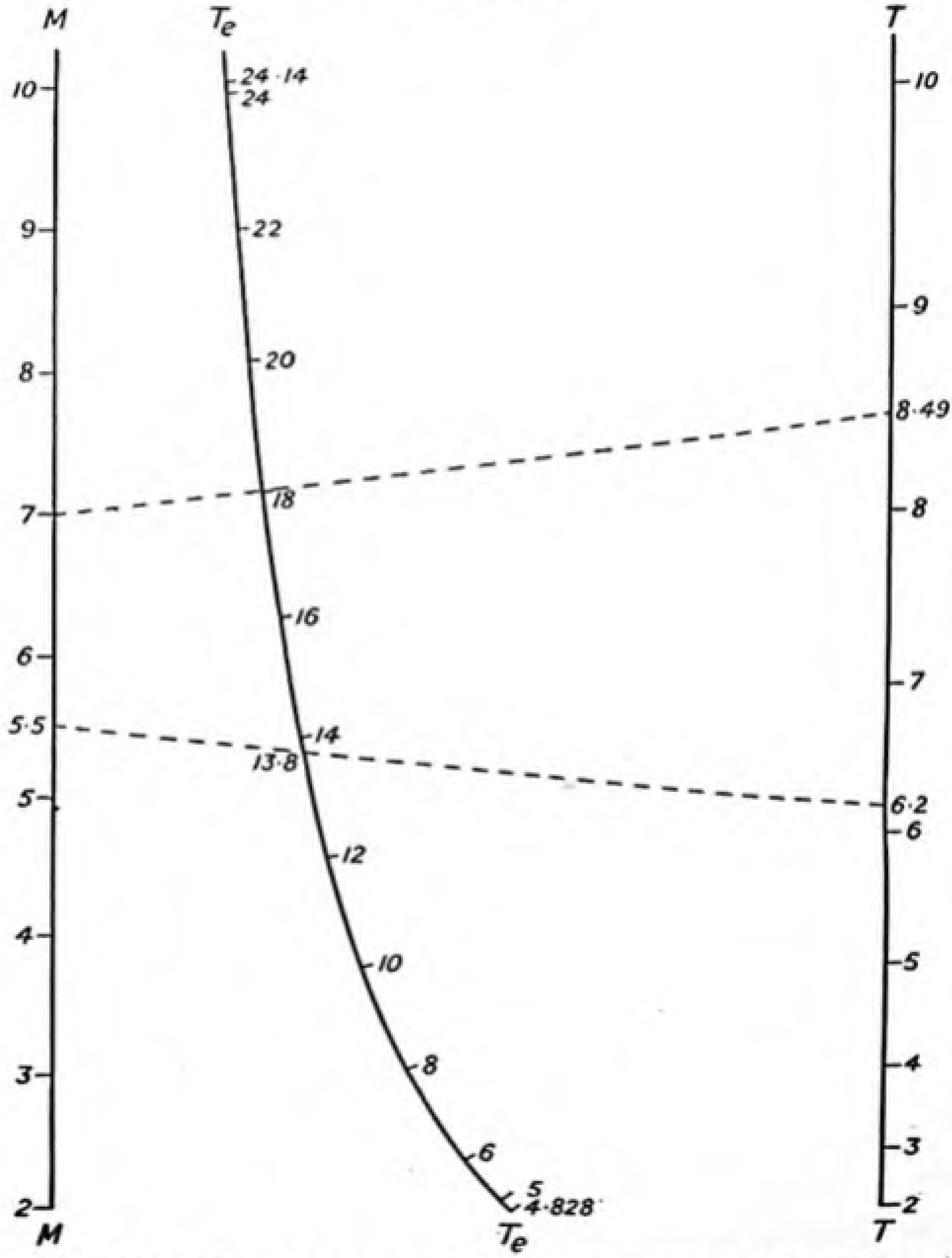


FIG. 101. NOMOGRAM FOR FORMULA $T_e = M + \sqrt{M^2 + T^2}$

In this case the base-line values on the e- and α-axes are both zero, so that from (XII.28),

$$\eta = -\frac{\lambda_1\lambda_3(-\theta)}{\lambda_1 + \lambda_3 \sin\theta}$$

$$= \frac{\frac{128}{\pi}(\theta)}{16 + \frac{8}{\pi}\sin\theta}$$

i.e.
$$\eta = \frac{8\theta}{\pi + 0{\cdot}5\sin\theta}$$

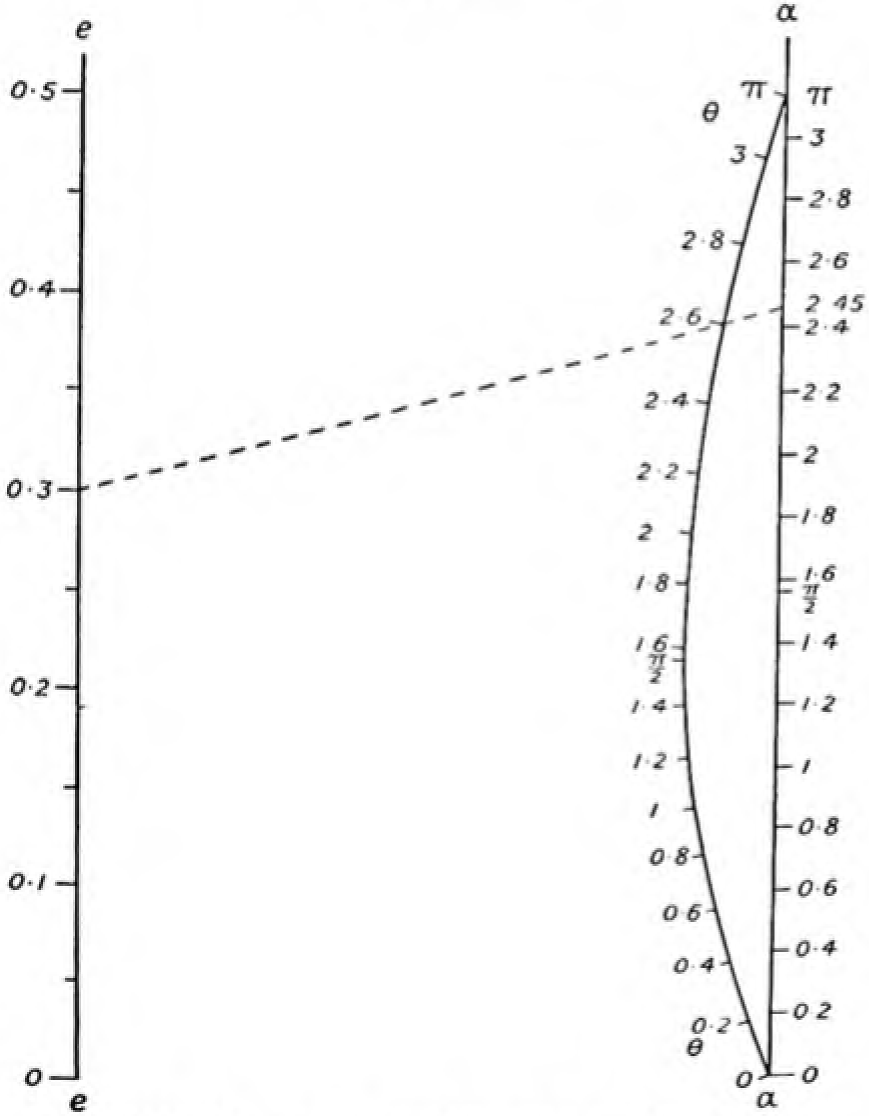


FIG. 102. NOMOGRAM FOR FORMULA $\alpha = \theta - e\sin\theta$

From these expressions for ξ and η values of these co-ordinates are calculated for values of θ between 0 and π, and the θ-axis is plotted and graduated accordingly.

It will be seen from the nomogram which is drawn in Fig. 102 that the θ-axis is a curve which cuts the α-axis in points at which the readings are 0 and π respectively on each of the two axes.

The broken line in the figure gives $\alpha = 2{\cdot}45$ when $e = 0{\cdot}3$ and $\theta = 2{\cdot}6$.

133. **Nomogram for Solving Equation of the Form** $af(x) + \phi(x) + b = 0$. The equation

$$af(x) + \phi(x) + b = 0 \qquad . \qquad . \text{(XII.29)}$$

where a and b are constants, and $f(x)$ and $\phi(x)$ are different functions of x, is in the form (XII.22), and by the method of Art.132 a nomogram can be constructed to give solutions of this equation for given ranges of values of a and b.

Many equations occurring in practice are in the form (XII.29) or can readily be put into that form, e.g. the quadratic equation $x^2 + ax + b = 0$ (written as $ax + x^2 + b = 0$), the cubic equation $x^3 + ax + b = 0$ (written as $ax + x^3 + b = 0$), and equations such as $a \sin\theta + \cos\theta + b = 0$, $a \log_e x + x^2 + b = 0$, $\tan^3 x + a\cos x + b = 0$, and so on.

In each case the method of construction of the nomogram is that explained in Art. 132. It will be sufficient to choose one of the equations noted above—say the cubic equation $x^3 + ax + b = 0$—and to construct a nomogram to represent this equation for given ranges of values of a and b.

The equation is written as

$$ax + x^3 + b = 0$$

and is then in the form (XII.22), where $X = a$, $Y' = x$, $Y'' = x^3$, and $Z = b$. Let the range of values for both a and b be -10 to $+10$. The distance between the extreme graduations on the a-axis and also on the b-axis is taken as 5 in., and the scale-factors λ_1 and λ_3 on these axes are then given by

$$\lambda_1[10 - (-10)] = 5$$

and

$$\lambda_3[10 - (-10)] = 5$$

so that

$$\lambda_1 = \lambda_3 = 0{\cdot}25$$

With the notation of Art.132

$$\xi = \frac{0{\cdot}25d}{0{\cdot}25 + 0{\cdot}25x}$$

i.e.
$$\xi = \frac{d}{1+x}$$

$$\eta = -\frac{0{\cdot}25 \times 0{\cdot}25(-10x + x^3 - 10)}{0{\cdot}25 + 0{\cdot}25x}$$

$$= -\frac{0{\cdot}25(x^3 - 10x - 10)}{1+x}$$

i.e.
$$\eta = 2{\cdot}5 - \frac{0{\cdot}25x^3}{1+x}$$

Values of ξ and η are calculated for values of x such as those shown on the diagram, and the corresponding points plotted to

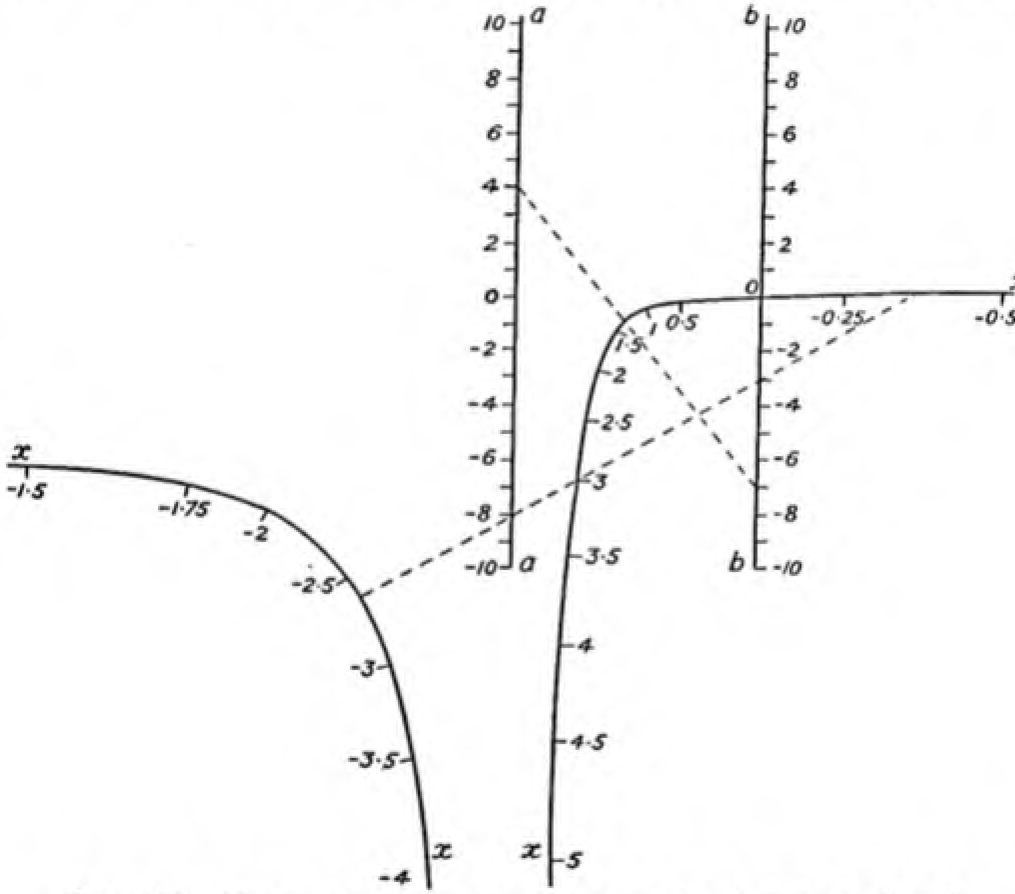


FIG. 103. NOMOGRAM FOR CUBIC EQUATION $x^3 + ax + b = 0$

determine the x-axis. It is seen from Fig. 103 that this axis consists of two branches of a curve. In order to keep the useful part of the curve within the limits of the paper, the length chosen for d must not be too great—this length is here 2 in. Careful graduation is necessary

to ensure that the nomogram gives the solutions of the equation to a reasonable degree of accuracy. Nevertheless, a nomogram such as that in Fig. 103, where the axes are not graduated closely, can furnish good approximations to the roots of the cubic equation. The broken lines in the figure, for example, show that

(1) the equation $x^3 + 4x - 7 = 0$ has only one real root, and this root is 1·25 nearly; and

(2) the equation $x^3 - 8x - 3 = 0$ has three real roots, one of which is 3, and the other two are approximately $-$ 2·6 and $-$ 0·4.

134. **Formula** $XY + Z = 0$. If in (XII.22) of Art. 132, $Y'' = 0$ and $Y' = Y$, the formula becomes

$$XY + Z = 0 \qquad \text{(XII.30)}$$

where X, Y, Z are functions of variables x, y, z respectively.

From (XII.26)

$$\xi = \frac{\lambda_1 d}{\lambda_1 + \lambda_3 Y} \qquad \text{(XII.31)}$$

and from (XII.27)

$$\eta = -\frac{\lambda_1\lambda_3(X_0 Y + Z_0)}{\lambda_1 + \lambda_3 Y} \qquad \text{(XII.32)}$$

If Y is eliminated between these two relations then

$$\eta = \left(\frac{\lambda_1 X_0 - \lambda_3 Z_0}{d}\right)\xi - \lambda_1 X_0$$

which is a linear relation between η and ξ; hence, the y-axis is a straight line.

If two values of y are substituted in (XII.31) and (XII.32), the corresponding pairs of values of ξ and η determine the position of the y-axis, a straight line.

Corresponding to any value $y = y_1$, the value of ξ from (XII.31) determines a point on the base-line, and the perpendicular drawn to that line from the point cuts the y-axis in the reading $y = y_1$.

If the y-axis is inclined to the other axes at a small angle, it is advisable to calculate values of η and plot these against ξ, so that the scale values of y may be accurately marked.

It will be noted that, when the base values X_0 and Z_0 are zero, then $\eta = 0$, and in that case the y-axis is perpendicular to the other axes. The example below illustrates the method of construction of a nomogram representing a formula of the type (XII.30), this method being similar to that of Art. 132.

EXAMPLE

Construct a nomogram of overall length 11 in. approx. to represent the formula $pv^n = 500$, the volume v to vary from 4 to 9, and n from 0·6 to 1·5.

The formula is transformed thus—

$$(\log v)n + \log \frac{p}{500} = 0$$

which is in the form (XII.30), where $X = \log v$, $Y = n$, and $Z = \log \frac{p}{500}$. At the base-line, $v = 4$ and $n = 0{\cdot}6$; hence, $p = \frac{500}{4^{0\cdot6}} = 217{\cdot}6$. Also, when $v = 9$ and $n = 1{\cdot}5$, $p = \frac{500}{9^{1\cdot5}} = 18{\cdot}52$.

The scale-factors λ_1 and λ_3 on the v- and p-axes respectively are given by

$$\lambda_1[\log 9 - \log 4] = 11 \text{ and } \lambda_3\left[\log \frac{18{\cdot}52}{500} - \log \frac{217{\cdot}6}{500}\right] = 11$$

so that
$$\lambda_1 = \frac{11}{\log 2{\cdot}25} = \frac{11}{0{\cdot}3522}, \text{ i.e. } \lambda_1 = 30, \text{ say}$$

and
$$\lambda_3 = \frac{11}{\bar{1}{\cdot}2676 - \bar{2}{\cdot}3377} = -\frac{11}{1{\cdot}0701}, \text{ i.e. } \lambda_3 = -10, \text{ say}$$

Distance (in inches) from base-line to any graduation v

$$= 30(\log v - \log 4)$$
$$= 30 \log \frac{v}{4}$$

Distance (in inches) from base-line to any graduation p

$$= -10\left[\log \frac{p}{500} - \log \frac{217{\cdot}6}{500}\right]$$
$$= 10(2{\cdot}3377 - \log p)$$

Tables of values for the graduation of the v- and p-axes can now be built up.

From (XII.31),
$$\xi = \frac{30d}{30 - 10n} = \frac{12}{3 - n}$$

where 4 in. is taken as a convenient value for d.

Thus, when $n = 0{\cdot}6$, $\xi_{\min} = 5$ in., and when $n = 1{\cdot}5$, $\xi_{\max} = 8$ in.

From (XII.32), $\eta = -\dfrac{30 \times -10[(\log 4)n + \log(217{\cdot}6/500)]}{30 - 10n}$

i.e.
$$\eta = \frac{30(0{\cdot}6021n - 0{\cdot}3613)}{3 - n}$$

When $n = 0{\cdot}6$, $\eta = 0$, and when $n = 1{\cdot}5$, $\eta = 10{\cdot}84$.

The two points given by ($\xi = 5$ in., $\eta = 0$ in.) and ($\xi = 8$ in., $\eta = 10{\cdot}84$ in.) fix the position of the n-axis, a straight line.

If values of ξ are calculated for various values of n between 0·6 and 1·5, and perpendiculars to the base-line are drawn from the corresponding points on that line, these perpendiculars cut the n-axis in points at which the readings are the respective values of n.

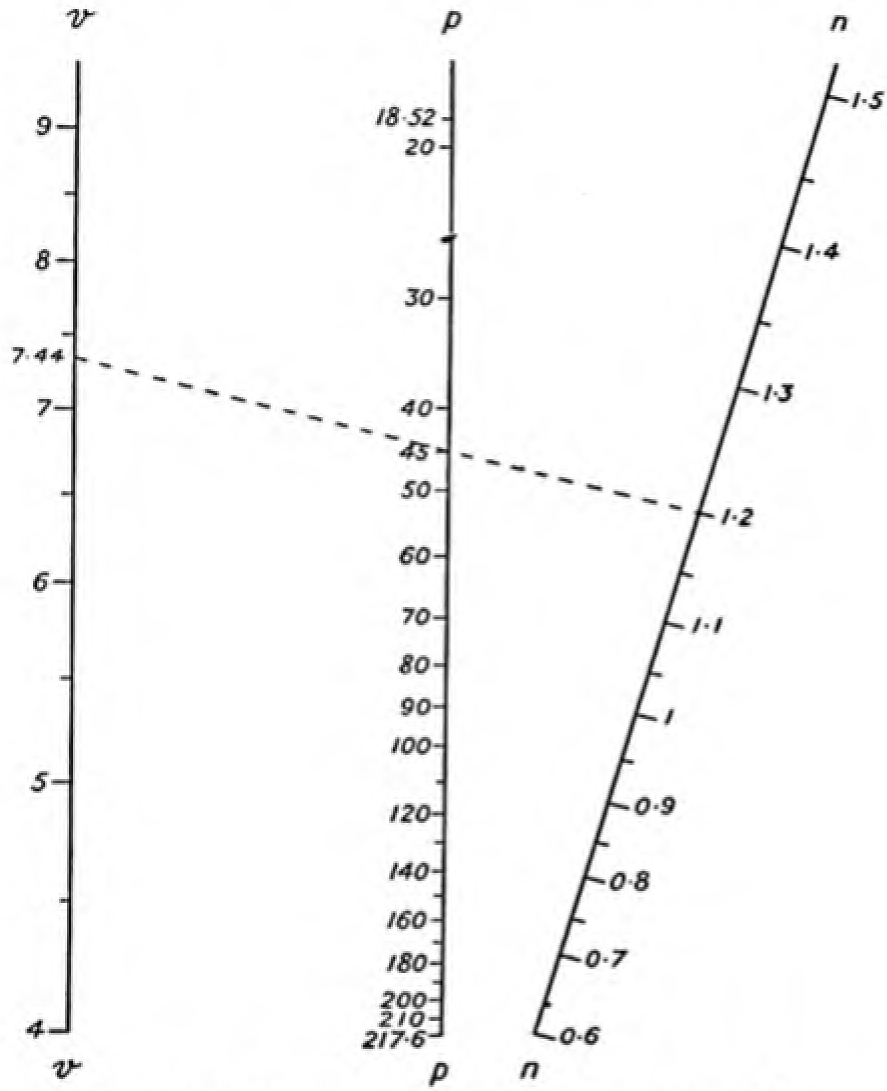


FIG. 104. NOMOGRAM FOR FORMULA $pv^n = 500$

Fig. 104 shows the nomogram on a reduced scale, and the broken line in the figure gives the result $v = 7{\cdot}44$ when $p = 45$ and $n = 1{\cdot}2$.

It will be noted that in this case ξ is greater than d, so that the n-axis is on the right of the p-axis.

It has already been noted that, if in the formula $XY + Z = 0$ the base values X_0 and Z_0 are both zero, then $\eta = 0$, as can be seen from (XII.32), and the y-axis is then perpendicular to the other axes.

The following Example is given as illustrative of this case.

EXAMPLE

Construct a nomogram to represent the formula $a^x = x^n$, where a is assumed to vary from 1 to 100 and n from 0 to 10.

By logarithmic transformation, $x \log a = n \log x$, and this is written as

$$(\log a) \times \frac{x}{\log x} + (-n) = 0$$

which is of the type (XII.30), where $X = \log a$, $Y = \dfrac{x}{\log x}$, and $Z = -n$.

The scale-factors λ_1 and λ_3 along the a- and n-axes respectively, are given by

$$\lambda_1(\log 100 - \log 1) = 5 \text{ and } \lambda_3[(-10) - 0] = 5$$

i.e.

$$\lambda_1 = 2{\cdot}5 \text{ and } \lambda_3 = -0{\cdot}5$$

the distance between the extreme graduations on the a- and n-axes being taken as 5 in. in each case.

From (XII.31),

$$\xi = \frac{2{\cdot}5d}{2{\cdot}5 - 0{\cdot}5 \times \dfrac{x}{\log x}}$$

i.e.

$$\xi = \frac{(5 \log x)d}{5 \log x - x}$$

The values of the functions $\log a$ and $(-n)$ at the base-line, i.e. when $a = 1$ and $n = 0$, are both zero, and from (XII.32) it is seen that $\eta = 0$ in this case. The x-axis, therefore, is a straight line perpendicular to the a- and n-axes.

Values of ξ are calculated for various values of x from 1 to 200, and the corresponding points plotted on the x-axis.

Fig. 105 shows the completed nomogram.

The expression for ξ can be put into the form $\xi = \dfrac{5d}{5 - \dfrac{x}{\log x}}$ so that obviously ξ has its greatest numerical value when the function $\dfrac{x}{\log x}$ is greatest, which occurs when $\dfrac{d}{dx}\left(\dfrac{x}{\log x}\right) = 0$, i.e. when $x = e$ (the base of Napierian logarithms).

Thus, there are two sets of graduations on the x-axis, the one set giving readings from 1 to e, and the other set giving readings in the opposite direction from e onwards.

From the broken line in the figure it is seen that, when $a = 10$ and $n = 7{\cdot}15$, the two possible values of x are 5 and $1{\cdot}77$ (nearly).

A formula of the type (XII.30) can be transformed into one of the type (XII.16). For example, the formula $pv^n = 500$ (for which a nomogram has been constructed by the method of Art. 134) may be expressed as

$$\log n - \log\log \frac{500}{p} + \log\log v = 0$$

by double logarithmic transformation.

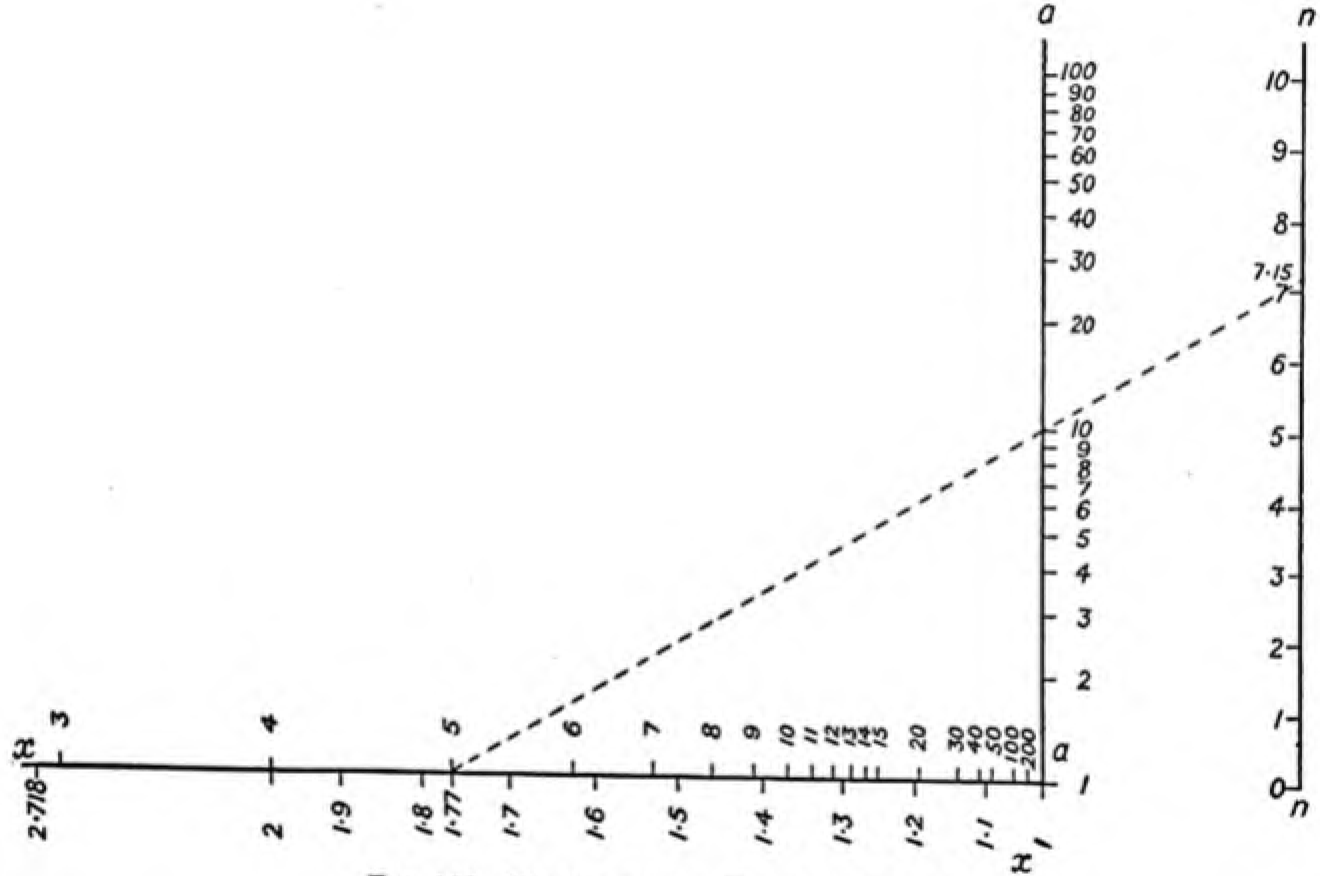


FIG. 105. NOMOGRAM FOR FORMULA $a^x = x^n$

Since this latter formula is in the form $X + Y + Z = 0$, it can be represented by a nomogram consisting of three parallel straight lines. In this particular case the method of Art. 134 is to be preferred as it obviates the necessity of using log log values.

135. **Formula** $\frac{Z'}{X} + \frac{Z''}{Y} = 1$. The formula

$$\frac{Z'}{X} + \frac{Z''}{Y} = 1 \quad . \quad . \quad . \quad (XII.33)$$

where X and Y are functions of variables x and y respectively, and Z' and Z'' are different functions of a variable z, can be represented by a nomogram consisting of two intersecting straight lines (the x- and y-axes) and a curve (the z-axis).

It is generally advisable to have the intersecting straight lines at right angles to each other, as in Fig. 106, and to take O, their point of intersection, as the point at which $X = 0$ and $Y = 0$. The x-, y-, z-axes must be so graduated that, if *any* straight line cuts them at the points P, Q, R respectively, then the readings at these points will satisfy the relation (XII.33).

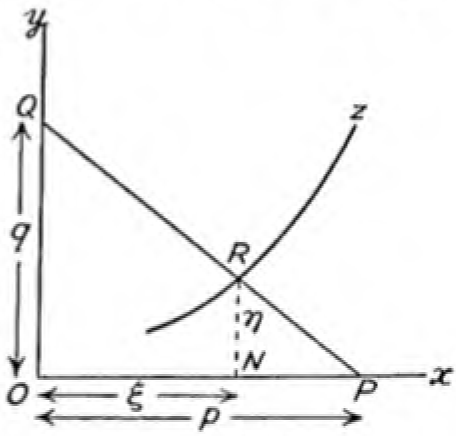


FIG. 106

Let the actual lengths OP and OQ be p and q respectively. Draw RN perpendicular to the x-axis, and let $ON = \xi$ and $NR = \eta$.

From the similar triangles NPR, OPQ,

$$\frac{NP}{OP} = \frac{NR}{OQ}$$

i.e.

$$\frac{p - \xi}{p} = \frac{\eta}{q}$$

i.e.

$$\frac{\xi}{p} + \frac{\eta}{q} = 1 \quad . \quad . \quad . \quad . \quad (XII.34)$$

Let λ_1 and λ_2 be scale-factors along the x- and y-axes respectively, such that

$$p = \lambda_1 X \text{ and } q = \lambda_2 Y$$

and substitute these values of p and q in (XII.34).

Then $$\frac{\xi/\lambda_1}{X} + \frac{\eta/\lambda_2}{Y} = 1 \quad . \quad . \quad . \quad \text{(XII.35)}$$

(XII.33) and (XII.35) are equivalent relations, so that

$$\frac{\xi}{\lambda_1} = Z' \text{ and } \frac{\eta}{\lambda_2} = Z''$$

i.e. $$\xi = \lambda_1 Z' \text{ and } \eta = \lambda_2 Z'' \quad . \quad . \quad \text{(XII.36)}$$

Thus, ξ and η are functions of z, and as z varies, the corresponding values of ξ and η determine points on a curve, the z-axis.

If Z'' is a linear function of Z', say $Z'' = m \times Z' + n$, where m and n are constants, the formula (XII.33) becomes

$$\frac{Z'}{X} + \frac{m \times Z' + n}{Y} = 1 \quad . \quad . \quad . \quad \text{(XII.37)}$$

From (XII.36)

$$\xi = \lambda_1 Z' \text{ and } \eta = \lambda_2(m \times Z' + n)$$

so that

$$\eta = \frac{m \times \lambda_2}{\lambda_1}\xi + n \times \lambda_2$$

Hence, in this case the z-axis is a straight line of gradient $\frac{m \times \lambda_2}{\lambda_1}$ which cuts the y-axis at distance $n \times \lambda_2$ from the origin O.

EXAMPLE

Construct a nomogram to represent the formula

$$\frac{z}{x-1} + \frac{z^2 - 2}{y^2} = 1$$

where x varies from 1 to 6, y from 0 to 4, and z is positive. Overall dimensions of nomogram, 7 in. by 7 in. approx.

Here $\lambda_1[(6-1)-(1-1)] = 6$, say, so that $\lambda_1 = 1{\cdot}2$

and $\lambda_2[4^2 - 0] = 6$, so that $\lambda_2 = 0{\cdot}375$

From (XII.36), $\xi = \lambda_1 z = 1{\cdot}2z$

and $\eta = \lambda_2(z^2 - 2) = 0{\cdot}375(z^2 - 2)$

The axes can now be drawn and graduated in the usual manner.

The nomogram is shown in Fig. 107.

From the broken line in the figure it is seen that $z = 2{\cdot}25$ when $x = 4$ and $y = 3{\cdot}5$.

Nomograms for the solutions of equations of the type

$$af(x) + \phi(x) + b = 0$$

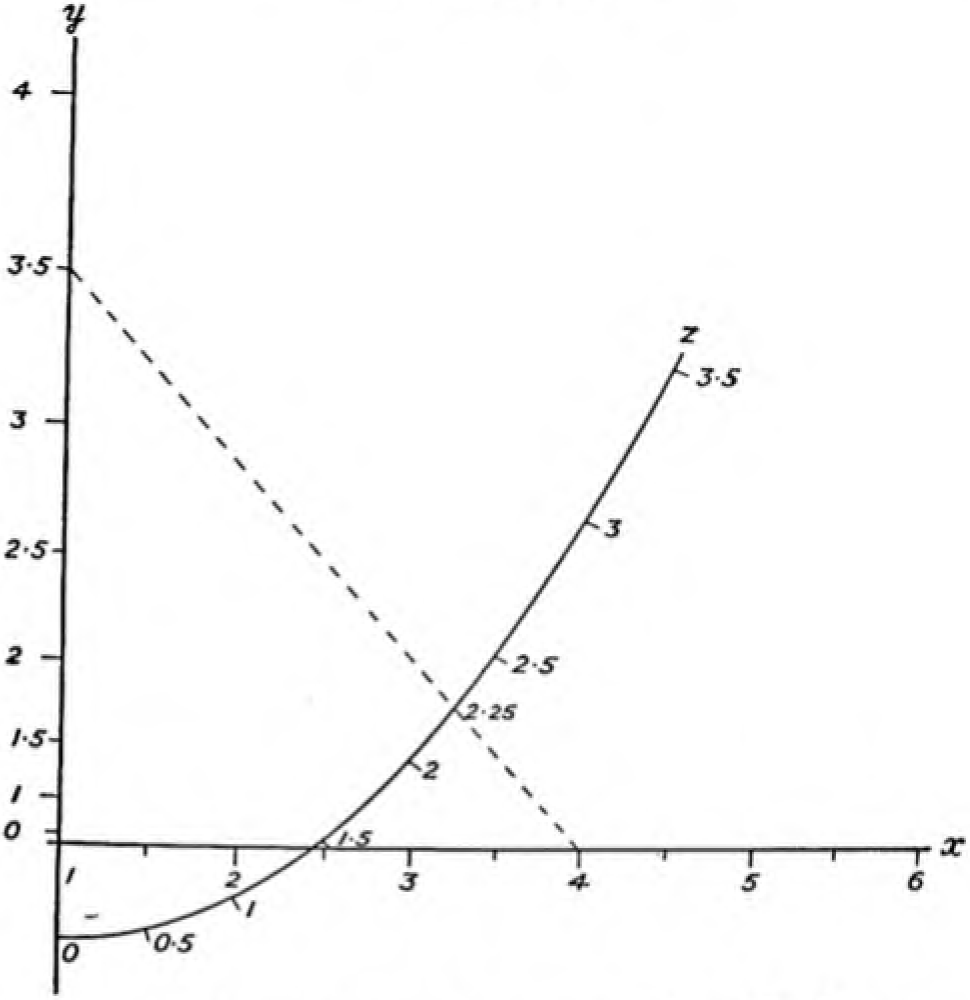


FIG. 107. NOMOGRAM FOR FORMULA $\dfrac{z}{x-1} + \dfrac{z^2-2}{y^2} = 1$

treated in Art. 133 can also be constructed by the method of this section. For the equation may be written in the form (XII.33), thus—

$$\frac{f(x)}{-\dfrac{b}{a}} + \frac{\phi(x)}{-b} = 1$$

i.e.

$$\frac{f(x)}{a'} + \frac{\phi(x)}{b'} = 1$$

where

$$a' = -\frac{b}{a} \text{ and } b' = -b$$

When suitable scale-factors λ_1 and λ_2 have been chosen for the a'- and b'-axes respectively, the x-axis is then given by the relations

$$\xi = \lambda_1 \times f(x) \text{ and } \eta = \lambda_2 \times \phi(x)$$

As examples of this method, the reader should construct nomograms to represent the equations $p \sin\theta + q\cos\theta = r$ and $px^2 + qx + r = 0$, first writing these equations in the respective forms $\frac{\sin\theta}{a} + \frac{\cos\theta}{b} = 1$, where $a = \frac{r}{p}$ and $b = \frac{r}{q}$, and $\frac{x}{a} + \frac{1/x}{b} = 1$, where $a = -\frac{q}{p}$ and $b = -\frac{q}{r}$

136. **Formula** $\frac{1}{Z} = \frac{1}{X} + \frac{1}{Y}$. In (XII.33) let $Z' = Z'' = Z$; the formula becomes then

$$\frac{1}{Z} = \frac{1}{X} + \frac{1}{Y} \qquad . \quad . \quad . \quad (XII.38)$$

From (XII.36) $\qquad \xi = \lambda_1 Z$ and $\eta = \lambda_2 Z$

so that

$$\frac{\eta}{\xi} = \frac{\lambda_2}{\lambda_1}$$

Hence, in this case the z-axis is a straight line of gradient $\frac{\lambda_2}{\lambda_1}$ passing through O (Fig. 108).

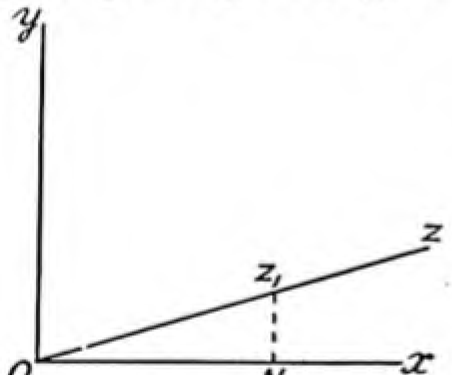


FIG. 108

Corresponding to any value $z = z_1$, the value of ξ from $\xi = \lambda_1 Z$ determines a point N on the x-axis, and the perpendicular drawn from N to the x-axis cuts the z-axis (i.e. the straight line at angle $\tan^{-1}\frac{\lambda_2}{\lambda_1}$ to the x-axis) in the reading $z = z_1$.

When the scale-factors along the x- and y-axes are equal, i.e. $\lambda_1 = \lambda_2$, the z-axis is inclined at 45° to each of the other two axes.

Probably the most important case of (XII.38) occurs when

$$X = x, \; Y = y, \text{ and } Z = z$$

The formula (XII.38) then takes the form

$$\frac{1}{z} = \frac{1}{x} + \frac{1}{y} \qquad . \quad . \quad . \quad . \quad (XII.39)$$

Since here $\xi = \lambda_1 z$, the reading at any point on the z-axis shows the x-co-ordinate of that point, and this fact provides the obvious method of graduating the z-axis for this particular formula.

The following example is illustrative of this case.

EXAMPLE

Construct a nomogram to represent the formula

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$$

where R is the joint resistance of two resistances r_1 and r_2 in parallel, r_1 and r_2 both to vary from 0 to 10 ohms, and the distance between the extreme readings for r_1 and r_2 to be 6 in. in each case.

Here the scale-factors λ_1 and λ_2 on the r_1- and r_2-axes respectively are equal and are given by

$$\lambda_1 = \lambda_2 = \frac{6}{10} = 0{\cdot}6$$

The R-axis thus bisects the angle between the r_1- and r_2-axes (Fig. 109).

Since the given formula is of the type (XII.39), the R-axis can be graduated as follows—

From any point on the r_1-axis whose reading is $r_1 = a$, say, a straight line is drawn perpendicular to the r_1-axis. This perpendicular cuts the R-axis in the reading $R = a$.

From Fig. 109 it is seen that $R = 3{\cdot}21$ ohms when $r_1 = 9$ ohms and $r_2 = 5$ ohms.

It is evident that (XII.38) can be expressed in the form (XII.16) thus—

$$\frac{1}{X} + \left(-\frac{1}{Z}\right) + \frac{1}{Y} = 0$$

and, therefore, can be represented by a nomogram consisting of three parallel straight lines bearing reciprocal scales. The method of this Section, however, has this advantage over the three parallel axes method, that no reciprocals of functions are involved in the calculations leading up to the construction of the nomogram. In general, a formula of the type (XII.38) is most conveniently represented by concurrent axes.

137. **Multi-variable Formula** $X + Y + Z + V + \ldots = 0$. In the case of a formula of the type

$$X + Y + Z + V = 0 \qquad \text{(XII.40)}$$

where X, Y, Z, V are functions of variables x, y, z, v respectively, the method of Art. 131 is applicable, a reference axis being used as connecting link.

Let α denote the sum of two of the functions X, Y, Z, V, say $\alpha = Y + Z$

Then $$X + \alpha + V = 0 \quad . \quad . \quad . \qquad \text{(i)}$$

and $$\alpha - Y - Z = 0 \quad . \quad . \quad . \qquad \text{(ii)}$$

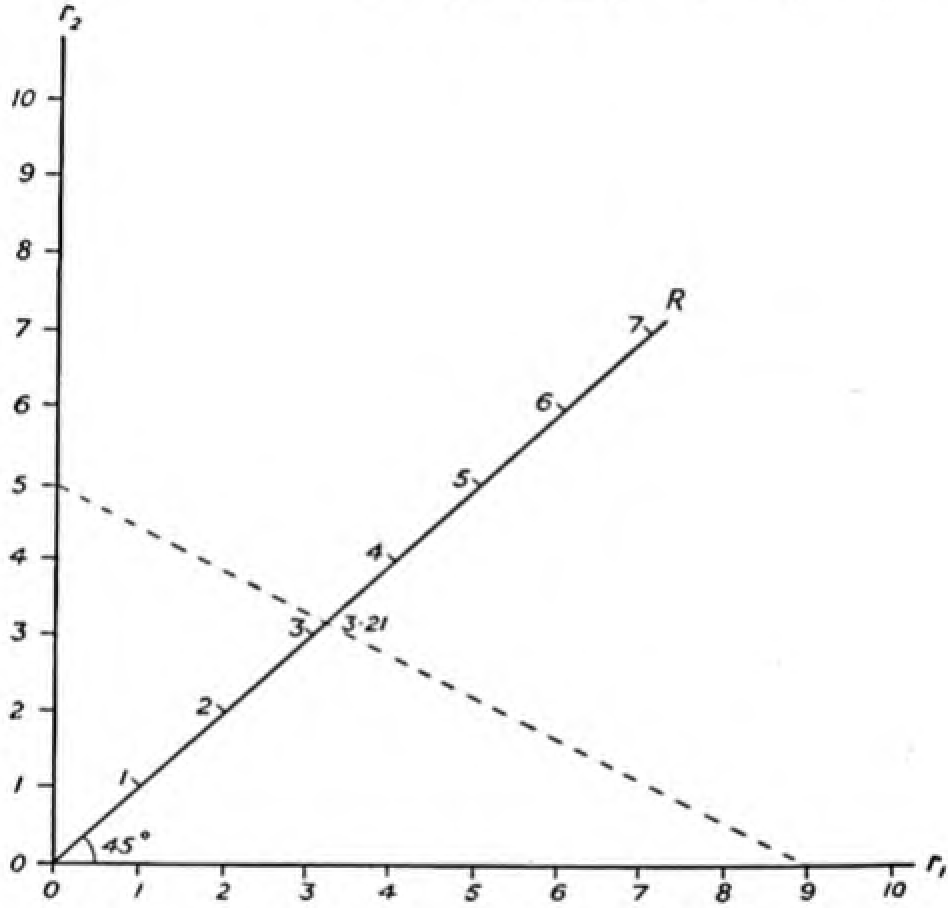


FIG. 109. NOMOGRAM FOR FORMULA $\dfrac{1}{R} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$

A nomogram consisting of three parallel straight lines can be constructed to represent the relation (i), the α-axis, which is the reference axis, being ungraduated.

Then, with the same α-axis, a nomogram of similar type can be constructed to represent the relation (ii), and the complete nomogram will thus contain five straight parallel axes one of which will be ungraduated.

The procedure is illustrated in the following Example.

EXAMPLE 1

The "centrifugal force" F (lb) acting on a body of weight W (lb) which is moving with velocity v (ft/sec) in a circular path of radius r (ft) is given by the formula

$$F = \frac{Wv^2}{gr}$$

where g is the acceleration due to gravity. A nomogram is to be constructed to represent this formula for the following ranges of values of W, v, and r—

W, 20 to 150; v, 5 to 35; r, 2 to 10

By logarithmic transformation the formula becomes

$$2 \log v - \log F + \log (W/g) - \log r = 0$$

Let $\alpha = - \log F + \log (W/g)$, so that

$$2 \log v + \alpha - \log r = 0 \; . \qquad . \quad . \quad . \quad . \quad \text{(i)}$$

and

$$\alpha + \log F - \log (W/g) = 0 \; . \qquad . \quad . \quad . \quad . \quad \text{(ii)}$$

For the relation (i), let λ_1, λ_2, λ_3 be the scale-factors along the v-, α-, r-axes respectively, a the distance between the v- and α-axes, and b the distance between the α- and r-axes. The α-axis is the reference axis and will not be graduated, but the value of the scale-factor λ_2 along that axis must be found as this value enters into the subsequent calculation.

Suppose the overall length of the nomogram is to be 7 in.

Then,
$$\lambda_1(2 \log 35 - 2 \log 5) = 7$$

so that
$$\lambda_1 = \frac{3 \cdot 5}{\log 7} = \frac{3 \cdot 5}{0 \cdot 8451} = 4, \text{ say}$$

$$\lambda_3[(- \log 10) - (- \log 2)] = 7$$

so that
$$\lambda_3 = \frac{7}{- \log 5} = - \frac{7}{0 \cdot 699} = - 10, \text{ say}$$

$$\lambda_2 = - \frac{\lambda_1 \lambda_3}{\lambda_1 + \lambda_3} = - \frac{4 \times - 10}{4 - 10} = - \frac{20}{3}$$

$$\frac{a}{b} = \frac{\lambda_1}{\lambda_3} = \frac{4}{- 10} = - \frac{2}{5}$$

The length $a + b$ will be determined later.

For the graduation of the v-axis, distance (in inches) from base-line

$$= 4(2 \log v - 2 \log 5) = 8 \log (v/5)$$

For the graduation of the r-axis, distance (in inches) from base-line

$$= - 10[(- \log r) - (- \log 2)] = 10 \log (r/2)$$

For the relation (ii), let λ_1', λ_2', λ_3' be the scale-factors along the α-, F-, W-axes respectively, a' the distance between the α- and F-axes, and b' the distance between the F- and W-axes.

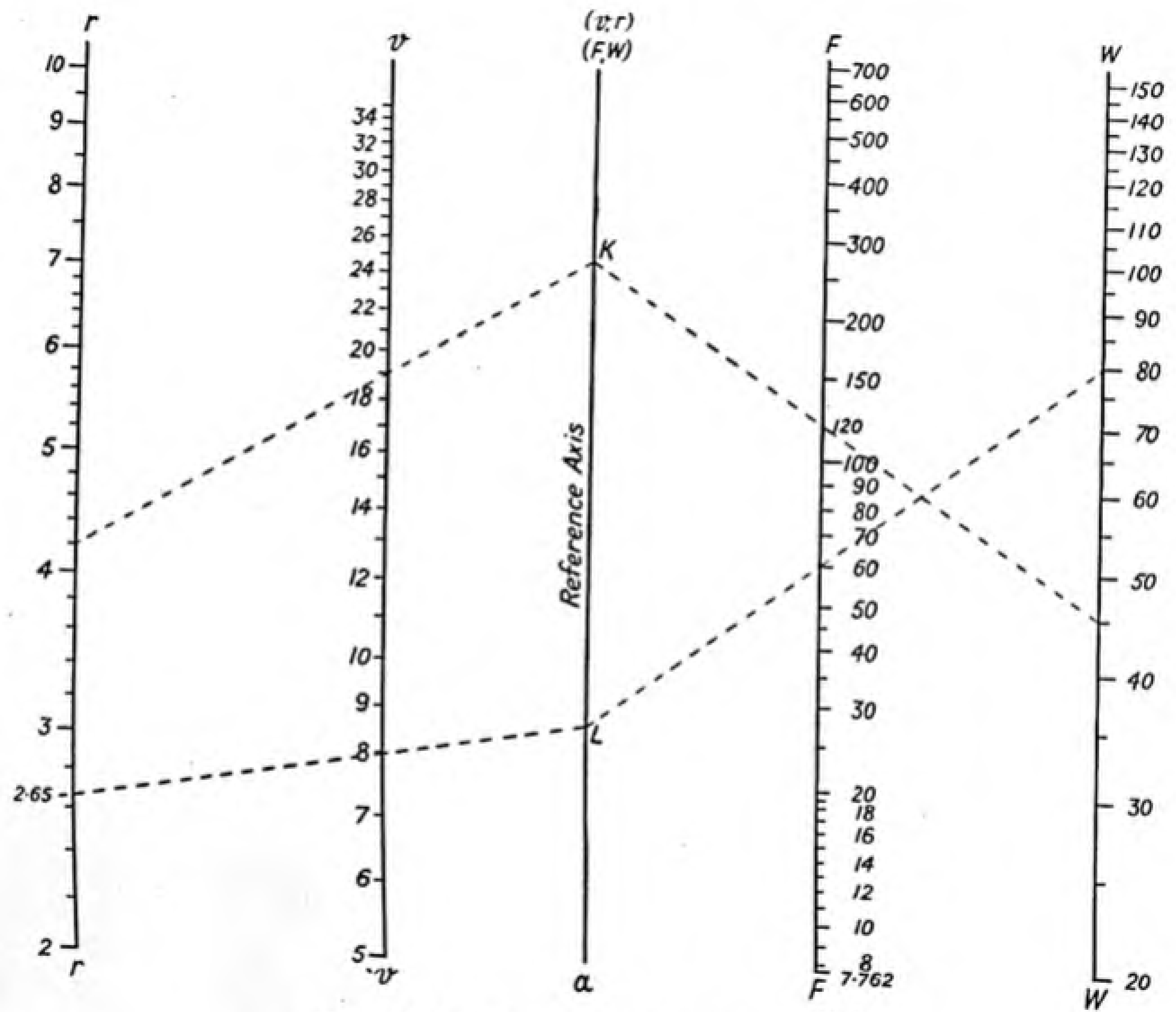


FIG. 110. NOMOGRAM FOR FORMULA $F = \frac{Wv^2}{gr}$

The α term in (ii) must have the same sign as in (i), and the value of the scale-factor λ_1' must be the same as that of λ_2.

Thus,
$$\lambda_1' = \lambda_2 = -\frac{20}{3}$$

$$\lambda_3'\left[\left(-\log\frac{150}{g}\right) - \left(-\log\frac{20}{g}\right)\right] = 7$$

so that
$$\lambda_3' = \frac{7}{-\log 7{\cdot}5} = -\frac{7}{0{\cdot}8751} = -8, \text{ say}$$

$$\lambda_2' = -\frac{-(20/3)\times -8}{-(20/3) - 8} = \frac{40}{11}$$

$$\frac{a'}{b'} = \frac{-(20/3)}{-8} = \frac{5}{6}$$

For the graduation of the F-axis, distance (in inches) from base-line

$$= \frac{40}{11}(\log F - \log 7{\cdot}762) = \frac{40}{11}(\log F - 0{\cdot}8900)$$

7·762 being the value of F at the base-line, i.e. when $W = 20$, $v = 5$, and $r = 2$.

For the graduation of the W-axis, distance (in inches) from base-line

$$= -8\left[\left(-\log\frac{W}{g}\right) - \left(-\log\frac{20}{g}\right)\right]$$
$$= 8\log\frac{W}{20}$$

In determining the distances between the axes, full advantage should be taken of the breadth of the paper on which the nomogram is to be constructed, and the axes kept as far apart as possible.

Suppose that the distance between the outside axes is to be 9 in. approx. Since the ratio $\frac{a}{b} = -\frac{2}{5}$ and the ratio $\frac{a'}{b'} = \frac{5}{6}$, the r- and α-axes will be on opposite sides of the v-axis, and the F- and W-axes will be on the same side of the α-axis. The axes may then be spaced conveniently as follows, the direction being from left to right—

r-axis to v-axis, 2·7 in.

v-axis to α-axis, 1·8 in.

α-axis to F-axis, 2 in.

F-axis to W-axis, 2·4 in.

This spacing gives 8·9 in. as the distance between the outside axes. The axes can be graduated by either of the methods indicated in Ex. 2, Art. 131.

Fig. 110 shows the completed nomogram.

When readings are taken from the nomogram, the reference axis must be used in conjunction with the v- and the r-axes, and with the F- and the W-axes, and not, for example, with the v- and the F-axes.

In Fig. 110, the point $r = 4{\cdot}2$ on the r-axis and the point $v = 19$ on the v-axis are joined by a straight line which cuts the reference axis at K. The straight line joining the point K to the point $W = 45$ on the W-axis cuts the F-axis in the reading $F = 120$.

Hence, when $W = 45$ lb, $r = 4{\cdot}2$ ft, and $v = 19$ ft/sec, then $F = 120$ lb. Again, the straight line joining the point $W = 80$ on the W-axis to the point $F = 60$ on the F-axis cuts the reference axis at L, and the straight line joining the point L to the point $v = 8$ on the v-axis cuts the r-axis in the reading $r = 2{\cdot}65$.

Hence, when $W = 80$ lb, $F = 60$ lb, and $v = 8$ ft/sec, then $r = 2{\cdot}65$ ft.

The method of construction employed in the case of the formula

$$X + Y + Z + V = 0$$

can be extended to suit a multi-variable formula of the type

$$X + Y + Z + V + W + \ldots = 0 \qquad \text{(XII.41)}$$

Ex. 2 below illustrates the process.

EXAMPLE 2

Given the formula

$$W = \frac{43\,260qH}{kN^3R^2}$$

where W (tons) is the weight of flywheel rim necessary for an engine indicating H horse-power at N r.p.m., R (ft) being the radius of gyration, q an energy fluctuation ratio, and k a speed fluctuation ratio, construct a nomogram to represent this formula, the ranges of values being as follows—

N, 120 to 300; R, 2 to 4; H, 40 to 80;

k, 0·010 to 0·020; and q, 0·08 to 0·32

By logarithmic transformation, the given formula becomes

$$3 \log N + \log W - \log (43\,260q) + \log k - \log H + 2 \log R = 0$$

Let $\qquad \alpha = \log W - \log (43\,260q) + \log k - \log H$

$\qquad \beta = \log W - \log (43\,260q) + \log k$

and $\qquad \gamma = \log W - \log (43\,260q)$

Then,

$$3 \log N + \alpha + 2 \log R = 0 \qquad \text{(i)}$$

$$\alpha - \beta + \log H = 0 \qquad \text{(ii)}$$

$$-\beta + \gamma + \log k = 0 \qquad \text{(iii)}$$

and

$$\gamma - \log W + \log (43\,260q) = 0 \qquad \text{(iv)}$$

Each of the relations (i) to (iv) is in the form (XII.16), and, therefore, can be represented by a nomogram consisting of three parallel straight lines. For the relation (i), let λ_1, λ_2, λ_3 be the scale-factors along the N-, α-, R-axes respectively, a the distance between the N- and α-axes, and b the distance between the α- and R-axes.

Then, $$\lambda_1[3 \log 300 - 3 \log 120] = 7{\cdot}2$$

where the overall length of the nomogram is taken as 7·2 in.

Hence, $$\lambda_1 = \frac{2{\cdot}4}{\log 2{\cdot}5} = \frac{2{\cdot}4}{0{\cdot}3979} = 6, \text{ say}$$

Also, $$\lambda_3[2 \log 4 - 2 \log 2] = 7{\cdot}2$$

so that $$\lambda_3 = \frac{3{\cdot}6}{\log 2} = \frac{3{\cdot}6}{0{\cdot}3010} = 12, \text{ say}$$

Then, $$\lambda_2 = -\frac{\lambda_1\lambda_3}{\lambda_1 + \lambda_3} = -\frac{6 \times 12}{6 + 12} = -4$$

Further, $$\frac{a}{b} = \frac{\lambda_1}{\lambda_3} = \frac{1}{2}$$

For the relation (ii), let λ_1', λ_2', λ_3' be the scale-factors along the α-, β-, H-axes respectively, a' the distance between the α- and β-axes, and b' the distance between the β- and H-axes.

Then, $$\lambda_1' = \lambda_2 = -4$$

$$\lambda_3'[\log 80 - \log 40] = 7{\cdot}2$$

so that $$\lambda_3' = \frac{7{\cdot}2}{\log 2} = \frac{7{\cdot}2}{0{\cdot}3010} = 24, \text{ say}$$

$$\lambda_2' = -\frac{-4 \times 24}{-4 + 24} = 4{\cdot}8$$

$$\frac{a'}{b'} = \frac{-4}{24} = -\frac{1}{6}$$

For the relation (iii), let λ_1'', λ_2'', λ_3'' be the scale-factors along the β-, γ-, k-axes respectively, a'' the distance between the β- and γ-axes, and b'' the distance between the γ- and k-axes.

Then, $$\lambda_1'' = \lambda_2' = 4{\cdot}8$$

$$\lambda_3''[\log 0{\cdot}020 - \log 0{\cdot}010] = 7{\cdot}2$$

so that $$\lambda_3'' = \frac{7{\cdot}2}{0{\cdot}3010} = 24, \text{ say}$$

$$\lambda_2'' = -\frac{4{\cdot}8 \times 24}{4{\cdot}8 + 24} = -4$$

$$\frac{a''}{b''} = \frac{4{\cdot}8}{24} = \frac{1}{5}$$

For the relation (iv), let λ_1''', λ_2''', λ_3''' be the scale-factors along the γ-, W-, q-axes respectively, a''' the distance between the γ- and W-axes, and b''' the distance between the W- and q-axes.

Then,
$$\lambda_1''' = \lambda_2'' = -4$$

$$\lambda_3'''[\log(43\,260 \times 0{\cdot}32) - \log(43\,260 \times 0{\cdot}08)] = 7{\cdot}2$$

so that
$$\lambda_3''' = \frac{7{\cdot}2}{\log 4} = \frac{7{\cdot}2}{0{\cdot}6021} = 12, \text{ say}$$

$$\lambda_2''' = -\frac{-4 \times 12}{-4 + 12} = 6$$

$$\frac{a'''}{b'''} = \frac{-4}{12} = -\frac{1}{3}$$

Since the scale-factors are now fixed, tables of values for the graduation of the axes can be built up. The α-, β-, γ-axes are the reference axes and do not require to be graduated.

The distances between the axes are chosen in accordance with the values found for the ratios $\frac{a}{b}$, $\frac{a'}{b'}$, $\frac{a''}{b''}$, and $\frac{a'''}{b'''}$, and with the breadth of paper on which the nomogram is to be constructed.

Fig. 111 shows the completed nomogram.

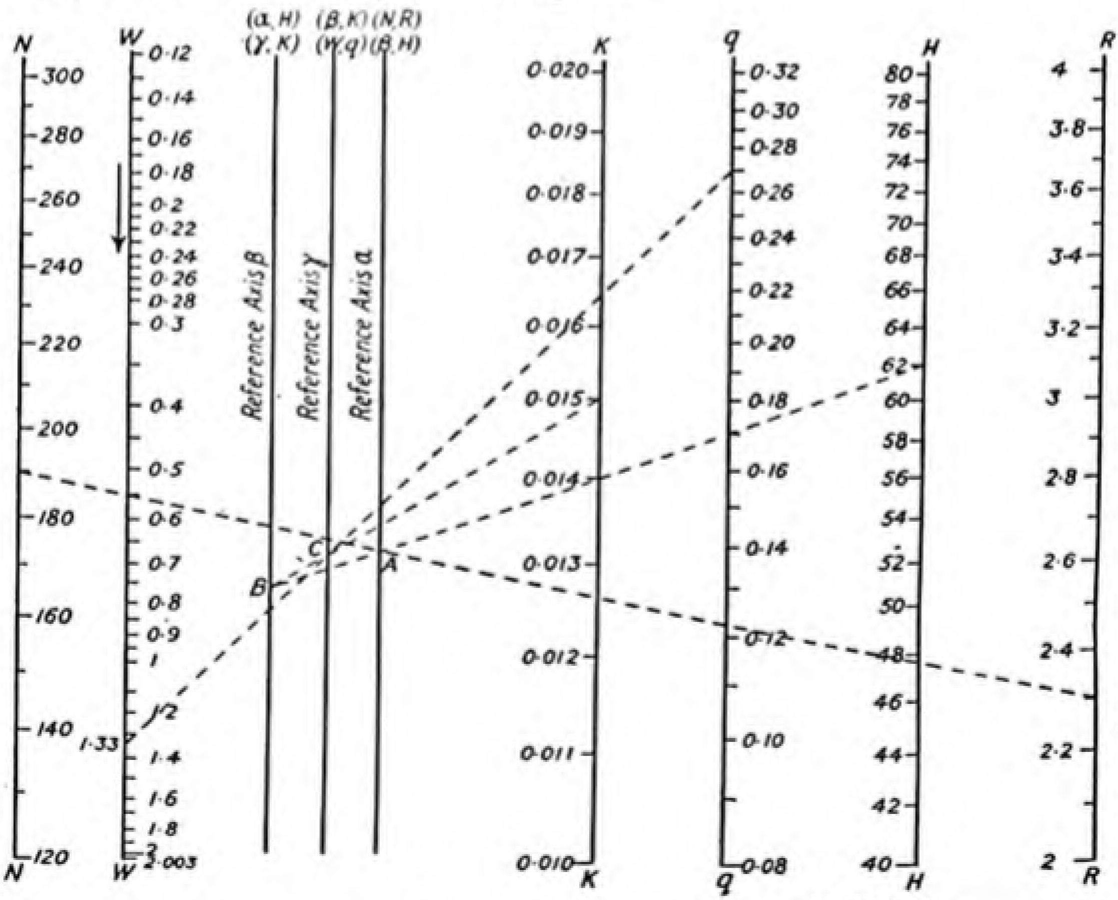


FIG. 111. NOMOGRAM FOR FORMULA $W = \frac{43\,260qH}{kN^3R^2}$

In this figure the distance between the extreme axes is 9 in., and it is to be noted that in the spacing of the axes full advantage is taken of this breadth of paper so as to have the axes as far apart as possible.

Suppose that it is desired to obtain from the nomogram the value of W when $N = 190$, $R = 2{\cdot}3$, $H = 62$, $k = 0{\cdot}015$, and $q = 0{\cdot}27$.

The point $N = 190$ on the N-axis is joined to the point $R = 2{\cdot}3$ on the R-axis, the straight line joining these points cutting the α-axis at A; then the point A is joined to the point $H = 62$ on the H-axis, the straight line joining these points cutting the β-axis at B; then the point B is joined to the point $k = 0{\cdot}015$ on the k-axis, the straight line joining these points cutting the γ-axis at C; and, finally, the point C is joined to the point $q = 0{\cdot}27$ on the q-axis, the straight line joining these points cutting the W-axis in the reading $W = 1{\cdot}33$.

Thus, the required weight of the flywheel rim is 1·33 tons.

138. **Some Other Multi-variable Formulae.** Further examples on the construction of nomograms for formulae containing four or five variables are given below.

EXAMPLE 1

Fig. 112 shows a nomogram representing the formula

$$y = ax^n \qquad \text{(XII.42)}$$

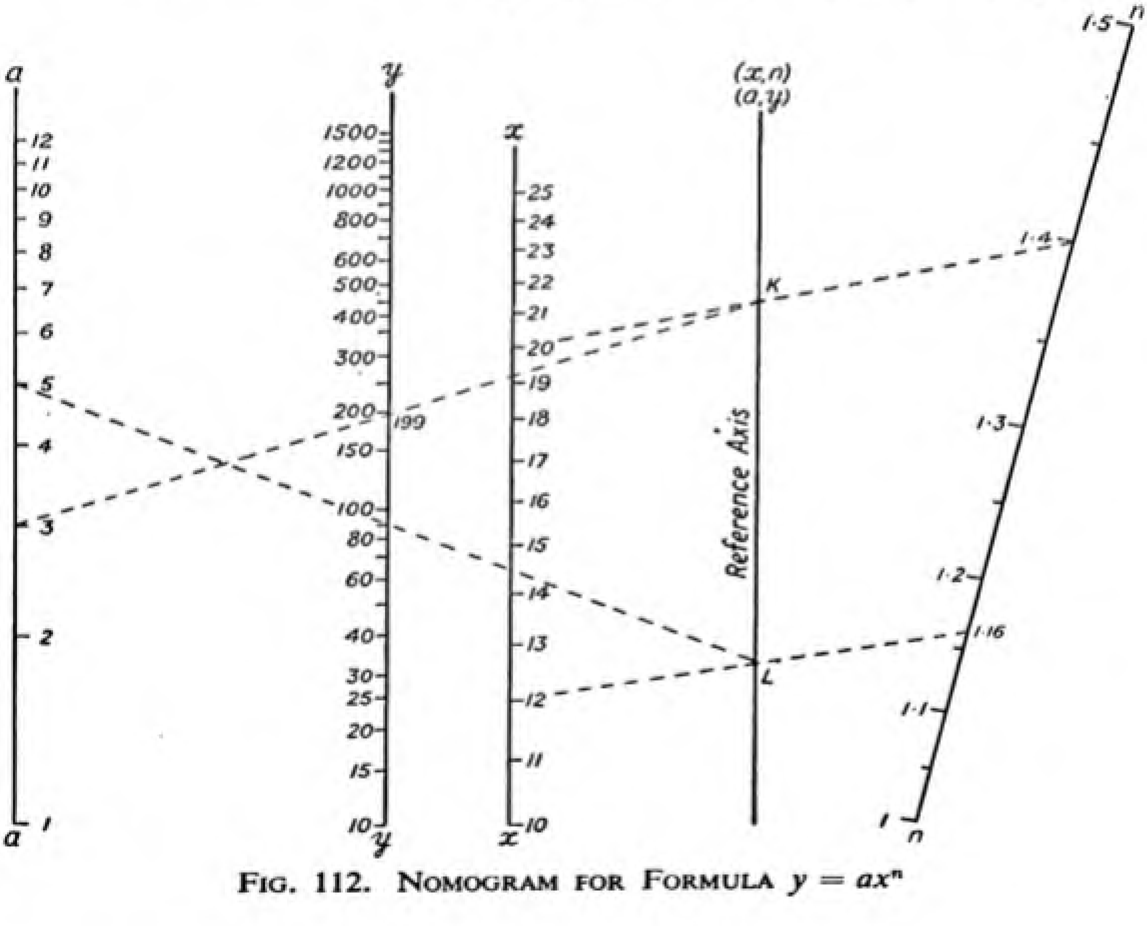


FIG. 112. NOMOGRAM FOR FORMULA $y = ax^n$

a varying from 1 to 12, x from 10 to 25, n from 1 to 1·5, and hence y varying from 10 to 12 × $25^{1\cdot5}$, i.e. from 10 to 1 500.

The method of constructing the nomogram is as follows—

By logarithmic transformation, the formula becomes

$$\log\frac{y}{a} = n\log x$$

and if R denotes $\log\frac{y}{a}$, this can be written as

$$(\log x)\,n - R = 0 \quad . \quad . \quad . \quad . \quad (\text{XII.43})$$

which is of the form (XII.30), where $X = \log x$, $Y = n$, and $Z = -R$. The nomogram for (XII.43) is constructed by the method of Art. 134. The R-axis is the reference axis and is not graduated.

Then, since $R = \log\frac{y}{a}$, this relation is written in the form

$$-R - \log a + \log y = 0 \quad . \quad . \quad . \quad (\text{XII.44})$$

the first term being made $-R$ in order that the function of $\frac{y}{a}$ should be the same in (XII.43) and (XII.44).

The nomogram for (XII.44) is then constructed by the method of Art. 131.

The completed nomogram thus contains four parallel axes (the a-, y-, x-axes and the reference axis) and one sloping axis (the n-axis).

In using the nomogram it is necessary to note that the reference axis must be taken in conjunction with the x- and n-axes, and with the a- and y-axes. Thus, to find y when $a = 3$, $x = 20$, and $n = 1{\cdot}4$, let the straight line joining the readings $x = 20$ and $n = 1{\cdot}4$ cut the reference axis at K, and then draw the straight line joining K to the reading $a = 3$. This latter line cuts the y-axis in the reading $y = 199$, which is the result required. Again, to find n when $y = 90$, $a = 5$, and $x = 12$, let the straight line joining the readings $a = 5$ and $y = 90$ cut the reference axis at L, and then draw the straight line joining L to the reading $x = 12$. This line cuts the n-axis in the reading $n = 1{\cdot}16$, and this is the value of n required.

The law $pv^n = C$, where p, v, n, and C all vary, can be treated in a similar manner.

EXAMPLE 2

In the formula

$$G = \frac{V}{r\log_e\frac{R}{r}} \quad . \quad . \quad . \quad . \quad (\text{XII.45})$$

there are four variables, and the nomogram representing the formula contains a reference axis.

The formula is written as

$$r\log_e\frac{R}{r} = \frac{V}{G} = A\text{, say}$$

i.e.
$$r(\log_e R - \log_e r) = A$$

i.e.
$$(-\log_e R)r + r\log_e r + A = 0 \quad . \quad . \quad . \quad (\text{XII.46})$$

This relation is in the form (XII.22), where $X = -\log_e R$, $Y' = r$, $Y'' = r \log_e r$, and $Z = A$, and accordingly a nomogram representing (XII.46) can be constructed by the method of Art. 132.

The A-axis is the reference axis and is not graduated.

Then, the relation $A = \frac{V}{G}$ is written in the form (XII.30), thus

$$AG - V = 0 \qquad . \quad . \quad . \quad . \quad (XII.47)$$

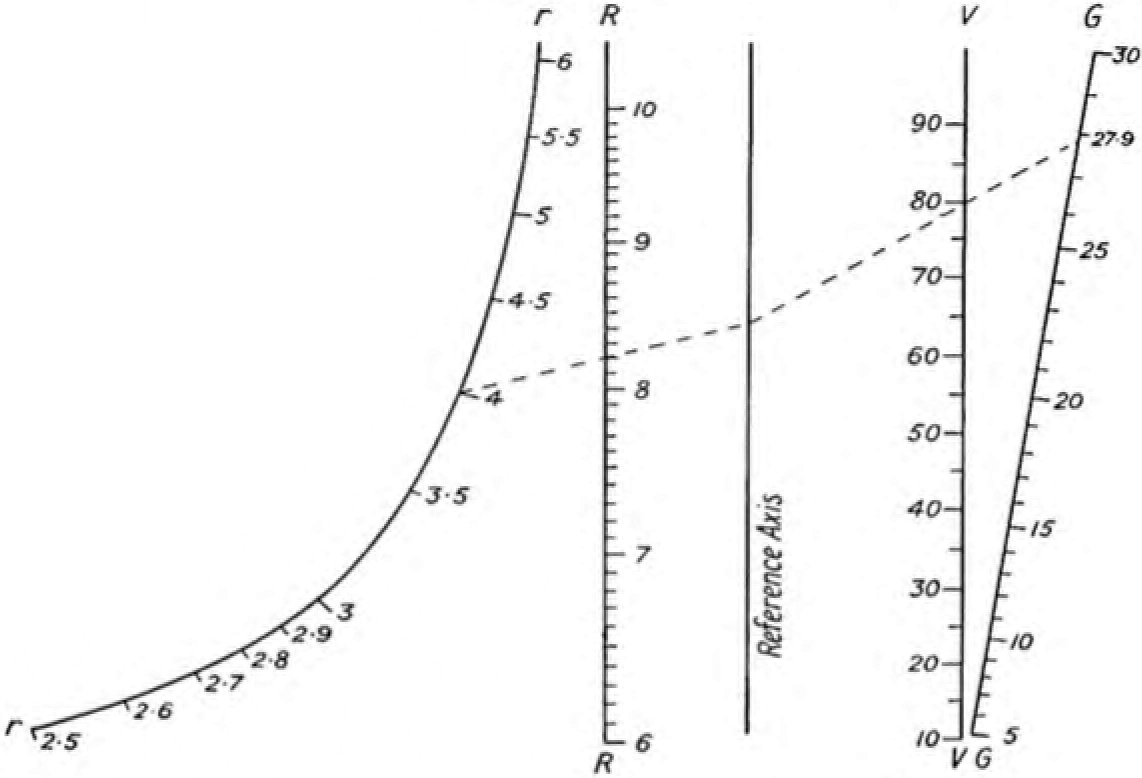


FIG. 113. NOMOGRAM FOR FORMULA $G = \dfrac{V}{r \log_e \dfrac{R}{r}}$

A nomogram representing (XII.47) can be constructed by the method of Art. 134. Although the reference axis is not graduated, the calculations require the value of the scale-factor along this axis, and it is essential that this scale-factor retain in the case of (XII.47) the same value as used in the case of (XII.46).

Fig. 113 shows a nomogram constructed to represent the formula (XII.45), where r varies from 2·5 to 6 cm, R from 6 to 10 cm, and V from 10 to 90 kV.

The broken line in the diagram gives the result—

$$G = 27{\cdot}9 \text{ kV when } r = 4 \text{ cm},\ R = 8{\cdot}2 \text{ cm, and } V = 80 \text{ kV}.$$

EXAMPLE 3

In the formula

$$W = \frac{\pi}{4}[D_1^2 - D_2^2]hw \qquad . \quad . \quad . \quad . \quad (XII.48)$$

there are five variables, and any nomogram representing the formula will include two reference axes.

The following method of construction can be adopted in this case.

The formula is written in the form

$$R_1 - D_1^2 + D_2^2 = 0 \quad . \quad . \quad . \quad . \quad \text{(XII.49)}$$

where $R_1 = \dfrac{4W}{\pi h w}$

By the method of Art. 131 a nomogram is constructed to represent the formula (XII.49), the R_1-axis being ungraduated.

Then, if $R_2 = \dfrac{4W}{\pi w}$, $R_1 h = R_2$, from above, and this relation is expressed in the form

$$R_1 h - R_2 = 0 \quad . \quad . \quad . \quad . \quad \text{(XII.50)}$$

The formula (XII.50) is of the type (XII.30), and the nomogram for this formula is constructed by the method of Art. 134, the R_2-axis being ungraduated.

Finally, since $R_2 = \dfrac{4W}{\pi w}$, then

$$-R_2 w + \frac{4W}{\pi} = 0 \quad . \quad . \quad . \quad . \quad \text{(XII.51)}$$

The formula (XII.51) is also of the type (XII.30), and the method of Art. 134 is again applied.

Thus, in the completed nomogram there will be five parallel straight lines, namely, the D_1-, D_2-, W-axes and the reference axes R_1 and R_2, and, in addition, other two straight lines, the h- and w-axes.

The system to be followed in reading the nomogram, i.e. in finding one of the five quantities W, D_1, D_2, h, w, given the values of the other four, can be indicated briefly thus—

To Find W (or w)

D_1 to D_2 to R_1; then R_1 to h to R_2; and then R_2 to w (or W) to W (or w).

To Find D_1 (or D_2)

w to W to R_2; then R_2 to h to R_1; and then R_1 to D_2 (or D_1) to D_1 (or D_2)

To Find h

D_1 to D_2 to R_1; then w to W to R_2; and then R_1 to R_2 to h

A nomogram representing the formula

$$W = \frac{\pi}{4}\,[D_1^2 - D_2^2]hw$$

is shown in Fig. 114, the ranges of values being as follows—

D_1, 0 to 12 in., D_2, 0 to 9 in., h, 3 to 8 in., and w, 0·1 to 0·3 lb/in.^{3}

It will be noted that the h- and w-axes are perpendicular to the other axes. The base values of D_1 and D_2 are both zero, so that the base values of R_1, R_2, and W are also all zero. It follows that, with the notation of Art. 134, $\eta = 0$

in the case of (XII.50) and also in the case of (XII.51) of this Section. Both the h- and the w-axes are thus perpendicular to the other axes.

In Fig. 114, the straight line joining $D_1 = 11$ to $D_2 = 8$ cuts the reference

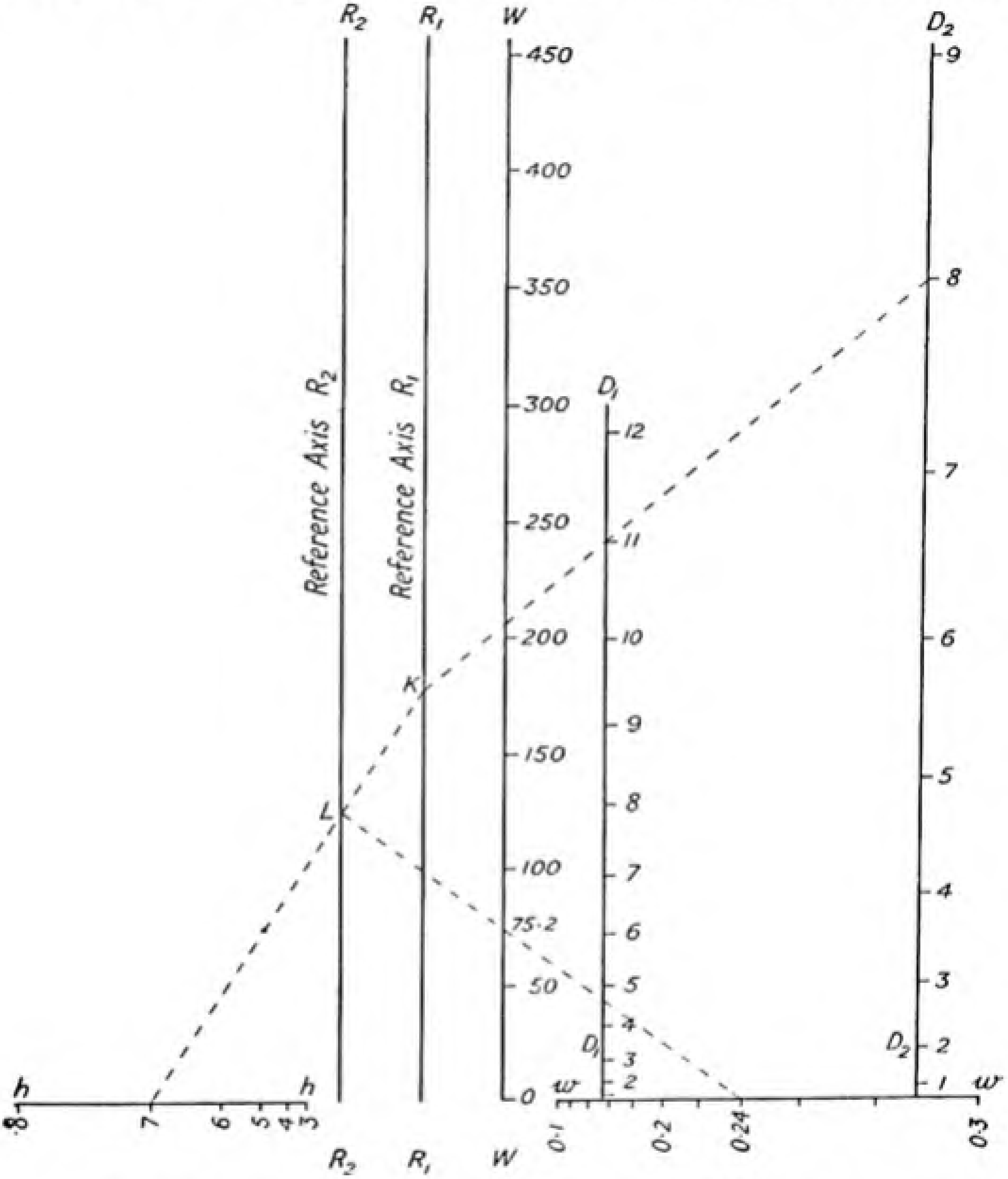


FIG. 114. NOMOGRAM FOR FORMULA $W = \frac{\pi}{4}(D_1^2 - D_2^2)hw$

axis R_1 at K; the straight line joining K to $h = 7$ cuts the reference axis R_2 at L; and, finally, the straight line joining L to $w = 0{\cdot}24$ cuts the W-axis at $W = 75{\cdot}2$.

Therefore, when $D_1 = 11$ in., $D_2 = 8$ in., $h = 7$ in., and $w = 0{\cdot}24$ lb/in.^{3}, $W = 75{\cdot}2$ lb.

EXAMPLES XII

NOTE. Test each nomogram at least twice by checking two readings against the corresponding calculated values.

Construct nomograms to represent the formulae (1) to (18) below, taking into account the dimensions of the paper on which each nomogram is to be drawn.

(1) $V^2 = 2gH$, where V is the velocity acquired by a body in falling vertically under gravity through a distance H from rest, H to vary from 0 to 16 ft. [Take $g = 32{\cdot}2$ ft/sec².]

(2) $C = 0{\cdot}03397D^2$, where C gallons is the capacity of a foot length of a pipe of internal diameter D in., D to vary from 0 to 24.

(3) $pv^{0{\cdot}88} = 360$, where p lb/in.² is the pressure of a mass of gas when the volume is v in.³, v to vary from 4 to 10.

(4) $C = \frac{5}{9}(F - 32)$, where C degrees Centigrade is the temperature equivalent to the temperature F degrees Fahrenheit, F to vary from 32 to 212.

(5) $V = \frac{4}{3}\pi(R^3 - r^3)$, where V is the volume of material forming a hollow sphere, external radius R and internal radius r, R to vary from 1 in. to 5 in. and r from 0 to 2 in.

(6) $I = \frac{\pi}{64}(D^4 - d^4)$, where I in.⁴ is the moment of inertia of a circular ring, outer diameter D in. and inner diameter d in., about a diameter, D to vary from 2 to 8 and d from 0 to 5.

(7) $746H = AV$, where H is the horse-power supplied to an electric motor, A is the amperage, and V is the voltage, A to vary from 2 to 12, and V from 100 to 250.

(8) $W = \frac{\pi}{6}d^3w$, where W is the weight of a solid sphere of diameter d and of weight w per unit volume, d to vary from 5 in. to 10 in. and w from 0·3 lb/in.³ to 0·6 lb/in.³

(9) $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$, where C is the electrostatic capacity equivalent to two capacities C_1 and C_2 in series, C_1 to vary from 0 to 0·2 μF and C_2 from 0 to 0·1 μF.

(10) $\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$, where f is the focal length of a lens and u and v are the distances of the object and image respectively from the lens, u to vary from 0 to 10 cm and v from $-$ 20 cm to 5 cm.

(11) $A = \frac{\pi DL}{144}$, where A ft² is the area of the curved surface of a solid cylinder of diameter D in. and length L in., D to vary from 2 to 8 and L from 5 to 20.

(12) $A = \left(1 + \frac{r}{100}\right)^n$, where A is the amount of £1 in n years at r per cent compound interest, r to vary from 1 to 5, and n from 1 to 20.

(13) $\log_e \frac{T_1}{T_2} = \frac{\mu\pi\phi}{180}$, where $\frac{T_1}{T_2}$ is the ratio of the tensions on the tight and slack

sides of a belt on a pulley, μ being the coefficient of friction between belt and pulley and ϕ the angle of lap in degrees, μ to vary from 0·15 to 0·25 and ϕ from 90 to 270.

(14) $pv^n = C$, v to vary from 0·5 ft³ to 2 ft³, n from 0·8 to 1·4, and C from 5 000 to 10 000.

(15) $W = \dfrac{t(D-t)}{0{\cdot}0826}$, where W lb is the weight per foot length of copper pipe of external diameter D in. and thickness of wall t in., D to vary from 0·5 to 3·5, and t from 0·1 to 0·25.

(16) $T^3 = 0{\cdot}0244\,\dfrac{D_1^4 - D_2^4}{D_1}$, a flange coupling formula for a hollow shaft, T in. being the thickness of flange, and D_1 in. and D_2 in. the external and internal diameters respectively of the shaft, D_1 to vary from 4 to 12, and D_2 from 1 to 5.

(17) $A = \dfrac{D^2}{8}(\theta - \sin\theta)$, where A is the area of a segment of a circle of diameter D and θ is the angle in radians subtended at the centre of the circle by the arc of the segment, D to vary from 1 in. to 6 in. and θ from 0 to π.

(18) $W = \dfrac{\pi}{12}d^2hw$. where W is the weight of a solid right-circular cone of height h, base diameter d, and weight w per unit volume, d to vary from 1 in. to 8 in., h from 1 in. to 12 in., and w from 0·2 lb/in.³ to 0·4 lb/in.³

(19) Show how Fig. 109 may be extended to represent the formula

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

and construct a nomogram to represent this formula for the following ranges of values of r_1, r_2, r_3—

r_1, 0 to 5 ohms, r_2, 0 to 4 ohms. r_3 = 0 to 2 ohms

(20) Explain the methods you would adopt in constructing nomograms to represent the following formulae—

(1) $I = A\left(\dfrac{d^2}{8} + h^2\right)$, where $A = \dfrac{\pi}{4}d^2$

(2) $A = \dfrac{\pi H}{4}(5D - H)$

(3) $F = \dfrac{wbtv^2}{g}$

(4) $L = \dfrac{\pi n(D + d)}{24}$

In (1) to (3) all the quantities are to be assumed variable. π and g, of course, are constants.

(21) Assuming a and b both to vary from -10 to $+10$, construct nomograms to give the solutions of the following equations—

(1) $x^4 + ax^3 + b = 0$

(2) $x^{1\cdot2} + ax + b = 0$

ANSWERS TO EXAMPLES

EXAMPLES I. Page 32

1. 0	**2.** 0	**3.** $4abc$
4. -30	**5.** 3 531	**6.** 5 040
7. 144	**8.** -103	**9.** -868
11. $-2\frac{2}{23}$	**12.** 1, 1·39, $-14{\cdot}39$	**13.** 1, $-2\frac{3}{70}$

14. 0·684, $-0{\cdot}742$ **15.** $-1{\cdot}75$

16. $\frac{1}{2}(1-s)\,[k^4+2(1+m^2+s)k^2+m^2(1+m^2)]$

17. 5 040 **18.** $x^2+y^2+2x+4y-8=0$

19. $x=1,\ y=3,\ z=2$ **20.** $x=3,\ y=5,\ z=2$

22. $G = \begin{vmatrix} p & -q & e \\ r & -s & o \\ b & b+q+s & E \end{vmatrix} \Bigg/ \begin{vmatrix} g & p & -q \\ -(g+r+s) & r & -s \\ s & b & b+q+s \end{vmatrix}$; $ps = qr$

24. $t = 0, 3$; $x = y = z$. When $t = 3$, the equations are identical.

26. $\begin{vmatrix} a & b & c & d & 0 \\ 0 & a & b & c & d \\ a' & b' & c' & 0 & 0 \\ 0 & a' & b' & c' & 0 \\ 0 & 0 & a' & b' & c' \end{vmatrix} = 0$; $x = 2,\ y = 3,\ z = -4$

28. $-3\,384$

31. $\lambda = 1,\ x = -5,\ y = 1$; $\lambda = -1,\ x = -\frac{1}{11},\ y = -\frac{15}{11}$; $\lambda = 12,\ x = \frac{1}{2},\ y = 1$

32. 0·085

33. 4·4318; 3·2288

34. $\bar{3}{\cdot}376854$

35. 15·5 in./sec; 6·13 in./sec^2

36. $1{\cdot}586 \times 10^{-2}$; $-1{\cdot}2874 \times 10^{-4}$

37. 6·65058

38. 0·32330; 0·03688

39. 15·70; 17·31 nautical miles. If d and h are expressed in the same units $\frac{d(d)}{dh} = 204{\cdot}5$ when $h = 250$ ft, and $\frac{d(d)}{dt} = 2\,072$ ft/sec

40. 2·598 in.^{2}

41. $1 + \frac{5n}{12} + \frac{53n^2}{24} - \frac{11n^3}{12} + \frac{7n^4}{24}$; 871

42. $\bar{1}\cdot70418$

43. $0\cdot3076$; $-0\cdot04698$

44. $3\cdot814$ radn/sec; $6\cdot725$ radn/sec^2

45. $6\cdot86441$

46. $1\cdot0511137$

EXAMPLES II. Page 61

1. $\frac{1}{2}ab(a+b)$ **2.** $\frac{1}{4}a^2b^2$ **3.** $\frac{1}{3}ab(a^2+b^2)$

4. πR^2 **5.** $\frac{1}{2}\pi R^4$ **6.** 18

8. -2 **9.** $e(e+1)(e-1)^2 = 29\cdot842$ **10.** $\frac{1}{2}b\pi$

11. 12 **12.** $\frac{1}{4}\pi r^2$ **13.** $\frac{1}{16}\pi r^4$

14. $\frac{1}{8}\pi r^4$ **15.** $\frac{3}{4}\pi a^2$ **16.** 30

17. $\frac{1}{3}abc(b^2+c^2)$ **18.** $\frac{1}{3}abc(a^2+b^2+c^2)$ **19.** $\frac{1}{10}\pi r^5$

20. Area $= \frac{1}{2}ab$; $I_{OX} = \frac{1}{12}ab^3$; centroid at point $\left(\frac{a}{3}, \frac{b}{3}\right)$

21. Area $= \frac{1}{2}\pi ab$; $I_{OX} = \frac{1}{8}\pi ab^3$; centroid at point $\left(0, \frac{4b}{3\pi}\right)$

22. Area $= \frac{3}{2}\pi a^2$; centroid at point $(r = \frac{5}{6}a, 0 = 0)$

23. $\frac{1}{4}\pi a^3b = I_{OY}$ for ellipse; $\frac{1}{4}\pi ab(a^2+b^2) = I$ for ellipse about centre.

24. $\frac{a^2b}{2}$; $\frac{a^3b}{3}$; $\frac{a^2b^2}{4}$

25. 0; $\frac{\pi a^4}{8}$; 0

26. $146\cdot3$; centroid at point $(2\cdot4, 0)$

27. C.G. at point $\left[\frac{a(4a+3b)}{6(a+b)}, \frac{b(3a+4b)}{6(a+b)}\right]$; mass $M = \frac{\sigma b}{2}(a+b)$;
$I_{OX} = M\frac{b^2(2a+3b)}{6(a+b)}$; $I_{OY} = M\frac{3a^2 - ab + 3b^2}{6}$

28. (i) $\frac{w_0x^3}{6a}$; (ii) $\frac{w_0x^4}{12a^2}$

31. $\frac{\pi q_0{}^2LR^2}{4N}$ **32.** As in No. 31

33. On central radius at distance $\frac{2r\sin\alpha}{3\alpha}$ from centre of circle.

34. $\dfrac{r^4}{6}(9\alpha - 8\sin\alpha)$

35. $\frac{2}{3}c^2$ **36.** $\frac{1}{3}a^5$ **37.** $\frac{1}{15}$

38. $0;\ \dfrac{\pi a^3 b^3}{24}$ **39.** $\dfrac{a^4}{8}$ **40.** $18\,880;\ \dfrac{\pi}{4}$

41. Centroid on central radius at distance $\dfrac{4r}{3\pi}$ from centre.

42. $\bar{x} = \dfrac{3k}{20}, \bar{y} = \dfrac{3k}{16}$ **43.** $\bar{x} = \dfrac{9a}{20}, \bar{y} = \dfrac{9a}{20}$ **44.** $\bar{x} = \dfrac{\pi}{2}, \bar{y} = \dfrac{\pi}{8}$

45. Radius of gyration $= \sqrt{\dfrac{3r^2}{5}} = 0{\cdot}7746r$

46. $\dfrac{a^3}{18}(3\pi - 4) = 0{\cdot}3014a^3;\ \dfrac{1}{ab}[\sin a\pi + \sin b\pi - \sin(a+b)\pi]$

47. Let $ABCD$ be the end, A being the highest point and AD being an edge of length 4 ft. Total fluid thrust on $ABCD = 12(2\sqrt{3} + 3)w = 4\,849$ lb, where w = weight of 1 ft³ of water = 62·5 lb. Taking the axes along AB and AD respectively, we find that $\bar{\xi}$ and $\bar{\eta}$, the co-ordinates of the centre of pressure, are given by

$$\bar{\xi} = 2\sqrt{3} = 3{\cdot}464 \text{ ft and } \bar{\eta} = \tfrac{2}{3}(7 - 2\sqrt{3}) = 2{\cdot}357 \text{ ft}$$

48. $\dfrac{1}{3};\ \dfrac{\pi}{4};$ Vol. $= c\displaystyle\int_0^b\int_0^{\frac{a\sqrt{b^2-y^2}}{b}}\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)dx\,dy = \frac{abc}{4}\left(\pi - \frac{13}{6}\right)$

49. Let the oblique plane be parallel to the x-axis; then

$$\bar{x} = 0,\ \bar{y} = \frac{a(p-q)}{4(p+q)},\ \bar{z} = \frac{5p^2 + 6pq + 5q^2}{16(p+q)}$$

50. Vol. $= \frac{4}{3}\pi r^3$

51. Centroid on radius perpendicular to circular base and at distance $\frac{3}{8}r$ from centre.

52. $4\frac{1}{8}$ in. **53.** $\dfrac{4(a^2 - b^2)}{3\pi b}$

54. (*a*) If x is the depth immersed, the metacentric height

$$= \frac{3}{4h^2}[x(r^2 + h^2) - h^3]$$

(*b*) Metacentric height $= 0$

(*c*) If h is the depth immersed, the metacentric height

$$= \frac{k}{2} - \frac{2}{3}(H - h)$$

55. $\bar{x} = \dfrac{4a}{3\pi}, \bar{y} = \dfrac{4b}{3\pi}$ **56.** $22{\cdot}5\pi$ **57.** $\sqrt{\dfrac{2}{5}a^2} = 0{\cdot}6325a$

58. Centre of pressure on vertical axis of symmetry and at depth $2\frac{2}{3}$ ft.

59. Centre of pressure is at a depth $\dfrac{r^2}{4h}\cos^2\alpha + h$ below the surface and lies on the line through the centre perpendicular to the line in which the plane of the circle cuts the surface.

60. $\sigma = 0{\cdot}5$. (i) Metacentric height $= \dfrac{l^2 - 4r^2}{3\pi r}$; (ii) Metacentric height $= 0$

64. $4\pi^2 ac$ **65.** $\frac{1}{3}a^4$ **66.** $\dfrac{\pi^2}{2}$

EXAMPLES III. Page 94

1. $1 - \frac{1}{2}a^2\sin^2 t - \frac{1}{8}a^4\sin^4 t - \frac{1}{16}a^6\sin^6 t = A + B\cos 2t + C\cos 4t + D\cos 6t$, where $A = \dfrac{1}{256}(256 - 64a^2 - 12a^4 - 5a^6)$; $B = \dfrac{1}{512}(128a^2 + 32a^4 + 15a^6)$; $C = -\dfrac{1}{512}(8a^4 + 6a^6)$; $D = \dfrac{1}{512}a^6$

2. $f(x) = \dfrac{a}{2} + \dfrac{2a}{\pi}\left(\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right)$

3. $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\dfrac{1}{1^2}\cos x + \dfrac{1}{3^2}\cos 3x + \dfrac{1}{5^2}\cos 5x + \ldots\right)$

4. $f(x) = \dfrac{4a}{\pi}\left(\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right)$

5. $$f(x) = 4\left[\frac{\pi^2}{3} + \frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \ldots - \pi\left(\tfrac{1}{1}\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \ldots\right)\right]$$

6. $$f(x) = \frac{1 - e^{-2\pi}}{\pi}\left[\frac{1}{2} + \frac{\cos x}{1^2+1} + \frac{\cos 2x}{2^2+1} + \frac{\cos 3x}{3^2+1} + \ldots + \frac{\sin x}{1^2+1} + \frac{2\sin 2x}{2^2+1} + \frac{3\sin 3x}{3^2+1} + \ldots\right]$$

7. $$f(x) = \tfrac{1}{3}a^2 - \frac{4a^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{a} - \frac{1}{2^2}\cos\frac{2\pi x}{a} + \frac{1}{3^2}\cos\frac{3\pi x}{a} - \ldots\right]$$

8. $$f(x) = (e^a - e^{-a})\left\{\frac{1}{2a} - a\left(\frac{1}{a^2+\pi^2}\cos\frac{\pi x}{a} - \frac{1}{a^2+4\pi^2}\cos\frac{2\pi x}{a} + \frac{1}{a^2+9\pi^2}\cos\frac{3\pi x}{a} - \ldots\right) - \pi\left(\frac{1}{a^2+\pi^2}\sin\frac{\pi x}{a} - \frac{2}{a^2+4\pi^2}\sin\frac{2\pi x}{a} + \frac{3}{a^2+9\pi^2}\sin\frac{3\pi x}{a} - \ldots\right)\right\}$$

$$e^x = 2\pi(1+e)\left[\frac{1}{1+1^2\pi^2}\sin \pi x + \frac{3}{1+3^2\pi^2}\sin 3\pi x\right.$$

$$\left. + \frac{5}{1+5^2\pi^2}\sin 5\pi x + \ldots\right]$$

$$+ 2\pi(1-e)\left[\frac{2}{1+2^2\pi^2}\sin 2\pi x + \frac{4}{1+4^2\pi^2}\sin 4\pi x\right.$$

$$\left. + \frac{6}{1+6^2\pi^2}\sin 6\pi x + \ldots\right]$$

10. $$e^{-x} = \frac{1}{a}(1-e^{-a}) + 2a(1+e^{-a})\left[\frac{1}{a^2+1^2\pi^2}\cos\frac{\pi x}{a}\right.$$

$$\left. + \frac{1}{a^2+3^2\pi^2}\cos\frac{3\pi x}{a} + \frac{1}{a^2+5^2\pi^2}\cos\frac{5\pi x}{a} + \ldots\right]$$

$$+ 2a(1-e^{-a})\left[\frac{1}{a^2+2^2\pi^2}\cos\frac{2\pi x}{a}\right.$$

$$\left. + \frac{1}{a^2+4^2\pi^2}\cos\frac{4\pi x}{a} + \frac{1}{a^2+6^2\pi^2}\cos\frac{6\pi x}{a} + \ldots\right]$$

11. $$f(x) = \frac{2c}{\pi}\left[\tfrac{1}{1}\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \ldots\right.$$

$$\left. + 2\left(\tfrac{1}{2}\sin 2x + \tfrac{1}{6}\sin 6x + \tfrac{1}{10}\sin 10x + \ldots\right)\right]$$

12. (i) $$f(x) = \frac{3\sqrt{3}l}{2\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{l} + \frac{1}{2^2}\sin\frac{2\pi x}{l} - \frac{1}{4^2}\sin\frac{4\pi x}{l} - \frac{1}{5^2}\sin\frac{5\pi x}{l}\right.$$

$$\left. + \frac{1}{7^2}\sin\frac{7\pi x}{l} + \frac{1}{8^2}\sin\frac{8\pi x}{l} - \ldots\right]$$

(ii) $$f(x) = \frac{l}{6} + \frac{l}{\pi^2}\left[\sum_{(n \text{ odd})}\left(\frac{1}{n^2}\right)\left(3\cos\frac{n\pi}{3} - 1\right)\cos\frac{n\pi x}{l}\right.$$

$$\left. + 3\sum_{(n \text{ even})}\left(\frac{1}{n^2}\right)\left(\cos\frac{n\pi}{3} - 1\right)\cos\frac{n\pi x}{l}\right]$$

13. $$54{\cdot}167 + 26{\cdot}928\cos\frac{\pi x}{6} + 16\cos\frac{\pi x}{3} + 14\cos\frac{\pi x}{2} + 13{\cdot}333\cos\frac{2\pi x}{3}$$

$$+ 13{\cdot}072\cos\frac{5\pi x}{6} + 13\cos\pi x - 44{\cdot}784\sin\frac{\pi x}{6} - 20{\cdot}784\sin\frac{\pi x}{3}$$

$$- 12\sin\frac{\pi x}{2} - 6{\cdot}928\sin\frac{2\pi x}{3} - 3{\cdot}216\sin\frac{5\pi x}{6}$$

14. $f(x) = 2\pi^3 + 12\pi\left[\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \ldots\right]$

$$+ 4\left[\left(\frac{3}{1^3} - \frac{2\pi^2}{1}\right)\sin x + \left(\frac{3}{2^3} - \frac{2\pi^2}{2}\right)\sin 2x + \left(\frac{3}{3^3} - \frac{2\pi^2}{3}\right)\sin 3x + \ldots\right]$$

15. (i) $f(x) = \frac{2c}{\pi}\left[\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots + 2\left(\frac{1}{2}\sin 2x + \frac{1}{6}\sin 6x + \frac{1}{10}\sin 10x + \ldots\right)\right]$

(ii) $f(x) = \frac{2c}{\pi}\left[\frac{\pi}{4} + \frac{1}{1}\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \ldots\right]$

16. $a_n = \frac{8c}{n^2\pi^2}\sin\frac{n\pi}{2}$, so that

$$y = \frac{8c}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l} + \frac{1}{5^2}\sin\frac{5\pi x}{l} - \ldots\right]$$

17. $f(x) = \frac{2c}{\pi}\left[\frac{1}{1}\sin x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \frac{1}{11}\sin 11x + \frac{1}{13}\sin 13x + \ldots - 2\left(\frac{1}{3}\sin 3x + \frac{1}{9}\sin 9x + \frac{1}{15}\sin 15x + \ldots\right)\right]$

18. $T = 7\,846\sin\theta + 1\,507\sin 2\theta - 9{\cdot}19\sin 3\theta + 11{\cdot}55\sin 4\theta - 13{\cdot}67\sin 5\theta + 13{\cdot}33\sin 6\theta \ldots$ All but 1st and 2nd terms negligible.

19. $y = 216{\cdot}8 - 76{\cdot}15\cos\frac{\pi x}{6} - 25\cos\frac{\pi x}{3} - 11{\cdot}83\cos\frac{\pi x}{2} - 5{\cdot}83\cos\frac{2\pi x}{3}$

$$- 6{\cdot}02\cos\frac{5\pi x}{6} - 4\cos\pi x - 19{\cdot}87\sin\frac{\pi x}{6} - 1{\cdot}44\sin\frac{\pi x}{3} + 0{\cdot}17\sin\frac{\pi x}{2} + 0{\cdot}58\sin\frac{2\pi x}{3} + 0{\cdot}04\sin\frac{5\pi x}{6}$$

20. $f(x) = 1{\cdot}571 - 1{\cdot}303\cos x - 0{\cdot}175\cos 3x - 0{\cdot}094\cos 5x - 0{\cdot}094\cos 7x \ldots$

21. $x = 3{\cdot}9375 + \cos\theta + 0{\cdot}0625\cos 2\theta$, where θ is the crank angle.

22. $48{\cdot}45 - 17{\cdot}70\cos x - 3{\cdot}12\cos 2x - 0{\cdot}35\cos 3x + 0{\cdot}30\cos 4x + 0{\cdot}75\cos 5x + 0{\cdot}73\cos 6x - 2{\cdot}23\sin x - 1{\cdot}67\sin 2x - 1{\cdot}75\sin 3x - 0{\cdot}95\sin 4x - 0{\cdot}52\sin 5x$

23. $6{\cdot}273 - 1{\cdot}987\cos x - 0{\cdot}671\cos 2x - 0{\cdot}208\cos 3x - 0{\cdot}038\cos 4x + 0{\cdot}005\cos 5x - 0{\cdot}048\cos 6x + 0{\cdot}715\sin x - 0{\cdot}694\sin 2x - 0{\cdot}307\sin 3x - 0{\cdot}160\sin 4x - 0{\cdot}131\sin 5x$

24. $-1{\cdot}475 + 6{\cdot}045\cos x + 0{\cdot}533\cos 2x + 0{\cdot}067\cos 3x - 0{\cdot}250\cos 4x - 0{\cdot}162\cos 5x - 0{\cdot}117\cos 6x - 4{\cdot}180\sin x + 0{\cdot}289\sin 2x + 0{\cdot}283\sin 3x - 0{\cdot}029\sin 4x + 0{\cdot}063\sin 5x$

25. $A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$; $A_r = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos rx\,dx$; $B_r = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin rx\,dx$; $f(x) = x = \pi - 2\,[\frac{1}{1}\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots]$

27. $f(x) = \frac{4}{\pi}\left[\frac{1}{2} - \frac{1}{1\times 3}\cos\frac{2\pi x}{c} - \frac{1}{3\times 5}\cos\frac{4\pi x}{c} - \frac{1}{5\times 7}\cos\frac{6\pi x}{c} - \ldots\right]$

28. $$F(x) = l\left[\frac{1}{4} + \frac{2}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \ldots\right) + \frac{1}{\pi}\left(\frac{1}{1}\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \ldots\right)\right]$$

29. $$f(\theta) = \frac{2\sqrt{2}h}{\pi}\,[\tfrac{1}{1}\sin\theta - \tfrac{1}{3}\sin 3\theta - \tfrac{1}{5}\sin 5\theta + \tfrac{1}{7}\sin 7\theta + \tfrac{1}{9}\sin 9\theta - \tfrac{1}{11}\sin 11\theta - \tfrac{1}{13}\sin 13\theta + \ldots]$$

32. $f(x) = \frac{4a}{\pi}\left(\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \ldots\right)$

34. $f(x) = \frac{a}{2} - \frac{16a}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{c} + \frac{1}{6^2}\cos\frac{6\pi x}{c} + \frac{1}{10^2}\cos\frac{10\pi x}{c} + \ldots\right)$

35. $f(x) = \frac{72}{\pi^3}\left(\sin\frac{\pi x}{3} + \frac{1}{3^3}\sin\frac{3\pi x}{3} + \frac{1}{5^3}\sin\frac{5\pi x}{3} + \ldots\right)$

36. $f(x) = -\frac{4a}{\pi}\left(\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \ldots\right)$

37. $f(x) = \frac{2}{\pi}\left(\sin x - \frac{2}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x - \ldots\right)$

38. $\pm\frac{1}{4p^2}$, according as p is even or odd.

39. $f(x) = \frac{2\sqrt{3}}{3}(\cos x - \frac{1}{5}\cos 5x + \frac{1}{7}\cos 7x - \frac{1}{11}\cos 11x + \frac{1}{13}\cos 13x - \ldots)$

40. $f(x) = \frac{8}{\pi}\left[\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \ldots\right]$

41. $f(x) = \frac{c}{2} + \frac{c}{\pi}(\sin\alpha\cos x + \frac{1}{2}\sin 2\alpha\cos 2x + \frac{1}{3}\sin 3\alpha\cos 3x + \ldots)$

42. $$x(x-1)(x-2) = \frac{12}{\pi^3}\left(\sin\pi x + \frac{1}{2^3}\sin 2\pi x + \frac{1}{3^3}\sin 3\pi x + \ldots + \frac{1}{n^3}\sin n\pi x + \ldots\right)$$

EXAMPLES IV. Page 140

4. (ii) $x = \dfrac{a(3 + \cos\theta)(7 + 3\cos\theta)}{2(5 + 3\cos\theta)}$; $y = \dfrac{3a\sin\theta\,(1 + \cos\theta)}{2(5 + 3\cos\theta)}$

5. (*a*) $\sqrt[6]{2}\left(\cos\dfrac{r\pi}{12} + i\sin\dfrac{r\pi}{12}\right)$ where $r = 1, 9,$ or 17; (*b*) $-2,\ 1 \pm i\sqrt{3}$

6. (i) $\dfrac{1}{\sqrt{2}}(\pm 1 \pm i)$; (ii) $\cos\dfrac{2n\pi}{5} + i\sin\dfrac{2n\pi}{5}$, where $n = 0, 1, 2, 3, 4$;

(iii) $-\frac{1}{2}, -\frac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$

7. $\sin ix = i\sinh x$, $\cos ix = \cosh x$

14. 1·023; 0·5282

15. 0·7646 [6° 54′]; 0·2299 [91° 58′]

16. (i) $\sin x\cosh y - i\cos x\sinh y$; (ii) $\cos x\cosh y + i\sin x\sinh y$;

(iii) $\dfrac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$; (iv) $\dfrac{\sinh 2x - i\sin 2y}{\cosh 2x + \cos 2y}$

(i) $u = 1{\cdot}298$, $v = -0{\cdot}6349$; (ii) $u = 0{\cdot}8335$, $v = 0{\cdot}9891$;

(iii) $u = 0{\cdot}2717$, $v = 1{\cdot}084$; (iv) $u = 1{\cdot}084$, $v = -0{\cdot}2717$

18. (i) $0{\cdot}6 + i0{\cdot}8$; (ii) $\pm\dfrac{1}{\sqrt{2}}(1 + i)$; (iii) $\dfrac{1}{\sqrt{2}}(\cosh\frac{1}{2} - i\sinh\frac{1}{2})$

Semi-axes $r \pm \dfrac{1}{r}$; when $\theta = \dfrac{\pi}{4}$, locus is rectangular hyperbola $x^2 - y^2 = 2$

19. (i) $\log 2 + i2n\pi$; (ii) $\log 2 + i\tan^{-1}\sqrt{3}$; (iii) $\log\sqrt{\pi^2 + 2} - i\tan^{-1}\dfrac{\sqrt{2}}{\pi}$

(iv) $\log\sqrt{5} + i(\pi - \tan^{-1} 2)$; (v) $\log 5 + i(\pi + \tan^{-1}\frac{4}{3})$;

(vi) $\sqrt{2}\,e^{(2n+\frac{1}{4})\pi}[\cos\{(2n + \frac{1}{4})\pi - \log\sqrt{2}\} + i\sin\{(2n + \frac{1}{4})\pi - \log\sqrt{2}\}]$;

(vii) $e^{-2n\pi}[\cos(\log 3) + i\sin(\log 3)]$

22. (*a*) $\dfrac{\sinh 2x + i\sin 2y}{\cosh 2x + \cos 2y}$

28. (1) Interior of circle, centre at origin, radius a;
(2) Interior of circle, centre at z_1, radius a;
(3) Circumference and interior of circle of unit radius, centre at point $2 + i0$;
(4) Half z-plane on positive side of y-axis, exclusive of that axis;
(5) Exterior of circle, centre at point $\frac{1}{4} + i0$, radius $\frac{3}{4}$

31. (i) Straight line from origin;
(ii) Circle, centre at origin, radius = given constant;
(iii) Circle, centre at point z_1, radius = given constant;
(iv) Ellipse with points z_1 and z_2 as foci;
(v) Circle, and if the constant has different values, a family of coaxial circles having z_1 and z_2 as limiting points. When constant = 1, locus is perpendicular bisector of straight line joining z_1 to z_2.

32. (i) $z_3 = 6 + i1$, $z_4 = 3 + i5$, or $z_3 = -2 - i5$, $z_4 = -5 - i1$
(ii) $z_2 = 0 - i3$, $z_4 = 4 + i5$

33. (i) Yes; (ii) No; (iii) No; (iv) Yes; (v) Yes.

34. (i) $Z = iz$; (ii) $Z = \cosh z$; (iii) $Z = z$; (iv) $Z = \dfrac{x - iy}{x^2 + y^2} = \dfrac{1}{z}$

36. (i) $u = ax + b$, $v = ay$; $u = c$ gives family of straight lines $x = \dfrac{c - b}{a}$ parallel to y-axis, and $v = k$ gives family of straight lines $y = \dfrac{k}{a}$ parallel to x-axis

(ii) $u = \dfrac{x - 1}{x^2 + y^2 - 2x + 1}$, $v = -\dfrac{y}{x^2 + y^2 - 2x + 1}$; $u = c$ gives family of circles $x^2 + y^2 - \left(2 + \dfrac{1}{c}\right)x + 1 + \dfrac{1}{c} = 0$, and $v = k$ gives family of circles $x^2 + y^2 - 2x + \dfrac{y}{k} + 1 = 0$

(iii) $u = \dfrac{x}{x^2 + y^2 + 2y + 1}$, $v = -\dfrac{y + 1}{x^2 + y^2 + 2y + 1}$; $u = c$ gives family of circles $x^2 + y^2 - \dfrac{x}{c} + 2y + 1 = 0$, and $v = k$ gives family of circles $x^2 + y^2 + \left(2 + \dfrac{1}{k}\right)y + 1 + \dfrac{1}{k} = 0$

38. Semicircular arc, radius 1, centre at origin, above real axis

39. $a = \dfrac{x(x^2 + y^2 - 2y - 1)}{x^2 + (y - 1)^2}$, $b = \dfrac{2x^2 + y(x^2 + y^2 - 1)}{x^2 + (y - 1)^2}$; outside of circle, radius $\sqrt{2}$, centre at point $0 + i1$

41. $r^2 = 2 \cos 2\theta$

50. The straight line $u = \log k$ between the points for which $v = -\pi$ and $v = +\pi$

51. $\dfrac{2ak}{a^2 + k^2}$

53. (i) $Z = \dfrac{1 - z + i}{z + i}$; (ii) $Z = (1 + i)\dfrac{z - 1}{z + 1}$

EXAMPLES V. Page 175

3. $\sqrt{3}$ units at $\tan^{-1}\dfrac{1}{\sqrt{3}}$ to BC

4. (i) 5 ft/sec at 53° 8′ to BA; (ii) 16 ft; (iii) 2·4 sec

5. $(1 - n)\mathbf{r}_1 + n\mathbf{r}_2$; $(1 - k)\mathbf{r}_2 + k\mathbf{r}_1$

6. Straight line through B and mid-point of diagonal AC

7. (i) 5·795; (ii) 766; (iii) 12·25

10. 3 501 ft-lb; 583·5 ft-lb/sec

11. 17 units of work

14. (i) 68° 12′, 56° 9′, 42° 2′; 68° 59′, 53° 18′, 44° 10′. (ii) 70° 33′

16. $\mathbf{a} = -\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$, $\mathbf{b} = -6\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$; -2, $20\mathbf{i} + 44\mathbf{j} + 16\mathbf{k}$

17. 11·18 ft/sec in direction $\tan^{-1}(-\frac{2}{11})$ to direction of $\mathbf{i}$

18. $7\mathbf{i} - 14\mathbf{j} + 14\mathbf{k}$; 70° 32′, 131° 49′, 48° 11′

19. $-3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$; $5\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$; $-2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$; 4·690, 7·681, 7

20. $\dfrac{5a}{14}$ from OD, and $\dfrac{9\sqrt{3}a}{14}$ from OA

21. $\bar{x} = \frac{1}{11}$, $\bar{y} = \frac{1}{3}$, $\bar{z} = 3\frac{2}{3}$

22. $\dfrac{7\sqrt{26}}{78} = 0{\cdot}4576$; $\dfrac{1}{\sqrt{185}}(7\mathbf{i} - 6\mathbf{j} - 10\mathbf{k})$

23. (i) 39·78; (ii) 0·8184

24. $6(2\mathbf{i} + 6\mathbf{j} + \mathbf{k})$

25. 3·307; 11·54

26. $-7\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$; $\dfrac{2\sqrt{3}}{3}$; $\dfrac{\sqrt{3}}{3}(-\mathbf{i} + 12\mathbf{j} - 11\mathbf{k})$

31. 10·96 ft/sec in direction of vector $3\mathbf{i} + 8\mathbf{j} + 10\mathbf{k}$

35. 151·4 ft/sec

EXAMPLES VI. Page 229

3. $28e^{-2x}$; $10x^3e^{4x}(26x^2 + 35x + 10)$; $-302{\cdot}9e^{-x}\sin(7x - 0{\cdot}5882)$; $-48e^{-3x}\cos 3x$

4. $949\cos 3t$; $-13\sin(3x + 1{\cdot}176)$

5. $0{\cdot}2236e^{4x}\sin(2x - 0{\cdot}4636)$; $\frac{1}{4}e^{2x}(2x^2 - 2x + 1)$;
$\dfrac{1}{625}e^{5x}(125x^3 - 75x^2 + 30x - 6)$; $\sqrt{\dfrac{a^2 + b^2}{k^2 + p^2}}\,e^{kt}\sin\left(pt + \tan^{-1}\dfrac{b}{a} - \tan^{-1}\dfrac{p}{k}\right)$

6. $\frac{7}{4}e^{-2x}$; $\frac{5}{32}e^{4x}(x - \frac{7}{20})$; $-0{\cdot}1193e^{-x}\sin(7x + 0{\cdot}5882)$; $-\frac{4}{3}e^{-3x}\cos 3x$

7. $\sqrt{13}e^{3x}\cos(2x + 3 + \tan^{-1}\frac{2}{3})$; $2\,197e^{3x}\cos(2x + 3 + 6\tan^{-1}\frac{2}{3})$;
$\frac{1}{2\,197}e^{3x}\cos(2x + 3 - 6\tan^{-1}\frac{2}{3})$; $\frac{1}{\sqrt{13}}e^{2x}\cos(3x - \tan^{-1}\frac{3}{2})$;
$\frac{1}{13\sqrt{13}}e^{2x}\cos(3x - 3\tan^{-1}\frac{3}{2})$; $13\sqrt{13}e^{2x}\cos(3x + \tan^{-1}\frac{3}{2})$;
$\frac{1}{27}(18x^2 + 9x - 4)\sin 3x + \frac{1}{9}(4x + 1)\cos 3x$

8. $y = Ae^{-x} + 2x - 2$; $y = Ae^{6x} - \frac{7}{6}$; $y = Ae^{2x} - \frac{6\sqrt{13}}{13}\sin(3x + \tan^{-1}\frac{3}{2})$;
$y = Ae^{-ax} + \frac{b}{a}$; $y = Ae^{-ax} + \frac{b}{\sqrt{a^2 + p^2}}\sin\left(px + q - \tan^{-1}\frac{p}{a}\right)$

9. $y = Ae^{2x} + Be^{3x}$; $y = Ae^{-x} + Be^{-4x}$; $x = Ae^{2t}\sin(2\sqrt{2}t + q)$;
$x = (At + B)e^{t}$; $y = (At + B)e^{-at}$

10. $y = (A + Bx)e^{-3x}$; $y = (A + Bx + Cx^2)e^{2x}$

11. $y = Ae^{-3x}\sin(2x + q)$; $y = Ae^{-2x} + B\sin(3x + q)$;
$y = Ae^{-3x} + Be^{-4x} + (Cx + E)e^{-5x}$

12. $y = (A + Bx)\sin 3x + (C + Ex)\cos 3x$; $x = Ae^{3t} + Be^{-3t} + Ce^{2t} + Ee^{-2t}$

13. $x = e^{3t} + \frac{15}{7}e^{-2t} + \frac{18}{7}e^{\frac{1}{2}t}$

14. (i) $y = (A - 0{\cdot}2\sin 2x - 0{\cdot}1\cos 2x)e^{-2x} + Be^{-3x}$;
(ii) $y = e^{-ax}\left(A + Bx + \frac{x^4}{12}\right)$

15. (i) $y = (1 - x^2)(C - \log_e\sqrt{1 - x^2})$; (ii) $y - 2x = c(y - x)^2$;
(iii) $xy^2 = c - \cos x$

16. (i) $y = (1 + C\cos x)\cos x$; (ii) $(x + y)^2(2x + y)^3 = c$

17. (*a*) $y = \frac{1}{5}e^{-x}(7\sin x + 12\cos x) + \frac{1}{5}(\sin x - 2\cos x)$;
(*b*) $y = e^{x}(2 - 3x + \frac{1}{2}x^2)$

18. $y = Re^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x + q\right) + 3\cos x + 4\cos 2x$

19. $y = (1 + 6t + t^4)e^{-5t}$; $y = A + B\sin(2t + q) - 2\cos t$

20. $x = Ae^{4t} + Be^{-\frac{1}{2}t} - \frac{1}{27}e^{t}(3t^2 - 2t + 2)$;
$y = e^{-2{\cdot}5x}(A\sin 5{\cdot}455x + B\cos 5{\cdot}455x) + \frac{\sqrt{2}e^{-2x}}{10}\sin\left(5x - \frac{\pi}{4}\right)$

21. (i) $y = A \sin 4x + B \cos 4x - \frac{1}{2}x \cos 4x$;

(ii) $\theta = Re^{-2t} \sin (2t + q) + \dfrac{3\sqrt{5}}{10} \cos (2t - \tan^{-1} 2)$

(iii) $y = (A - \frac{3}{7}x)e^{-6x} + (B + \frac{1}{7}x)e^{x}$

22. (i) $x = y(c - \log_e x)$; (ii) $y = \frac{5}{6}x^3 + e^x + ax + b$; (iii) $y^2 = \dfrac{c}{x^3} - x$

23. (i) $y = A \sin (3x + \alpha) + \frac{1}{81} (9x^2 + 9x + 7)$;
(ii) $y = c \sec x + \frac{1}{2}e^x(1 + \tan x)$

24. (a) $13x = 8(e^{-t} - 1) \cos 4t + \left(1 - e^{\frac{\pi}{8} - t}\right) \sin 4t$;
(b) $y = Ae^x + Be^{3x} - (x^2 + 7)e^{2x}$

25. (i) $y = Ae^{-x} + Be^{-2x} - 0{\cdot}1e^{-x} (2 \sin 2x + \cos 2x)$; (ii) $y = \tan x$

26. (i) $y = \tan \left(x + \dfrac{\pi}{4}\right)$; (ii) $y \sin x = \frac{1}{6} (3 \sin x - \sin 3x - 4)$;
(iii) $y = e^{-\beta x} [(1 + x)^{\beta} - 1]$

27. (i) $y = A + B \sin 2x + C \cos 2x - 2 \cos x$;
(ii) $y = Ae^x + Be^{-x} + C \sin x + D \cos x - \frac{1}{5} \cos x \cosh x$;
(iii) $y = e^{-ax}(1 + ax + x^4)$

28. (i) $xy - 2 = (x + 1) [C + 2 \log_e (1 + x)]$;
(ii) $y = \dfrac{(3 - 8x^2) \sin 2x - 8x \cos 2x}{256}$

29. $y = \sin 2x + \cos 2x - \cos x$

30. (i) $y = 1 + x + \dfrac{c}{x^3}$; (ii) $y^2 + 4x + 2 = Ce^{2x}$

31. (i) $xy = C + \log_e x$; (ii) $x^2 - 2xy - y^2 + 4x + 4y + c = 0$

32. (i) $9y = 6x - 2 + ce^{-3x}$; (ii) $x^2 + xy + y^2 - 2x - 4y + c = 0$

33. (i) $y = e^x(A + Bx - \cos x)$;
(ii) $y = Ax + B\sqrt{x^2 - 1} + \frac{1}{2}\sqrt{x^2 - 1} \cosh^{-1}x$

35. (i) $y = Ae^{4x} + Be^{-0{\cdot}5x} - \frac{1}{7}e^{3x}$; (ii) $\tan^{-1} \dfrac{y}{x} + \log_e(k\sqrt{x^2 + y^2}) = 0$

36. (i) $y = Ae^{-3x} + (B - 10x)e^{-4x}$; (ii) $y = e^{2x}(Ax + B) + \frac{1}{8}(2x^2 + 4x + 3)$

37. (i) $y = Ae^{-x} + Be^{-2x} - \dfrac{e^{-x}}{\sqrt{2}} \cos \left(x + \dfrac{\pi}{4}\right)$;
(ii) $y = Ce^{-\tan x} - 3(1 - \tan x)$

38. $\dfrac{2\pi l}{d}\sqrt{\dfrac{Wl}{3gbdE}}$ sec; $\pi\sqrt{\dfrac{Wl}{3gbdE}}$ sec

39. $\dfrac{30}{\pi k}\sqrt{\dfrac{C_0 g}{W}}$ vibrations per min

42. 0·505 ft; 0·245 radian

43. $M = 6g$; $c = \frac{1}{12}$

44. 0·453 sec; displacement $= 0{\cdot}403 \sin 7t - 0{\cdot}203 \sin 13{\cdot}90t$

45. $x = 0{\cdot}375 \sin (7t - 0{\cdot}3732)$

47. $x = 0{\cdot}8(2 \sin 4t - \cos 4t)$

48. $\dfrac{5W}{g} \sin 3t$ lb; $0{\cdot}0511 \sin (3t - 0{\cdot}377)$ ft

51. (2) is the boundary between cases (1) and (3) and represents critical damping.

52. $i = \dfrac{\omega C V_0}{1 - \omega^2 LC} \cos \omega t$

53. $y = A \sin (nt + \alpha) + \dfrac{a}{n^2 - p^2} \sin (pt + q)$

55. (i) $I = Ae^{\frac{1}{2L}(-R+r)t} + Be^{\frac{1}{2L}(-R-r)t} + \dfrac{pEC}{\sqrt{(1-p^2CL)^2 + C^2R^2p^2}}$

$$\cos\left(pt - \tan^{-1}\frac{pCR}{1-p^2CL}\right), \text{ where } r^2 = \frac{CR^2 - 4L}{C}$$

(ii) $I = Ae^{-\frac{R}{2L}t} \sin\left(\sqrt{\dfrac{4L - CR^2}{C}}\, t + \alpha\right)$ + last term as in (i)

58. $$x = A \sin\left\{\sqrt{\left(\frac{3-\sqrt{5}}{2}\right)\frac{p}{m}}\cdot t + \alpha\right\} + B \sin\left\{\sqrt{\left(\frac{3+\sqrt{5}}{2}\right)\frac{p}{m}}\cdot t + \beta\right\}$$

$$y = \frac{\sqrt{5}-1}{2} A \sin\left\{\sqrt{\left(\frac{3-\sqrt{5}}{2}\right)\frac{p}{m}}\cdot t + \alpha\right\} - \frac{\sqrt{5}+1}{2} B \sin\left\{\sqrt{\left(\frac{3+\sqrt{5}}{2}\right)\frac{p}{m}}\cdot t + \beta\right\}$$

x and $x + y$ give the displacements of the upper and lower masses respectively from their equilibrium positions.

61. (i) $y = x(A \log_e x + B)$; (ii) $y = x(A \log_e x + B) + \frac{1}{4}x^3$

62. $u = kr(a^2 - r^2)/8$

63. (i) $y = \frac{1}{2r}[(a^2 - b^2)r^2 - a^2b^2 - r^4]$; (ii) $y = Ar + \frac{B}{r} + \frac{1}{5}r^4 - \frac{2}{r}\log_e r$

64. (i) $y = \frac{1}{x}[A \sin(\log_e x) + B \cos(\log_e x)] + \frac{5}{2}$

(ii) $y = \frac{A}{x} + \frac{B}{x^4} + \frac{1}{4}\log_e x - \frac{5}{16}$

(iii) $y = x(A \log_e x + B) + \frac{x}{6}(\log_e x)^3$

(iv) $y = x(A \log_e x + B) + \frac{C}{x^2}$

65. (i) $y = A\sqrt{x}\cos[\sqrt{3}\log_e(k\sqrt{x})] + \frac{5}{13}x^4$

(ii) $y = \frac{1}{72}\left(12x^2 \log_e x + \frac{9}{x^2} - \frac{4}{x^4} - x^2\right)$

66. $y = x\left(A + Be^{-\frac{1}{2x}}\right)$

67. $n = -2$; $x^2y = A + Be^{-4x} + \frac{1}{17}(4 \sin x - \cos x)$

68. (i) $x^2 + y^2 = A(x + y)$; (ii) $y = A \sec^2(x + \alpha)$

69. $y = Ar + \frac{B}{r}$; $y = Ar + \frac{B}{r} + \frac{1}{5}r^4 - \frac{2}{r}\log_e r$

70. (i) $x = t^2$; (ii) $y = A + B\log_e x + C(\log_e x)^2 + \frac{x^3}{27}(\log_e x - 1)$

72. (*a*) $y = \frac{1}{4}\left(x^3 + \frac{3}{x}\right)$; (*b*) $(x^2 - y + c)(x^3 + 3y + 3c) = 0$

73. (i) $(y - cx)(x^2y - c) = 0$; (ii) $(x^2 - 2y - c)(x + y - ce^{-x} - 1) = 0$

74. (i) $[4y - x^2 - x\sqrt{x^2 - 4} + 4\log_e(x + \sqrt{x^2 - 4}) - c] \times$
$[4y - x^2 + x\sqrt{x^2 - 4} - 4\log_e(x + \sqrt{x^2 - 4}) + c] = 0$

(ii) $y = \dfrac{3 + e^{\sqrt{2}x}}{3 - e^{\sqrt{2}x}}$

75. (i) With p as parameter, $x = cp^{-3}e^{\frac{1}{2p^2}}$ and $y = cp^{-2}(1 + p^2)e^{\frac{1}{2p^2}}$

(ii) $y = \frac{1}{16}[(2x^2 - 3)\cos x - 6x \sin x]$

77. $31{\cdot}62 \sin(2t + 1{\cdot}322)$; $\cos(t + 2{\cdot}927)$; $3{\cdot}606 \sin(4t + 0{\cdot}588)$; $\sin(3t + 2{\cdot}983)$

78. $1{\cdot}280E_m \sin(500t + 0{\cdot}0189)$

EXAMPLES VII. Page 283

1.

x	1·0	1·1	1·2	1·3	1·4	1·5
y	2·000	2·200	2·431	2·697	3·001	3·347

2. 2·000, 2·216, 2·466, 2·753, 3·081, 3·455;

$y = 28e^{\frac{1}{4}(x-1)} - 2x^2 - 8x - 16$;

3·455 by modified method, 3·453 by exact method.

3. 1·300, 1·620, 1·959, 2·319, 2·699, 3·100, 3·523, 3·967, 4·434, 4·924;
1·310, 1·640, 1·991, 2·362, 2·754, 3·168, 3·604, 4·063, 4·545, 5·051

4.

x	1·0	1·2	1·4	1·6	1·8	2·0
y	1·000	1·640	2·362	3·169	4·064	5·052

5. 2·465, 3·080, 3·877, 4·892, 6·165

6.

	x	2·0	2·1	2·2	2·3	2·4	2·5	2·6	2·7	2·8	2·9	3·0
(1)	y	4·000	4·400	4·820	5·260	5·719	6·198	6·697	7·216	7·755	8·314	8·892
(2)	y	4·000	4·410	4·840	5·290	5·760	6·250	6·760	7·290	7·840	8·410	9·000

7. $y = x(2 + \log_e x)$

x	1·0	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0
y (formula)	2·000	2·305	2·619	2·941	3·271	3·608	3·952	4·302	4·658	5·020	5·386
y (Euler)	2·000	2·300	2·609	2·926	3·251	3·583	3·922	4·267	4·618	4·975	5·337

8. $x = 1·5$, $y = 3·600$; $x = 2$, $y = 5·371$. Yes, since the graph of y against x is flat and not steep.

9. $y = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{17}{48}x^3 + \frac{17}{384}x^4 + \frac{17}{3\,840}x^5 + \frac{17}{46\,080}x^6 + \ldots$

10. $y = x - \frac{x^3}{3} + \frac{x^5}{15} - \frac{x^7}{105} + \frac{x^9}{945} - \frac{x^{11}}{10\,395} + \ldots$; 0·460, 0·7248

11. $y = 0·7248 + 0·2752(x-1) - 0·5(x-1)^2 + 0·0749(x-1)^3$
$+ 0·1063(x-1)^4 - 0·0362(x-1)^5 + 0·0116(x-1)^6 \ldots$; 0·764, 0·752

12.

x	0	0·2	0·4	0·6	0·8	1·0
y	0	0·197	0·380	0·533	0·650	0·724

13. $y = a_1 \left(x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|2}\underline{|3}} + \frac{x^4}{\underline{|3}\underline{|4}} + \frac{x^5}{\underline{|4}\underline{|5}} + \ldots \right)$

14. $y = a_0 \left(1 + \frac{x^3}{\underline{|3}} + \frac{4x^6}{\underline{|6}} + \frac{4 \,.\, 7x^9}{\underline{|9}} + \ldots \right)$

$$+ a_1 \left(x + \frac{2x^4}{\underline{|4}} + \frac{2 \,.\, 5x^7}{\underline{|7}} + \frac{2 \,.\, 5 \,.\, 8x^{10}}{\underline{|10}} + \ldots \right)$$

17. $y = a_0 \left(1 - \frac{x^2}{\underline{|2}} - \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{4x^5}{\underline{|5}} + \frac{3x^6}{\underline{|6}} - \frac{9x^7}{\underline{|7}} + \ldots \right)$

$$+ a_1 \left(x - \frac{x^3}{\underline{|3}} - \frac{2x^4}{\underline{|4}} + \frac{x^5}{\underline{|5}} + \frac{6x^6}{\underline{|6}} + \frac{9x^7}{\underline{|7}} + \ldots \right);$$

$n(n-1)a_n + a_{n-2} + a_{n-3} = 0$

18. $y = 1 + (x+1) - \frac{1}{\underline{|3}}(x+1)^3 - \frac{2}{\underline{|4}}(x+1)^4 + \frac{4}{\underline{|6}}(x+1)^6$

$$+ \frac{2 \,.\, 5}{\underline{|7}}(x+1)^7 - \frac{4 \,.\, 7}{\underline{|9}}(x+1)^9 - \frac{2 \,.\, 5 \,.\, 8}{\underline{|10}} x^{10} + \frac{4 \,.\, 7 \,.\, 10}{\underline{|12}} x^{12} \ldots$$

20. $y = a_0 \left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \,.\, 4^2} - \frac{x^6}{2^2 \,.\, 4^2 \,.\, 6^2} + \frac{x^8}{2^2 \,.\, 4^2 \,.\, 6^2 \,.\, 8^2} - \ldots \right)$

21. $y = a_0 \left(1 - \frac{4x^2}{\underline{|2}} + \frac{4 \,.\, 10x^4}{\underline{|4}} - \frac{4 \,.\, 10 \,.\, 16x^6}{\underline{|6}} + \ldots \right)$

$$+ a_1 \left(x - \frac{7x^3}{\underline{|3}} + \frac{7 \,.\, 13x^5}{\underline{|5}} - \frac{7 \,.\, 13 \,.\, 19x^7}{\underline{|7}} + \ldots \right);$$

$n(n-1)a_n + (3n-2)a_{n-2} = 0$

26. $y = 0{\cdot}47308$

28. $x = Ae^{6t} \cos(t + \varepsilon)$; $y = 3Ae^{6t} [\sin(t + \varepsilon) + \cos(t + \varepsilon)]$

29. $x = \frac{1 + \sqrt{2}}{2} Ae^{(\sqrt{2} - 4)t} + \frac{1 - \sqrt{2}}{2} Be^{-(\sqrt{2} + 4)t} + \frac{1}{178}(41 \sin 2t$
$- 30 \cos 2t)$

$y = Ae^{(\sqrt{2} - 4)t} + Be^{-(\sqrt{2} + 4)t} + \frac{1}{89}(5 \sin 2t - 8 \cos 2t)$

30. $x = A \sin(3t + \alpha) - B \sin(3\sqrt{2}t + \beta)$; $y = B \sin(3\sqrt{2}t + \beta)$

31. $x = (A + Bt)e^{t} + (C + Dt)e^{-t}$
$y = (A - B + Bt)e^{t} + (C + D + Dt)e^{-t}$

32. $x = 0{\cdot}1408,\ y = 1\ 415$

33. $x = \frac{1}{2}(e^{-3t} + e^{-t}),\ y = \frac{1}{2}(e^{-3t} - e^{-t})$

34. $x = Ae^{\frac{1}{3}t} + Be^{-\frac{1}{3}t},\ y = -\ 2Ae^{\frac{1}{3}t} - Be^{-\frac{1}{3}t} + \frac{1}{2}e^{t}$

35. $x = \frac{1}{80}\,(36 + 10e^{t} - 45e^{-t} - e^{-5t}),\ y = \frac{1}{80}\,(24 + 20e^{t} - 45e^{-t} + e^{-5t})$

36. $x = -\ \frac{1}{27}e^{-3t}(6t + 1) + \frac{1}{27}(3t + 1),\ y = -\ \frac{2}{27}e^{-3t}(3t + 2)$
$+\ \frac{2}{27}(2 - 3t)$

37. (i) $y = e^{x}(A\sin x + B\cos x) - \frac{2}{5}e^{x}\cos 2x$

(ii) $y = A + Be^{-2x} + e^{x},\ z = A - Be^{-2x} + e^{x}$

38. $x = 2\sqrt{2}b\sin\dfrac{t}{\sqrt{2}} + 2a\cos\dfrac{t}{\sqrt{2}},\ y = -\ \sqrt{2}b\sin\dfrac{t}{\sqrt{2}} - a\cos\dfrac{t}{\sqrt{2}}$

39. $z = R_1\sin(3x + \alpha_1) + R_2\sin(4x + \alpha_2)$,
$y = -\ 2 - \frac{1}{2}R_1\sin(3x + \alpha_1) + 3R_2\sin(4x + \alpha_2)$

40. $y = 4(\cos 6t - \cos 3t),\ z = 3(8\cos 6t + \cos 3t)$

41. $x = \dfrac{b}{a^2}(1 - \cos at),\ y = \dfrac{b}{a^2}(at - \sin at)$

42. $x = \dfrac{V}{3}(2t + \sin t),\ y = \dfrac{2V}{3}(t - \sin t);\ \ t = \pi,\ \dfrac{dx}{dt} = \dfrac{V}{3},\ \dfrac{dy}{dt} = \dfrac{4V}{3}$

43. $x = 2(e^{2t} - e^{t}),\ y = 5e^{2t} - 4e^{t}$

EXAMPLES VIII. Page 338

1. (i) $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial y}{\partial z} = 0$

(ii) $5\dfrac{\partial z}{\partial x} + 4\dfrac{\partial z}{\partial y} + 2 = 0$

(iii) $cx\dfrac{\partial z}{\partial x} + dy\dfrac{\partial z}{\partial y} = 0$

2. (i) $\dfrac{\partial^2 z}{\partial y^2} = 16\dfrac{\partial^2 z}{\partial x^2}$

(ii) $u\dfrac{\partial^2 u}{\partial s\partial t} = \dfrac{\partial u}{\partial s}\dfrac{\partial u}{\partial t}$

(iii) $a^2\dfrac{\partial^2 u}{\partial s^2} - 2a\dfrac{\partial^2 u}{\partial s\,\partial t} + \dfrac{\partial^2 u}{\partial t^2} = 0$

3. (i) $2\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial t} + u = 0$

(ii) $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = 0$

(iii) $2\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x\,\partial y} - 2\dfrac{\partial^2 z}{\partial y^2} = 0$

5. (i) $z = f(7x - 10y)$

(ii) $z = f(7x^2 - 10y^2)$

(iii) $z = f(xy)$

6. (i) $z = f\left(\dfrac{y^{10}}{x^7}\right)$

(ii) $z = f_1(x) + f_2(y)$

(iii) $u = f_1(x + ay) + f_2(x - ay)$

(iv) $u = f_1(3x + 2y) + f_2(x - y)$

7. (i) $u = f_1(5x - 2y) + x f_2(5x - 2y)$

(ii) $z = f_1(t + ix) + f_2(t - ix)$

(iii) $z = f_1(\overline{2 + i}\,x + y) + f_2(\overline{2 - i}\,x + y)$

8. (i) $u = f_1(x + y) + f_2(x - y) + f_3(2x - y)$

(ii) $z = f_1(x - y) + x f_2(x - y) + f_3(2ix + y) + f_4(2ix - y)$

(iii) $u = f_1(y - 2ix) + x f_2(y - 2ix) + f_3(y + 2ix) + x f_4(y + 2ix)$

9. $z = f_1(y + cx) + f_2(y - cx)$

12. $z = \frac{1}{3}xy(x^2 + y^2)$; $z = f_1(x) + f_2(y) + \frac{1}{3}xy(x^2 + y^2)$

13. $V = \frac{1}{12}(x^4 - y^4)$

14. (i) $z = f(5x - 3y) + 6(x + y) + \frac{1}{16}(x + y)^2$

(ii) $z = f(7x - 6y) + 2(x + y) + \frac{1}{14}y^2 + \frac{1}{18}x^3$. See Note in Ex. 2, Art. 78.

(iii) $y = \sin\left(\dfrac{n^2}{c^2}t + \alpha\right)(A \sin nx + B \cos nx + C \sinh nx + D \cosh nx)$

$+ \dfrac{a}{q^4 - p^4c^4}\sin qx \sin pt$ where n, α, A, B, C, D, are arbitrary constants

15. $y = f_1(2x - y) + f_2(3x + 2y) + x f_3(3x + 2y) + x^2 f_4(3x + 2y)$

Of many possible particular integrals the following are obvious—

$$y = \tfrac{1}{6}x^4,\ y = \tfrac{2}{5}t^2 \text{ and } y = \tfrac{1}{12}x^4 + \tfrac{1}{5}t^2$$

These are found by inspection. As the two terms are not of the same order the methods of Art. 78 cannot be used.

16. $z = z_0 e^{-a\sqrt{0{\cdot}5p}\,x} \sin (pt - a\sqrt{0{\cdot}5p}\,x)$

17. $z = Ae^{-p^2t} \sin apx$ is a solution where $p = n\pi/al$, n being any positive integer.

$$z = \frac{4z_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{n^2\pi^2 t}{a^2l^2}} \sin n\pi x/l$$

Flow of heat inside a plate of large area whose faces are suddenly cooled to zero temperature and kept at that temperature.

18. Angular frequency $= \dfrac{n^2\pi^2}{l^2}\sqrt{\dfrac{gEI}{w}}$; $n = 1, 2, 3$, etc.

19. Angular frequency $= m^2\sqrt{\dfrac{gEI}{w}}$, $ml = 1{\cdot}88, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}$, etc., approximately.

22. As in Art. 82.

23. As in p. 321 with the necessary change of symbols

24. $y = \dfrac{32h}{\pi^3}\left[\sin\dfrac{\pi x}{l}\cos ct + \dfrac{1}{3^3}\sin\dfrac{3\pi x}{l}\cos 9\,ct + \dfrac{1}{5^3}\sin\dfrac{5\pi x}{l}\cos 25ct + \ldots\right]$

where $c = \sqrt{\dfrac{12gT}{w}}$

26. $y = \dfrac{4\theta_0}{\pi}\left[e^{-\frac{c^2\pi^2t}{d^2}}\sin\dfrac{\pi x}{d} + \tfrac{1}{3}e^{-\frac{9c^2\pi^2t}{d^2}}\sin\dfrac{3\pi x}{d} + \tfrac{1}{5}e^{-\frac{25c^2\pi^2t}{d^2}}\sin\dfrac{5\pi x}{d} + \ldots\right]$

where $c = \sqrt{\dfrac{k}{ws}}$

27. $V = 20\left(1 - \dfrac{x}{l}\right) + \dfrac{24}{\pi}\left[e^{-\frac{\pi^2t}{Rcl^2}}\sin\dfrac{\pi x}{l} - \tfrac{1}{2}e^{-\frac{4\pi^2t}{Rcl^2}}\sin\dfrac{2\pi x}{l} + \ldots\right]$

29. $0{\cdot}960\sqrt{\dfrac{gE}{W}}, 0{\cdot}961\sqrt{\dfrac{gE}{W}}, \sqrt{\dfrac{gE}{W}}$

30. $x^2\dfrac{\partial^2 X}{\partial x^2} + x\dfrac{\partial X}{\partial x} = p^2X;\ \dfrac{\partial^2 Y}{\partial y^2} = -p^2Y;\ u = \dfrac{a^3\cos 2y}{2x^2}$

EXAMPLES IX. Page 387

1. $\frac{2}{7}\sqrt{14W}$ lb, $\sqrt{14}\,\dfrac{W}{14}$ lb **2.** 245 radn/sec², 22·8 ft/sec²

3. 3 310 lb

4. (*a*) $\dfrac{3v}{4a}$ for AB, $\dfrac{9v}{14a}$ for BC, $\dfrac{3v}{14a}$ for CD

(*b*) $\dfrac{3v}{4a}$ for AB, 0 for BC, $-\dfrac{3v}{4a}$ for CD

7. $\frac{2}{3}$ width from hinge. Opposite centre of percussion.

8. (1) $\dfrac{5\sqrt{3}a^4}{16}$; (2) the same **10.** $\frac{3}{8}wa$, $\frac{11}{16}wa$, $\frac{11}{16}wa$, $\frac{3}{8}wa$: $\dfrac{13wa^4}{1\,920EI}$

11. $\frac{11}{28}wa$, $\frac{5}{7}wa$, $\frac{13}{14}wa$, $\frac{5}{7}wa$, $\frac{11}{28}wa$

13. 1 485 lb, 2 076 lb-ft

14. 50 580 lb-ft, 3 380 lb

16. $\frac{a}{2}$ and $\frac{2a}{3}$ depths of C.P. of triangle and square.

17. $\omega = \frac{3v}{8a}$

18. (*a*) $\frac{83W}{128}$; $\frac{15g}{16a}$; (*b*) $\frac{403W}{448}$, $\frac{15g}{56a}$

19. $\frac{1}{3}M(a^2 + b^2)$

25. $h = 1{\cdot}4$ in.

27. Centre of mass on vertical through point of contact at a height $< a \cos \alpha$ for stability.

28. $\omega = \frac{3}{4a}\sqrt{2gh}$ radn./sec, $\frac{W}{4}\sqrt{\frac{17h}{g}}$ 59° 2′ to horizontal, $\frac{3}{8}$ Wh ft-lb

30. $\frac{3Wa^2}{8l(2l + 3a)}$

34. 122

38. $\frac{5a^2s}{24h}$; $h < a\sqrt{\frac{5s}{12(1 - s)}}$

40. 97·5 lb compression

41. $\frac{900g}{\pi^2 N^2}$ ft

42. (*a*) 28·8 r.p.m., (*b*) 3 770 ft-lb
(*c*) 1 937 ft-lb

43. 0·867 ft

44. $Z = 0{\cdot}204r^2$, units in inches; 416 in.^{3}

46. 1 045$\frac{1}{3}$; $-$ 14·5

47. 258$\frac{2}{3}$; 191$\frac{1}{5}$

51. 2$\frac{2}{3}$

52. 0

53. 49·1°, 63·5°, 4·309 ft

54. 76·0 ft, 28·9 ft

56. 545 ft, $\frac{185}{546}$ or nearly $\frac{1}{3}$, 1·002 to 1

EXAMPLES X. Page 435

1. (i) 10·49, 30° 55′, 68° 12′; (ii) 9·110, 39° 47′, $-$ 120° 58′;
(iii) 4·123, 104° 2′, 0°; (iv) 11·58, 30° 17′, $-$ 30° 58′

2. (i) 3, 5·196, 10·39; (ii) $-$ 2·654, 1·532, 2·571; (iii) 1·314, 0·5851, $-$ 1·389

3. For OP, $l = 0{\cdot}8137$, $m = -0{\cdot}3487$, $n = 0{\cdot}4650$; for OQ, $l = 0{\cdot}3075$, $m = 0{\cdot}8447$, $n = 0{\cdot}4384$

4. If A, B, C are the vertices of the triangle in the order given, length $AB = 3{\cdot}742$, $BC = 15{\cdot}30$, $CA = 11{\cdot}83$. Angles between the sides and the axes of x, y, z respectively are: For AB, 74° 30′, 57° 41′, 36° 42′; for BC, 62° 46′, 74° 51′, 31° 50′; for CA, 59° 31′, 80° 16′, 32° 18′

5. (i) 25·50; (ii) 6, 5, 3

6. $x - 3 = \dfrac{y + 4}{3} = \dfrac{z - 15}{-4}$

7. 0·5657; 45°, 115° 6′, 55° 33′

8. $2x - 2y - 3z = 14$; 3·395; 0·4850, − 0·4850, − 0·7276

9. $2x + 7y - 9z - 6 = 0$; $8x - y + z - 12 = 0$

10. 0·2721; $\dfrac{x + 3}{7} = \dfrac{y - 1}{-1} = \dfrac{z - 5}{2}$; − 3·259, 1·037, 4·926

11. $x + 8y - 6z + 16 = 0$; the acute angles between the given plane and the co-ordinate planes are 53° 20′, 84° 17′, 37° 13′

12. (i) 0·1720; (ii) 0·3884

13. $AD = 32{\cdot}09$ ft, $BD = 29{\cdot}32$ ft, $CD = 27{\cdot}49$ ft; $\widehat{ADB} = 34° 9′$, $\widehat{BDC} = 41° 19′$, $\widehat{CDA} = 41° 33′$

14. − 0·5774, 0·8083, 0·1155; 90°

15. $21x + 4y - 13z = 3$; $\dfrac{x}{21} = \dfrac{y}{4} = \dfrac{13z + 3}{457}$

16. $\dfrac{7\sqrt{30}}{90} = 0{\cdot}4260$; $-\dfrac{11\sqrt{30}}{90} = -0{\cdot}6694$; $\dfrac{\sqrt{30}}{9} = 0{\cdot}6086$

17. $\dfrac{11\sqrt{42}}{126} = 0{\cdot}5658$; $\dfrac{8\sqrt{42}}{63} = 0{\cdot}8229$; $-\dfrac{\sqrt{42}}{126} = -0{\cdot}0514$: 21° 47′

18. (i) 86° 23′; (ii) 58°

19. Let h = height of pyramid and a = side of base, and let the positive direction of the z-axis be taken from O towards the base. Then $2h = a\sqrt{\cot^2\dfrac{\alpha}{2} - 1}$; the equations of the faces are $\sqrt{2}h(x + y) \pm az = 0$, $\sqrt{2}h(x - y) \pm az = 0$; the equations of the edges are $\dfrac{\sqrt{2}x}{\pm a} = \dfrac{y}{0} = \dfrac{z}{h}, \dfrac{x}{0} = \dfrac{\sqrt{2}y}{\pm a} = \dfrac{z}{h}$; the angle between a pair of opposite edges $= \cos^{-1}\left(\dfrac{2h^2 - a^2}{2h^2 + a^2}\right)$

20. $3\frac{3}{7}$ in.

21. 60° 39′; $\dfrac{x - 3}{0{\cdot}5299} = \dfrac{y + 3}{0{\cdot}4901} = \dfrac{z - 7}{-0{\cdot}6947}$

22. 18° 10′

23. $\dfrac{11}{\sqrt{65}} = 1{\cdot}364$; $x = 2\frac{54}{65}$, $y = -2\frac{11}{13}$, $z = 8\frac{32}{65}$

24. 28° 53′

25. $x - 4y + 2z + 4 = 0$

26. 43° 3′; 5·746

27. 2·873

29. 1·946

30. − 0·6052, 0·6774, 0·4179

31. $x = 2, y = -1, z = 2$; 0·2063 **32.** 153·8 ft^2

33. 5·5 **34.** $7\frac{1}{6}$

35. If P, Q, R, S are the angular points of the tetrahedron in the order given, then length $PQ = \sqrt{2}$; $PR = \sqrt{5}$; $PS = \sqrt{13}$; $QR = 1$; $QS = 3$; $RS = \sqrt{6}$. The direction-cosines of PQ are $-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$; of PR, $-\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}$; of PS, $0, \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}$; of QR, $0, 0, 1$; of QS $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$; of RS, $\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$. The equation of the face PQR is $y = 0$; of the face PRS, $4x - 3y + 2z = 4$; of the face PSQ, $2x - 3y + 2z = 2$; of the face QRS, $2x - y = 0$. The volume of the tetrahedron $= \frac{1}{3}$.

36. 24

37. $x^2 + y^2 + z^2 + 12x - 2y - 6z + 30 = 0$; $\frac{40\pi}{3} = 41{\cdot}89$

38. $x + 2 = 0$; $y = 1, z = 3$

39. $19(x^2 + y^2 + z^2) - 33x + 9y + 15z = 374$

40. Centre at point (2, 1, 2); radius $= \sqrt{7} = 2{\cdot}646$; area of projection on xy-plane $= \frac{14\pi}{3} = 14{\cdot}66$

41. $4x + 3y + 4z = 98$; $\frac{x-2}{4} = \frac{y-6}{3} = \frac{z-18}{4}$

42. (i) If (x, y, z) is any point on the cylindrical surface, then by (X.38) $4 = (x-1)^2 + y^2 + (z-3)^2 - \frac{1}{6}\{2(x-1) + y - (z-3)\}^2$, which reduces to $2x^2 + 5y^2 + 5z^2 - 4xy + 2yz + 4zx - 16x - 2y - 34z + 35 = 0$. This is the equation of the surface. (ii) $2, 4\sqrt{3}$

43. $x = -4, y = 7, z = -14$

44. $16x - 36y + 36z = 61$; $4x - 9y + 9z \pm 3\sqrt{61} = 0$

45. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\left(\frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} - 1\right) = \left(\frac{px}{a^2} + \frac{qy}{b^2} + \frac{rz}{c^2}\right)^2$

46. $6x + 15y - 10z = 0$

47. $2x + 4y + 4z = 3$; $4x + 2y - 4z + 3 = 0$; $2x - 2y + z = 3$

48. $2x - y + z + 2\sqrt{34} = 0$

49. $x = 3, y = -1, z = 1$; $2\sqrt{3}, 3\sqrt{3}, \sqrt{3}$

50. $2x + 3y - 4z = \pm\sqrt{17}$ **51.** $2\frac{\sqrt{5}}{5} = 0{\cdot}8944$

52. $x + y + z = 7$ **53.** $\frac{2x}{3} = 2y + 1 = \frac{2z + 13}{5}$

54. $x^2 + y^2 + z^2 - 8x - 2y - 13z + 47 = 0;$
$16x + 4y + 26z = 237 \pm 7\sqrt{237}$

55. $(x - \alpha)(x_1 - \alpha) + (y - \beta)(y_1 - \beta) + (z - \gamma)(z_1 - \gamma) = r^2;$
$(x + \frac{5}{4})^2 + (y - \frac{25}{8})^2 + (z - \frac{1}{4})^2 = \frac{729}{64}$

56. $\frac{x}{2} = \frac{y}{6} = \frac{z}{5}$; $\cos^{-1}\frac{16}{29} = 56^\circ\ 31'$

57. $3(2 \pm \sqrt{2})x + (2 \mp 3\sqrt{2})y + 2(1 \pm 3\sqrt{2})z = 10;$

$5x + 3y = 10;\ \frac{5\sqrt{34}}{17} = 1{\cdot}716$

58. $\sqrt{(f - \alpha)^2 + (g - \beta)^2 + (h - \gamma)^2 - \{l(f - \alpha) + m(g - \beta) + n(h - \gamma)\}^2};$
$x^2 + 7y^2 + z^2 + 8xy - 16zx + 8yz = 0$

59. $x^2 + y^2 + (z - 3)^2 = 13$

60. Shortest distance from z-axis $= c$

EXAMPLES XI. Page 461

3. $a = 66^\circ\ 36',\ B = 54^\circ\ 20',\ C = 61^\circ\ 3'$
4. $a = 82^\circ\ 55',\ b = 80^\circ\ 54',\ c = 39^\circ\ 0'$
5. $A = 69^\circ\ 17',\ a = 59^\circ\ 49',\ b = 40^\circ\ 35'$
6. $b = 36^\circ\ 3',\ c = 11^\circ\ 24',\ C = 19^\circ\ 38'$
7. $B = 70^\circ\ 38',\ C = 76^\circ\ 5',\ c = 75^\circ\ 14'$
8. $A = 69^\circ\ 26',\ b = 42^\circ\ 9',\ c = 68^\circ\ 47'$
10. $a = 104^\circ\ 29',\ B = 26^\circ\ 33',\ C = 63^\circ\ 25'$; $0{\cdot}5229 : 4\pi = 0{\cdot}0416 : 1$
12. $c = 40^\circ\ 0'\ 0'',\ A = 121^\circ\ 36'\ 26'',\ B = 42^\circ\ 15'\ 0''$
13. $4'\ 1''$
14. 3 hr 45 min (nearly) after noon at the place.
15. 50° nearly.
16. $a = 55^\circ\ 12',\ b = 48^\circ\ 19',\ c = 43^\circ\ 8'$
17. $A = 61^\circ\ 32',\ b = 9^\circ\ 12',\ c = 17^\circ\ 36'$
18. $a = 53^\circ\ 9',\ c = 120^\circ\ 18',\ C = 129^\circ\ 6'$
19. $A = 90^\circ\ 36',\ B = 54^\circ\ 40',\ C = 49^\circ\ 5'$
20. $A = 58^\circ\ 39',\ B = 66^\circ\ 11',\ c = 62^\circ\ 44'$
21. $A = 66^\circ\ 44',\ B = 57^\circ\ 51',\ C = 102^\circ\ 8'$
22. $a = 65^\circ\ 32',\ B = 72^\circ\ 3',\ C = 92^\circ\ 36'$
23. $a = 33^\circ\ 11',\ b = 54^\circ\ 10',\ c = 60^\circ\ 4'$
24. $B = 79^\circ\ 27',\ A = 40^\circ\ 34',\ a = 34^\circ\ 47'$
25. 442 miles nearly.
26. $3{\cdot}411 \times 10^6$ square miles.

INDEX

(The numerals refer to the pages)